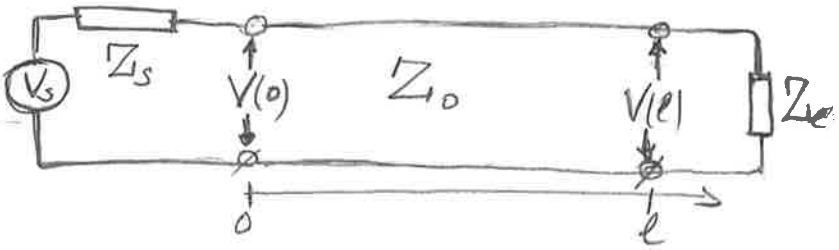
Resonators from transmission lines

• Review from last lecture:



$$\Gamma_V^{\ell} = \frac{Z_\ell - Z_0}{Z_\ell + Z_0} = -\Gamma_I^{\ell}$$

$$T_V^{\ell} = 1 + \Gamma_V^{\ell} \qquad T_I^{\ell} = 1 + \Gamma_I^{\ell}$$

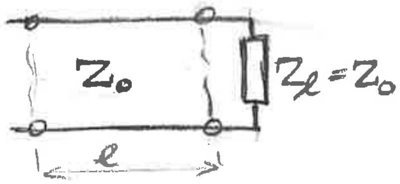
$$\overline{P}_\ell = \frac{1}{2Z_0} \left|V^+ e^{-\gamma\ell}\right|^2 \left(1 - |\Gamma_V^{\ell}|^2\right)$$

$$VSWR = \frac{1 + |\Gamma_V^{\ell}|}{1 - |\Gamma_V^{\ell}|} \qquad Z_{in} = Z_0 \cdot \frac{Z_\ell + Z_0 \tanh \gamma\ell}{Z_0 + Z_\ell \tanh \gamma\ell}$$

EXAMPLES OF LOADS (TERMINATIONS)

1) Matched load

$Z_\ell = Z_0$



$$\Rightarrow \Gamma_V^{\ell} \equiv \frac{Z_\ell - Z_0}{Z_\ell + Z_0} = 0$$

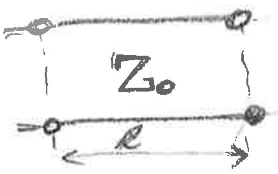
No reflection!

$VSWR = 1$, $P_\ell = \frac{1}{2Z_0} |V^+|^2 e^{-2\alpha\ell}$ — powered delivered is maximal, it is only attenuated if $\alpha \neq 0$

$Z_{in} = Z_0$

2) Open-circuit

$Z_\ell = \infty$



$$\Gamma_V^{\ell} = \frac{Z_\ell - Z_0}{Z_\ell + Z_0} = 1$$

$VSWR = \infty$

$P_\ell = 0$ — compare this with the DC-case where all input power is delivered!

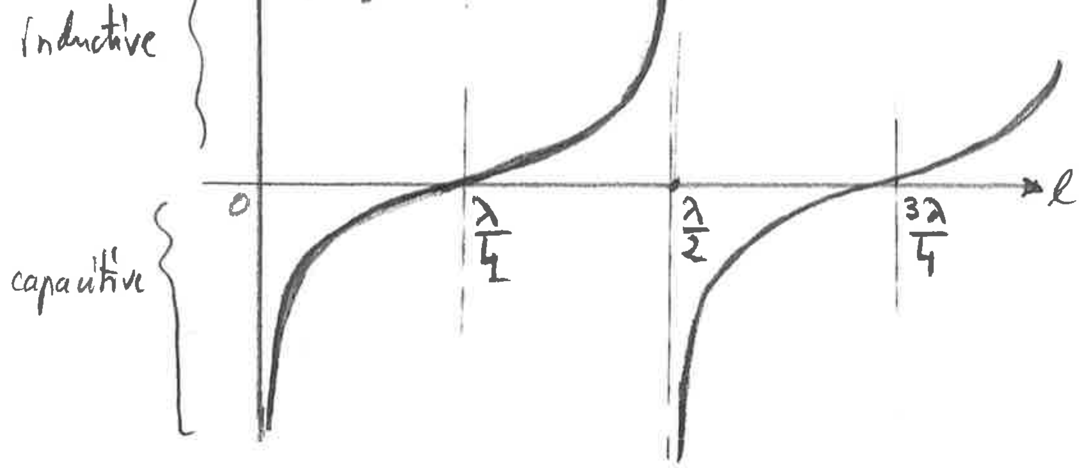
$$Z_{in} = Z_0 \coth \gamma\ell$$

For $\alpha = 0$ (lossless)
$Z_{in} = -i Z_0 \cot\!an \frac{2\pi\ell}{\lambda}$

if $\ell = \frac{\lambda}{4}$, $Z_{in} = 0$
so the open line will look as a shortcut!

$$\frac{Z_{in}}{iZ_0} = -\cot(\beta l)$$

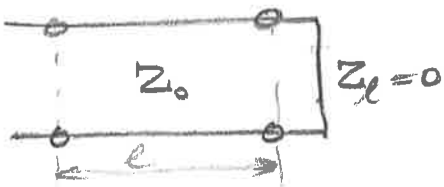
$\beta = \frac{2\pi}{\lambda}$



3) Short-circuit

$Z_l = 0$

$\Gamma_v = \dfrac{Z_l - Z_0}{Z_l + Z_0} = -1$



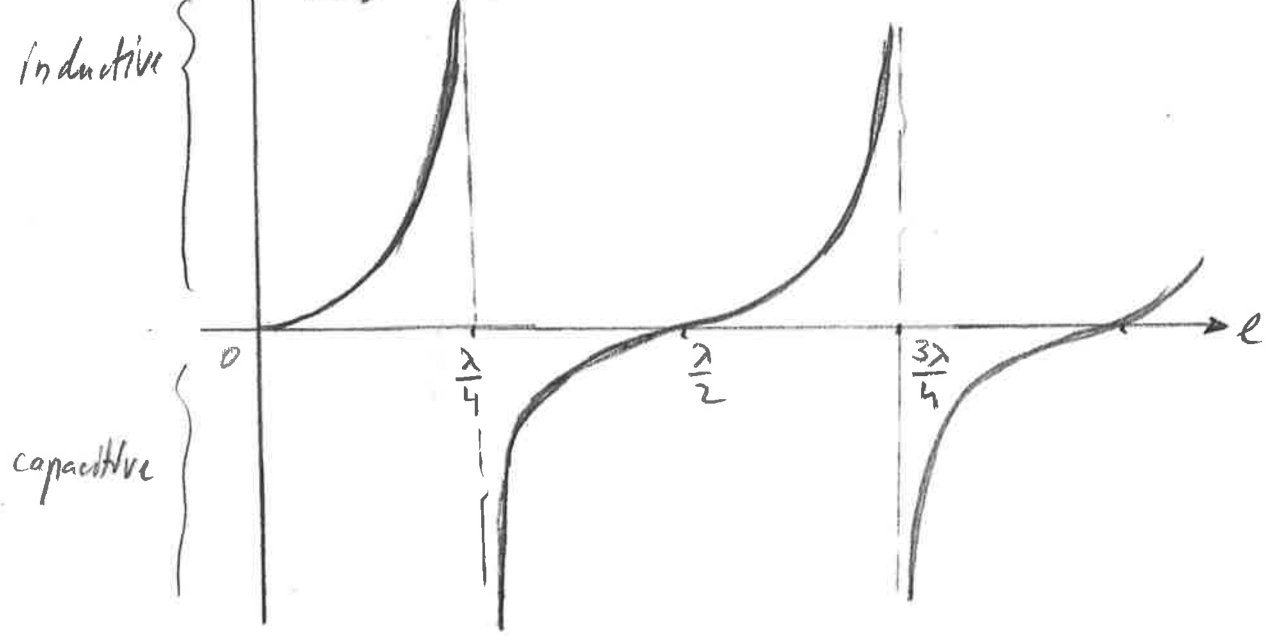
VSWR $= \infty$ $\quad P_l = 0$

$Z_{in} = Z_0 \tanh \gamma l$ $\quad$ For $\alpha = 0$ (lossless) $\quad \beta = \frac{2\pi}{\lambda}$

$$Z_{in} = i Z_0 \tan \frac{2\pi l}{\lambda}$$

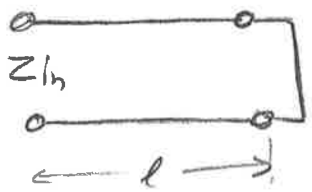
if $l = \frac{\lambda}{4}$, $Z_{in} = \infty$
so the shorted line looks like an infinite impedance to a source!
(even if the resistance of the wires is zero!)

$$\frac{Z_{in}}{iZ_0} = \tan(\beta l)$$



Resonators from transmission lines

- It is possible to make resonators from transmission lines, 3D cavities, etc.
- The most usual case is the short-circuited transmission-line resonator.



$$Z_\ell = 0$$

$$Z_{in} = Z_0 \tanh(\alpha\ell + i\beta\ell)$$

$$= Z_0 \frac{\tanh\alpha\ell + i\tan\beta\ell}{1 + i\tan\beta\ell\tanh\alpha\ell}$$

If losses are not too large, $\alpha\ell \ll 1$, we have $\tan\alpha\ell \simeq \alpha\ell$

so
$$Z_{in} = Z_0 \frac{\alpha\ell + i\tan\beta\ell}{1 + i\alpha\ell\tan\beta\ell}$$

Now, recall that $\beta = \frac{\omega}{v_p} = \omega\sqrt{L'C'}$ $\quad v_p = \frac{1}{\sqrt{L'C'}}$ $\quad Z_0 \simeq \sqrt{\frac{L'}{C'}}$ $\quad \alpha \simeq \frac{R'}{2}\sqrt{\frac{C'}{L'}}$

see Lecture 3

we also take $G' = 0$

We will consider $\underline{\beta_0\ell = \pi}$ or $\ell = \frac{\lambda_0}{2}$

as the resonance condition, leading to a resonance frequency ω_0 $\quad \frac{\omega_0}{v_p}\ell = \omega_0\sqrt{L'C'}\,\ell = \pi$

$$\text{so } \omega_0 = \frac{\pi}{\ell\sqrt{L'C'}}$$

You can see that $\omega_0 = 2\pi \times \nu_0$ where $\nu_0 = \frac{v_p}{\lambda_0} = \frac{v_p}{2\ell}$ so everything is consistent.

ν_0 = oscillation frequency

We can expand around this point

$$\tan \beta l \simeq \tan\left(\pi + \pi \frac{\omega-\omega_0}{\omega_0}\right) = \tan \pi \frac{\omega-\omega_0}{\omega_0} \simeq \pi \frac{\omega-\omega_0}{\omega_0}$$

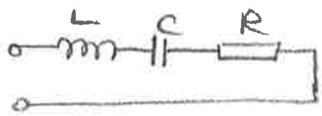
if $|\omega-\omega_0| \ll \omega_0$

So

$$Z_{in} = Z_0 \frac{\alpha l + i\pi \frac{\omega-\omega_0}{\omega_0}}{1+\alpha l \pi \frac{\omega-\omega_0}{\omega_0}} \simeq Z_0 \left(\alpha l + i\pi \frac{\omega-\omega_0}{\omega_0}\right)$$

$$= \sqrt{\frac{L'}{C'}}\left(\frac{l}{2}R'\sqrt{\frac{C'}{L'}} + il\sqrt{L'C'}\,(\omega-\omega_0)\right) = \frac{1}{2}R'l + iL'l\,(\omega-\omega_0)$$

- Suppose now that we look back at a series RLC circuit



$$Z = R + i\frac{L}{\omega}(\omega^2-\omega_0^2)$$

$$\simeq R + 2iL(\omega-\omega_0) \text{ near resonance }, \omega \simeq \omega_0$$

Therefore we can identify $R = \frac{1}{2}R'l$ and $L = \frac{1}{2}L'l$

Quality factor $Q = \frac{\omega_0 L}{R} = \frac{\omega_0 L'}{R'} = \frac{\beta_0}{2\alpha}$

- Interesting question to think about: why do we get the factor 1/2 in the RLC equivalent above?

Because the current in the short-circuited line is half a sinusoid, therefore we obtain only half of the total resistance and inductance of the full length l.

To see this explicitly, let us write the solution

$$\begin{cases} V(x) \simeq V^+ e^{-i\beta x} + V^- e^{i\beta x} & \text{— here we neglect } \alpha \\ I(x) \simeq -\frac{\beta}{\omega L}\left(-V^+ e^{-i\beta x} + V^- e^{i\beta x}\right) \end{cases}$$

Since $I(0) = 0 \Rightarrow V^+ \equiv V^-$ at this point

(also you can see that $\Gamma_V \equiv \frac{V^-}{V^+} e^{+2i\beta_0 l} = -1$ and $\beta_0 l = \pi$)

So $\begin{cases} V(x) = 2V^+ \cos\beta_0 x \\ I(x) = -\dfrac{2i\beta}{\omega L} V^+ \sin\beta_0 x = I^+ \sin\beta_0 x \end{cases}$

Therefore the magnetic field energy

$$\overline{W}_{L'} = \int_0^{\lambda_0/2} dx \cdot \frac{1}{4} L' |I(x)|^2 = \frac{1}{4} |I^+|^2 L' \int_0^{\frac{\lambda_0}{2}} \sin^2\beta_0 x \, dx$$

$$= \frac{\lambda_0}{16} \cdot |I^+|^2 L'$$

$\overline{W}_{C'} = \overline{W}_{L'}$ at resonance so $\overline{W} = \overline{W}_{C'} + \overline{W}_{L'} = \dfrac{\lambda_0}{8} L' |I^+|^2$

$$\overline{P} = \frac{1}{2} \int_0^{\lambda_0/2} dx \cdot R' |I(x)|^2 = \frac{R'}{2} |I^+|^2 \int_0^{\frac{\lambda_0}{2}} \sin^2\beta_0 x \, dx$$

so $\overline{P} = \dfrac{\lambda_0}{8} R' |I^+|^2$

Therefore $\underline{Q = \dfrac{\omega_0 \overline{W}}{\overline{P}} = \dfrac{\omega_0 L'}{R'}}$

or: $Q = \dfrac{\pi}{\ell R'} \sqrt{\dfrac{L'}{C'}} = \dfrac{\pi Z_0}{\ell R'} = \dfrac{\pi}{2\ell\alpha}$

where we used $\alpha \simeq \dfrac{R}{2Z_0}$, $\omega_0 = \dfrac{\pi}{\ell\sqrt{L'C'}}$

REFERENCES

- David M. Pozar – Microwave Engineering
- R. E. Collin – Foundations for Microwave Engineering