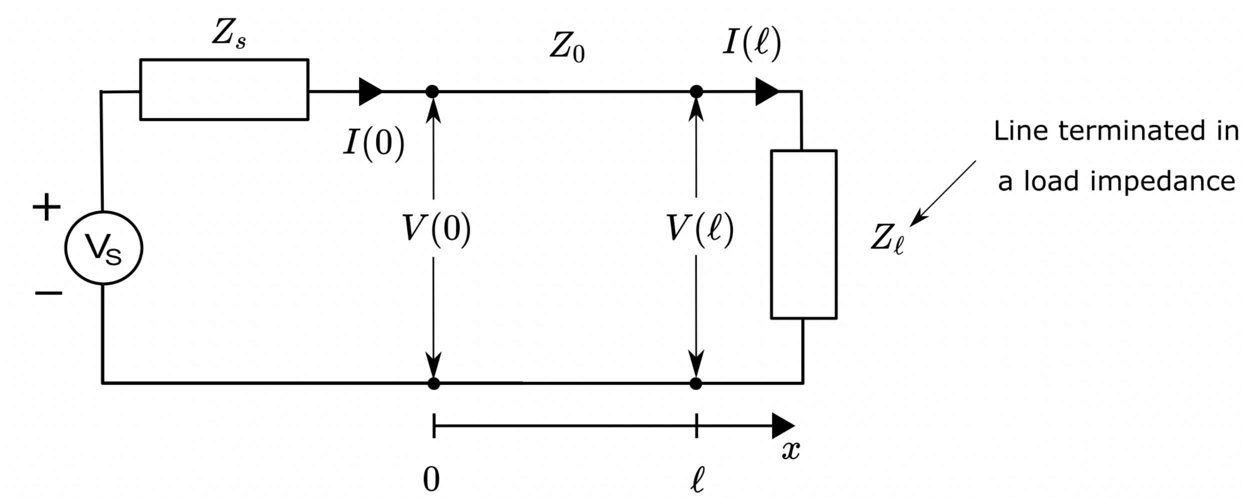
Lecture 4

Lecturer: G. S. Paraoanu

Department of Applied Physics, School of Science,

Aalto University, P.O. Box 15100, FI-00076 AALTO, Finland

I. RECAP FROM PREVIOUS LECTURE



- Reflection and transmission coefficients

$$\Gamma_V = \frac{Z_l - Z_0}{Z_l + Z_0} = -\Gamma_l \tag{1}$$

- Average power delivered to the load

$\overline{P_\ell} = \frac{1}{2}\text{Re}[V(\ell)I^*(\ell)]$, where the 1/2 comes from the fact that the field is harmonic.

$$\overline{P_\ell} = \frac{1}{2Z_0} \cdot |V^+ e^{-\gamma\ell}|^2 (1 - |\Gamma_V|^2) \ . \tag{2}$$

- VSWR

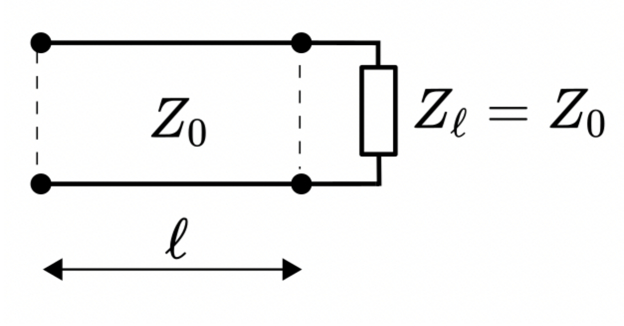
$$VSWR = \frac{1 + |\Gamma_V|}{1 - |\Gamma_V|}. \tag{3}$$

- Input impedance

$$Z_{in} = Z_0 \cdot \frac{Z_\ell + Z_0 \tanh \gamma\ell}{Z_0 + Z_\ell \tanh \gamma\ell}. \tag{4}$$

II. EXAMPLES OF LOADS (TERMINATIONS)

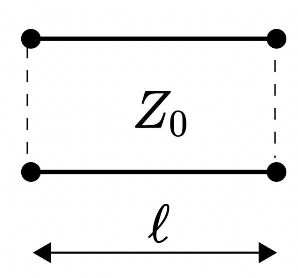
1. Matched Load

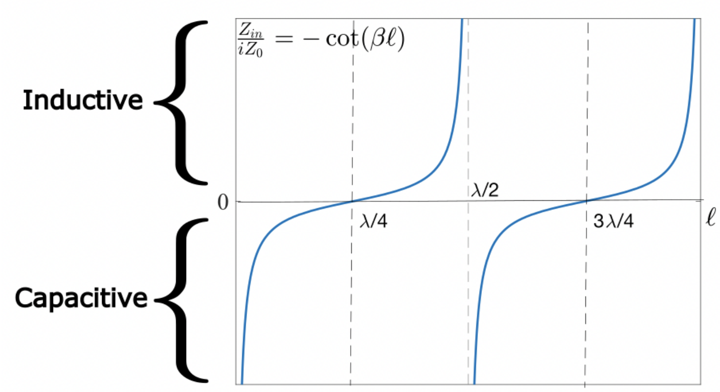


$Z_\ell = Z_0 \implies \Gamma_V \equiv \frac{Z_\ell - Z_0}{Z_\ell + Z_0} = 0$ No reflection!

VSWR $= 1$, $Z_{in} = Z_0$, $P_\ell = \frac{1}{2Z_0}|V^+|^2 e^{-2\alpha\ell}$ — power delivered is maximum. This is only obtained if $\alpha \neq 0$.

2. Open-Circuit $Z_\ell = \infty \implies \Gamma_V = \frac{Z_\ell - Z_0}{Z_\ell + Z_0} = 1$

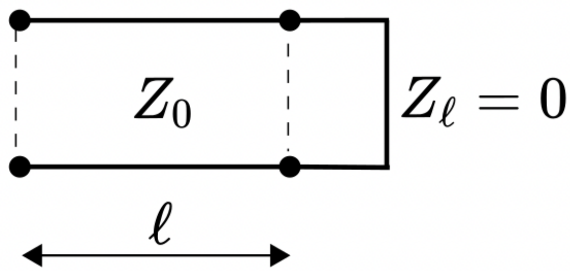


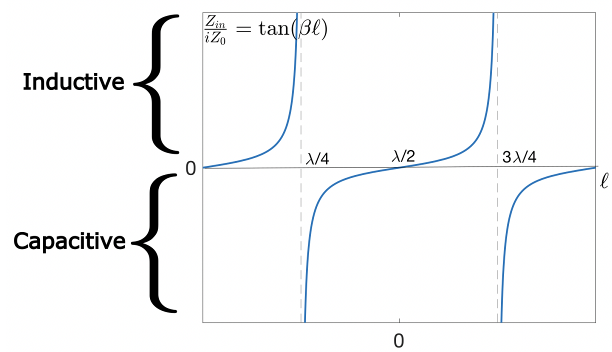


VSWR $= \infty$, $Z_{in} = Z_0 \coth \gamma\ell$, $P_\ell = 0$ — Compare this with the DC-case where all the input power is delivered!

For $\alpha = 0$ (lossless), $Z_{in} = -iZ_0 \cot \frac{2\pi\ell}{\lambda}$ if $\ell = \frac{\lambda}{4}$, $Z_{in} = 0$, so the open line will look as a shortcut!

3. Short-circuit $Z_\ell = 0 \implies \Gamma_V = \frac{Z_\ell - Z_0}{Z_\ell + Z_0} = -1$,



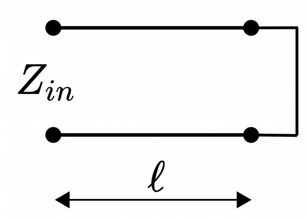


VSWR $= \infty$, $\quad Z_{in} = Z_0 \tanh \gamma\ell$, $\quad P_\ell = 0$.

For $\alpha = 0$ (lossless), $\quad \beta = \frac{2\pi}{\lambda}$ quad $Z_{in} = iZ_0 \tan \frac{2\pi\ell}{\lambda}$ — If $\ell = \frac{\lambda}{4}$, $\quad Z_{in} = \infty$, so the shorted line looks like an infinite impedance to a source! (even if the resistance of the wire is zero!)

III. RESONATORS FROM TRANSMISSION LINES

It is possible to make resonators from transmission lines, 3D cavities, etc. – The most usual case is the short-circuited transmission-line resonator.



$Z_\ell = 0, \quad Z_{in} = Z_0 \tanh(\alpha\ell + i\beta\ell) = Z_0 \frac{\tanh\alpha\ell + i\tan\beta\ell}{1 + i\tan\beta\ell\tanh\alpha\ell}$.

If losses are not too large, $\alpha\ell \ll 1$, we have $\tan\alpha\ell \approx \alpha\ell$, so

$$Z_{in} = Z_0 \frac{\alpha\ell + i\tan\beta\ell}{1 + i\alpha\ell\tan\beta\ell} \ . \tag{5}$$

Now, recall from the previous lecture that $\beta = \omega/v_p = \omega\sqrt{L'C'}, \quad v_p = 1/\sqrt{L'C'}, \quad Z_0 = \sqrt{L'/C'}$,
$\alpha = \frac{R'}{2}\sqrt{C'/L'}$. We also take $G' = 0$.

We will consider $\underline{\beta_0\ell = \pi}$, or $\ell = \lambda_0/2$ as the resonance condition, leading to a resonance frequency ω_0.

We can find this frequency from $\frac{\omega_0}{v_p}\ell = \omega_0\sqrt{L'C'}\ell = \pi$, so $\omega_0 = \frac{\pi}{\ell\sqrt{L'C'}}$. Let's check that everything is O.K.: so we get $\omega_0 = 2\pi \times \nu_0$, where $\nu_0 = v_p/\lambda_0 = v_p/(2l)$ = frequency of oscillation = inverse of period of oscillation.

We can expand this around this point: $\tan\beta\ell \simeq \tan\left(\pi + \pi\frac{\omega-\omega_0}{\omega_0}\right) = \tan\pi\frac{\omega-\omega_0}{\omega_0} \simeq \pi\frac{\omega-\omega_0}{\omega_0}$, if $|\omega - \omega_0| \ll \omega_0$.

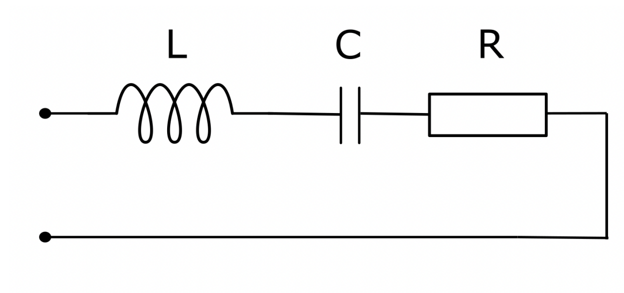
So, $\quad Z_{in} = Z_0 \frac{\alpha\ell + i\pi\frac{\omega-\omega_0}{\omega_0}}{1 + \alpha\ell\pi\frac{\omega-\omega_0}{\omega_0}} \simeq Z_0(\alpha\ell + i\pi\frac{\omega-\omega_0}{\omega_0})$
$= \sqrt{L'/C'}(\frac{\ell}{2}R'\sqrt{C'/L'} + i\ell\sqrt{L'C'}(\omega - \omega_0)) = \frac{1}{2}R'\ell + iL'\ell(\omega - \omega_0)$.

Suppose now that we look back at the series RLC circuit

$Z = R + i\frac{L}{\omega}(\omega^2 - \omega_0^2) \simeq R + 2iL(\omega - \omega_0)$ near resonance, $\omega \simeq \omega_0$.

Therefore, we can identify $\underline{R = \frac{1}{2}R'\ell}$ and $\underline{L = \frac{1}{2}L'\ell}$.

$$\text{Quality Factor} \quad Q = \frac{\omega_0 L}{R} = \frac{\omega_0 L'}{R'} = \frac{\beta_0}{2\alpha}. \tag{6}$$



Interesting question to think about: Why do we get the factor 1/2 in the RLC equivalent above?

–Answer: Because the current in the short-circuited line is half a sinusoid , therefore we obtain only half of the total resistance and inductance of the full length ℓ.

To see this explicitly, let us write the solution

$$\begin{cases} V(x) \simeq V^+ e^{-i\beta x} + V^- e^{i\beta x} \text{ — here we neglect } \alpha. \\ I(x) \simeq -\frac{\beta}{\omega L}(-V^+ e^{-i\beta x} + V^- e^{i\beta x}) \end{cases} \tag{7}$$

Since $I(0) = 0 \implies V^+ \equiv V^-$ at this point (also you can see that $\Gamma_V \equiv \frac{V^-}{V^+} e^{2i\beta_0 \ell} = -1$ and $\beta_0 \ell = \pi$).

So

$$\begin{cases} V(x) = 2V^+ \cos \beta_0 x \\ I(x) = -\frac{2i\beta}{\omega L} V^+ \sin \beta_0 x = I^+ \sin \beta_0 x \ . \end{cases} \tag{8}$$

Therefore the magnetic-field energy:

$$\overline{W_{L'}} = \int_0^{\lambda_0/2} dx \cdot \frac{1}{4} L^{'} |I(x)|^2 = \frac{1}{4} |I^+|^2 L' \int_0^{\lambda_0/2} \sin^2 \beta_0 x dx = \frac{\lambda_0}{16} \cdot |I^+|^2 L'. \tag{9}$$

At resonance: $\overline{W_{C'}} = \overline{W_{L'}}$, so

$$\overline{W} = \overline{W_{C'}} + \overline{W_{L'}} = \frac{\lambda_0}{8} L' |I^+|^2 \ . \tag{10}$$

$\overline{P} = \frac{1}{2}\int_0^{\lambda_0} dx \cdot R'|I(x)|^2 = \frac{R'}{2}|I^+|^2 \int_0^{\lambda_0} \sin^2 \beta_0 x dx,$

so

$$\overline{P} = \frac{\lambda_0}{8} R'|I^+|^2 \ . \tag{11}$$

Therefore,

$$Q = \frac{\omega_0 \overline{W}}{\overline{P}} = \frac{\omega L^{'}}{R'} \ , \tag{12}$$

or: $\quad Q = \frac{\pi}{\ell R'}\sqrt{\frac{L'}{C'}} = \frac{\pi Z_0}{\ell R'} = \frac{\pi}{2\ell\alpha}$, where we used $\alpha \simeq \frac{R}{2Z_0}$, $\quad \omega_0 = \frac{\pi}{\ell\sqrt{L'C'}}$.

References

- David M. Pozar — Microwave Engineering.
- R.E. Collin — Foundations for Microwave Engineering.