6 Finite element methods for the Euler–Bernoulli beam problem
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2. Energy methods and basic 1D finite element methods
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6 Finite element methods for the Euler–Bernoulli beam problem

Contents
1. Strong and weak forms for Euler–Bernoulli beams
2. Finite element methods for Euler–Bernoulli beams

Learning outcome
A. Understanding of the basic properties of the Euler–Bernoulli beam problem and ability to derive the basic formulations related to the problem
B. Basic knowledge and tools for solving Euler–Bernoulli beam problems by finite element methods – with $C^1$ elements, in particular

References
Lecture notes: chapter 9.1
Text book: chapters 1.16, 3.1, A1.I
6.0 Motivation for the Euler-Bernoulli beam element analysis

Beam structures (frames, trusses, beams, arches) are the most typical structural parts in modern structural engineering — and the Euler–Bernoulli beam element is the most typical one in commercial FEM software.
6.1 Strong and weak forms for Euler–Bernoulli beams

Let us consider a thin straight beam structure subject to such a loading that the deformation state of the beam can be modeled by the bending problem in a plane. The basic *kinematical dimension reduction assumptions* of a *thin beam*, called *Euler–Bernoulli beam* (1751), i.e.,

(K1) normal fibres of the beam axis remain straight during the deformation
(K2) normal fibres of the beam axis do not stretch during the deformation
(K3) material points of the beam axis move in the vertical direction only
(K4) normal fibres of the beam axis remain as normals during the deformation

\[ w(x) = \frac{dw(x)}{dx} = \beta(x) \]
6.1 Strong and weak forms for Euler–Bernoulli beams

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(K4) normal fibres of the beam axis remain as normals during the deformation

come true if the displacements are presented as

\[ v(x, y) = v(x, 0) - y(1 - \cos(w'(x))) \approx v(x, 0) =: w(x), \]
\[ u(x, y) = -y \sin(w'(x)) \approx -yw'(x), \]

with \( w \) denoting the *deflection* of the beam (central or neutral axis), appearing as the only variable of the problem (with the current assumptions) and, furthermore, depending on the \( x \) coordinate only.
6.1 Strong and weak forms for Euler–Bernoulli beams

Considering linear deformations the displacement field above implies the axial strain

$$\varepsilon_x (x, y) = \frac{\partial u(x, y)}{\partial x} = -yw''(x).$$
6.1 Strong and weak forms for Euler–Bernoulli beams

Linear deformations for the displacement field above implies the axial strain (alone)

\[ \varepsilon_x(x, y) = \frac{\partial u(x, y)}{\partial x} = -yw''(x). \]

Defining the bending moment as (without specifying the stresses at the moment)

\[ M(x) := M_z(x) := \int_{A(x)} \sigma_x(x, y, z) y \, dA \]
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the energy balance of the principle of virtual work can be written in the form

\[ 0 = \delta W_{\text{int}} + \delta W_{\text{ext}} \]

\[ = -\int_V \sigma : \delta \varepsilon \, dV + \int_V b \cdot \delta u \, dV + \int_{s_l} t \cdot \delta u \, dS \quad \text{3D elasticity theory!} \]

\[ = -\int_L \int_A \sigma_x \delta \varepsilon_x \, dA \, dx + \int_{s_l} t_y \delta v \, dS \quad \text{1D + 2D} \]
6.1 Strong and weak forms for Euler–Bernoulli beams

\[ \int_0^L (\int_0^L \sigma_x y dA) \delta w'' dx + \int_0^L f \delta w dx = \int_0^L M \delta w'' dx + \int_0^L f \delta w dx \quad \forall \delta w, \quad 1D, \text{ i.e., } x \text{ alone} \]

\[ 1D + 2D \]
6.1 Strong and weak forms for Euler–Bernoulli beams

\[
= \int_{0}^{L} (\int_{A} \sigma_{x} y dA) \delta w'' \, dx + \int_{0}^{L} f \delta w \, dx
\]

\[
= \int_{0}^{L} M \delta w'' \, dx + \int_{0}^{L} f \delta w \, dx \quad \forall \delta w,
\]

where the beam is assumed to be subject to a vertical distributed surface loading \( t_{y} = t_{y}(x, y, z) \) acting on the upper and lower surfaces of the beam, defining a resultant loading

\[
f(x) := \int_{Z_{t}^{+}(x)} t_{y}(x, t/2, z) \, dz + \int_{Z_{t}^{-}(x)} t_{y}(x, -t/2, z) \, dz,
\]

where the integrals are taken along lines \( Z_{t}^{+}(x) \) and \( Z_{t}^{-}(x) \) in the \( z \) direction for each \( x \) on upper and lower surfaces \( S_{t}^{+} \) and \( S_{t}^{-} \), respectively (for collecting the physical load from surfaces onto the beam axis).

**Remark.** Other loading types could be considered as well.
Integration by parts in the term for the internal virtual work gives the form

\[ 0 = M (\delta w)'_0^L - \int_0^L M' (\delta w)' dx + \int_0^L f \delta w dx \]

\[ = M (\delta w)'_0^L - M' \delta w |_0^L + \int_0^L (M'' + f) \delta w dx \quad \forall \delta w \]
6.1 Strong and weak forms for Euler–Bernoulli beams

Integration by parts in the term for the internal virtual work gives the form

\[ 0 = M (\delta w)^{L}_{0} - \int_{0}^{L} M' (\delta w)' \, dx + \int_{0}^{L} f (\delta w) \, dx \]

\[ = M (\delta w)^{L}_{0} - M' \delta w \bigg|_{0}^{L} + \int_{0}^{L} (M'' + f) \delta w \, dx \quad \forall \delta w \]

implying the force balance and boundary conditions, i.e., the strong form, as

\[- M''(x) = f(x) \quad \forall x \in \Omega = (0, L) \quad \text{(EB - M)} \]

\[
\begin{align*}
M(0) &= 0 \quad \vee \quad w'(0) = 0 \\
M(L) &= 0 \quad \vee \quad w'(L) = 0 \\
M'(0) &= 0 \quad \vee \quad w(0) = 0 \\
M'(L) &= 0 \quad \vee \quad w(L) = 0 \\
\end{align*}
\]

or

\[
\begin{align*}
M(0) &= M_{0} \quad \vee \quad w'(0) = \beta_{0} \\
M(L) &= M_{L} \quad \vee \quad w'(L) = \beta_{L} \\
M'(0) &= Q_{0} \quad \vee \quad w(0) = w_{0} \\
M'(L) &= Q_{L} \quad \vee \quad w(L) = w_{L} \\
\end{align*}
\]
Integration by parts in the term for the internal virtual work gives the form

\[ 0 = M \left( \frac{\delta w}{\delta} \right)^0_L - \int_0^L M' \left( \frac{\delta w}{\delta} \right) dx + \int_0^L f \delta w dx \]

\[ = M \left( \frac{\delta w}{\delta} \right)^0_L - M' \delta w^0_0 + \int_0^L (M'' + f) \delta w dx \quad \forall \delta w \]

implying the force balance and boundary conditions, i.e., the strong form, as

\[-M''(x) = f(x) \quad \forall x \in \Omega = (0, L) \quad \text{(EB - M)}\]

\[ M(0) = 0 \quad \vee \quad w'(0) = 0 \]

\[ M(L) = 0 \quad \vee \quad w'(L) = 0 \quad \text{or} \]

\[ M'(0) = 0 \quad \vee \quad w(0) = 0 \]

\[ M'(L) = 0 \quad \vee \quad w(L) = 0 \]

The shear force is determined by the moment equilibrium: \( Q(x) = M'(x) \quad \forall x \in \Omega. \)
Taking into account the *linearly elastic constitutive relations* in the form

\[
\sigma_x(x, y) = (E \varepsilon_x)(x, y) = -y(Ew'')(x)
\]
6.1 Strong and weak forms for Euler–Bernoulli beams

Taking into account the *linearly elastic constitutive relations* in the form

\[ \sigma(x, y) = (E \varepsilon_x)(x, y) = -y(E w')(x) \]

the strong form can be written as a *displacement formulation* as follows: For a given loading \( f : \Omega \rightarrow R \), find the deflection \( w : \overline{\Omega} \rightarrow R \) such that

\[
\begin{align*}
(Elw')(x) &= f(x) \quad \forall x \in \Omega = (0, L), \\
(Elw')(0) &= M_0 \quad \lor \quad w'(0) = \beta_0 \\
(Elw')(L) &= M_L \quad \lor \quad w'(L) = \beta_L \\
(Elw')(0) &= Q_0 \quad \lor \quad w(0) = w_0 \\
(Elw')(L) &= Q_L \quad \lor \quad w(L) = w_L.
\end{align*}
\]
6.1 Strong and weak forms for Euler–Bernoulli beams

Taking into account the *linearly elastic constitutive relations* in the form

\[ \sigma_x(x, y) = (E\varepsilon_x)(x, y) = -y(Ew'')(x) \]

the strong form can be written as a *displacement formulation* as follows: For a given loading \( f : \Omega \to R \), find the deflection \( w : \overline{\Omega} \to R \) such that

\[
(EIw'')(x) = f(x) \quad \forall x \in \Omega = (0, L), \tag{EB - w}
\]

\[
(EIw)''(0) = M_0 \quad \lor \quad w'(0) = \beta_0
\]

\[
(EIw)''(L) = M_L \quad \lor \quad w'(L) = \beta_L
\]

\[
(EIw)'''(0) = Q_0 \quad \lor \quad w(0) = w_0
\]

\[
(EIw)'''(L) = Q_L \quad \lor \quad w(L) = w_L.
\]

The moment and shear force are given in terms of the deflection as

\[ M(x) = -(EIw'')(x), \]

\[ Q(x) = -(EIw'')(x), \quad I(x) := I_z(x) := \int_{A(x)} y^2 \, dA, \quad \forall x \in \Omega. \]
6.1 Strong and weak forms for Euler–Bernoulli beams

The corresponding weak form is obtained from the virtual work expressions above or, as usual, by multiplying the strong form by a test function (variational function), integrating over the domain and finally integrating by parts:

\[
\int_0^L f\hat{w}dx = \int_0^L (EI\hat{w}''')''\hat{w}dx = (EI\hat{w}')'\hat{w}|_0^L - \int_0^L (EI\hat{w}')'\hat{w}'dx
\]

\[
= (EI\hat{w}')'\hat{w}|_0^L - (EI\hat{w}')\hat{w}|_0^L + \int_0^L EI\hat{w}''\hat{w}'dx \quad \forall \hat{w} \in W.
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\[ \int_0^L f \hat{w} dx = \int_0^L (EIw'')'' \hat{w} dx = (EIw'')' \hat{w}\big|_0^L - \int_0^L (EIw'')' \hat{w}' dx = (EIw'')' \hat{w}\big|_0^L - (EIw'') \hat{w}\big|_0^L + \int_0^L EIw'' \hat{w}'' dx \quad \forall \hat{w} \in W. \]

This equation gives the energy balance with respect to the variational space as

\[ \int_0^L EIw'' \hat{w}'' dx = \int_0^L f \hat{w} dx \quad \forall \hat{w} \in W, \]

and the essential boundary conditions – for a cantilever beam, for instance – as

\[ w(0) = 0 \quad \land \quad w'(0) = 0; \quad \hat{w}(0) = 0 \quad \land \quad \hat{w}'(0) = 0. \]
6.1 Strong and weak forms for Euler–Bernoulli beams

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\int_0^L f \hat{w} dx = \int_0^L (EIw'')'' \hat{w} dx = (EIw'')' \hat{w}|_0^L - \int_0^L (EIw'')' \hat{w}' dx
\]

\[
= (EIw'')' \hat{w}|_0^L - (EIw'') \hat{w}|_0^L + \int_0^L EIw'' \hat{w}'' dx \quad \forall \hat{w} \in W.
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w(0) = 0 \quad \land \quad w'(0) = 0; \quad \hat{w}(0) = 0 \quad \land \quad \hat{w}'(0) = 0.
\]

**Remark.** In addition, the trial and test function spaces are determined by the weak form, as usual – although in this case the space \( H^1(\Omega) \) is not an appropriate choice anymore due to the second order derivatives present in the bilinear form.
6.1 Strong and weak forms for Euler–Bernoulli beams

The weak form of the Euler–Bernoulli beam problem: Let us consider a cantilever beam subject to a distributed load \( f \in L^2(\Omega) \), \( \Omega = (0, L) \). Find \( w \in W \) s.t.

\[
a(w, \hat{w}) = l(\hat{w}) \quad \forall \hat{w} \in W,
\]

with the bilinear form, load functional and the variational space

\[
a(w, \hat{w}) = \int_{\Omega} EIw'' \hat{w}'' \, d\Omega,
\]

\[
l(\hat{w}) = \int_{\Omega} f\hat{w} \, d\Omega,
\]

\[
W = \{ v \in H^2(\Omega) \mid v(0) = 0, \ v'(0) = 0 \} \subset H^2(\Omega).
\]
6.1 Strong and weak forms for Euler–Bernoulli beams

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\[
a(w, \hat{w}) = \int_{\Omega} EI w'' \hat{w}'' \, d\Omega,
\]

\[
l(\hat{w}) = \int_{\Omega} f \hat{w} \, d\Omega,
\]

\[
W = \{ \ v \in H^2(\Omega) \mid v(0) = 0, \ v'(0) = 0 \ \} \subset H^2(\Omega).
\]

Remark. For the first time, the variational space is a subspace of Sobolev space \( H^2(\Omega) \), which will essentially influence the finite element space. Accordingly, continuity, coercivity and error estimates will be formulated with respect to the \( H^2 \) norm – basic principles for the analysis will remain the same, however.
6.1 Strong and weak forms for Euler–Bernoulli beams

Show that the bilinear form of the Euler–Bernoulli beam problem is (i) elliptic and (ii) continuous with respect to the $H^2(\Omega)$ norm:

\[
(i) \quad a(v,v) = \int_{\Omega} EIv''v''\,d\Omega \geq \ldots \geq \alpha \|v\|^2_2 \quad \forall v \in W,
\]

\[
(ii) \quad a(v,\hat{u}) = \int_{\Omega} EIv''\hat{u}''\,d\Omega \leq \ldots \leq C \|v\|_2 \|\hat{u}\|_2 \quad \forall v,\hat{u} \in W.
\]

For which type of values of the cross sectional quantities $E$ and $I$ the quotient $C/\alpha$ appearing in the corresponding error estimates will be large/small?
6.2 Finite element formulation for Euler–Bernoulli beams

**Conformity.** It is now clear that a piecewise linear continuous finite element approximation is not an appropriate choice for the current beam problem. Instead, we have to find out which kind of conditions for the polynomial order and continuity across the elements will satisfy the *conformity condition*

\[ W_h \subset W \subset H^2(\Omega). \]

How about a second order \((k = 2)\) piecewise linear continuous approximation?
Finite element formulation for Euler–Bernoulli beams

Conformity. It is now clear that a piecewise linear continuous finite element approximation is not an appropriate choice for the current beam problem. Instead, we have to find out which kind of conditions for the polynomial order and continuity across the elements will satisfy the conformity condition

\[ W_h \subset W \subset H^2(\Omega). \]

How about a second order (\( k = 2 \)) piecewise linear continuous approximation? Previously, the conformity (subspace) condition was of the form

\[ V_h \subset V \subset H^1(\Omega) \]

and it was satisfied by simply defining the discrete space as

\[ V_h = \{ \ n \in H^1(\Omega) \ | \ n|_K \in P_k(K) \ \}. \]
6.2 Finite element formulation for Euler–Bernoulli beams

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and it was satisfied by simply defining the discrete space as

\[ V_h = \{ \, v \in H^1(\Omega) \mid v|_K \in P_k(K) \, \}. \]

In practice, we have previously used a piecewise linear approximation which is globally continuous (from element to element). In general, is this a sufficient property for satisfying the condition \( u_h \in H^1(\Omega) \) (cf. Coffee exercise 5.1)?
Continuity. Is a continuous function an $H^1$ function, or even an $H^2$ function?
6.2 Finite element formulation for Euler–Bernoulli beams

**Continuity.** Is a continuous function an $H^1$ function, or even an $H^2$ function?

It can be shown that continuity across the element edges is a sufficient condition for the existence of the weak derivative as long as the function has a weak derivative locally in each element ($d = 1, 2, 3$):

$$\Omega \subset \mathbb{R}^d, \quad \nu \mid_K \in H^1(K) \quad \text{and} \quad \nu \in C(\overline{\Omega}) \quad \Rightarrow \quad \nu \in H^1(\Omega).$$
6.2 Finite element formulation for Euler–Bernoulli beams

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$$\Omega \subset \mathbb{R}^d, \quad v|_K \in H^1(K) \quad \text{and} \quad v \in C(\overline{\Omega}) \quad \Rightarrow \quad v \in H^1(\Omega).$$

Since a finite element approximation is often a function which is a polynomial in each element, $v|_K \in P_k(K)$, and hence infinitely smooth in each element – due to the fact that $P_k(K) \subset C^\infty(K) \subset H^1(K)$ – it means that continuity across the element edges is an essential condition to be required from the approximation.
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Accordingly, this means that continuity of the derivative of a function across the element edges is a sufficient condition for the existence of the second weak derivative as long as the function has a second order weak derivative locally in each element:

$$\Omega \subset \mathbb{R}^d, \quad v|_K \in H^2(K) \quad \text{and} \quad v \in C^1(\overline{\Omega}) \quad \Rightarrow \quad v \in H^2(\Omega).$$
### 6.2 Finite element formulation for Euler–Bernoulli beams

**Conforming finite element method for the Euler–Bernoulli beam problem:** Let us consider a cantilever beam subject to a loading \( f \in L^2(\Omega), \quad \Omega = (0, L). \) Find \( w_h \in W_h \) such that

\[
\begin{align*}
    a(w_h, \hat{w}) &= l(\hat{w}) \quad \forall \hat{w} \in W_h, \\
    a(v, \hat{w}) &= \int_{\Omega} EI v'' \hat{w}'' \, d\Omega \\
    l(\hat{w}) &= \int_{\Omega} f \hat{w} \, d\Omega
\end{align*}
\]

\( W_h \subset W = \{ v \in H^2(\Omega) \mid v(0) = 0, \ v'(0) = 0 \}, \)

\( W_h = \{ v \in C^1(\overline{\Omega}) \mid v(0) = 0, \ v'(0) = 0, \ v|_K \in P_3(K) \}. \)
6.2 Finite element formulation for Euler–Bernoulli beams

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Find \( w_h \in W_h \) such that

\[
\begin{align*}
    a(w_h, \hat{w}) &= l(\hat{w}), \quad \forall \hat{w} \in W_h, \\
    a(v, \hat{w}) &= \int_{\Omega} EIv'' \hat{w}'' \, d\Omega, \\
    l(\hat{w}) &= \int_{\Omega} f\hat{w} \, d\Omega
\end{align*}
\]

where \( W_h \subseteq W = \{ \ v \in H^2(\Omega) \ | \ v(0) = 0, \ v'(0) = 0 \ \}, \)

\[
W_h = \{ \ v \in C^1(\Omega) \ | \ v(0) = 0, \ v'(0) = 0, \ v|_K \in P_3(K) \ \}.
\]

**Remark.** \( C^1 \) continuity, i.e., continuity of the function and its derivatives accross the element edges, will be satisfied by applying the third order Hermite shape functions which essentially differ from the previously used Lagrange shape functions.
Previously, for the continuous piecewise polynomial finite element approximation of order $k$, the degrees of freedom were the nodal values (Lagrange interpolation):

$$u_h(x) = \sum_{i=1}^{n} \phi_i(x) d_i, \quad \phi_i(x_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \Rightarrow u_h(x_j) = d_j \ \forall j.$$
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\[
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 1, & i = j \\
 0, & i \neq j
\end{cases} \quad \Rightarrow \quad u_h(x_j) = d_j \quad \forall j.
\]

Now, a piecewise cubic (third order, i.e., \( k = 3 \)) finite element approximation will be used, with nodal values of both the function and its derivatives taken as degrees of freedom (Hermite interpolation):

\[
 w_h(x) = \sum_{i=1}^{n} \phi_{2i-1}(x) d_{2i-1} + \phi_{2i}(x) d_{2i}, \\
 \phi_{2i-1}(x_j) = \begin{cases} 
 1, & 2i - 1 = j \\
 0, & 2i - 1 \neq j
\end{cases} \quad \land \quad \phi_{2i-1}'(x_j) = 0 \quad \forall j,
\]

![Image of finite element formulation for Euler–Bernoulli beams](image-url)
6.2 Finite element formulation for Euler–Bernoulli beams

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Now, a piecewise cubic (third order, i.e., $k = 3$) finite element approximation will be used, with nodal values of both the function and its derivatives taken as degrees of freedom (*Hermite interpolation*):

$$w_h(x) = \sum_{i=1}^{n} \phi_{2i-1}(x) d_{2i-1} + \phi_{2i}(x) d_{2i},$$

$$\phi_{2i-1}(x_j) = \begin{cases} 1, & 2i - 1 = j \\ 0, & 2i - 1 \neq j \end{cases} \wedge \phi_{2i-1}'(x_j) = 0 \ \forall j,$$

$$\phi_{2i}'(x_j) = \begin{cases} 1, & 2i = j \\ 0, & 2i \neq j \end{cases} \wedge \phi_{2i}(x_j) = 0 \ \forall j.$$ 

$$\Rightarrow \ w_h(x_j) = d_{2j-1}, \quad w_h'(x_j) = d_{2j} \ \forall j.$$
6.2 Finite element formulation for Euler–Bernoulli beams

In this approach, four shape functions will be related to each element; two to each end of each interval \((x_i, x_{i+1})\):

\[
\phi_{2i-1}(x) = \frac{-(x - x_{i+1})^2(-h_i + 2(x_i - x))}{h_i^3} \quad \leftrightarrow \quad w_h(x_i), \quad h_i = x_{i+1} - x_i
\]

\[
\phi_{2i}(x) = \frac{(x - x_i)(x - x_{i+1})^2}{h_i^2} \quad \leftrightarrow \quad w_h'(x_i)
\]

\[
\phi_{2i+1}(x) = \frac{(x - x_i)^2(h_i + 2(x_{i+1} - x))}{h_i^3} \quad \leftrightarrow \quad w_h(x_{i+1})
\]

\[
\phi_{2(i+1)}(x) = \frac{(x - x_i)^2(x - x_{i+1})}{h_i^2} \quad \leftrightarrow \quad w_h'(x_{i+1})
\]
6.2 Finite element formulation for Euler–Bernoulli beams

In this approach, four shape functions will be related to each element; two to each end of each interval \((x_i, x_{i+1})\):

\[ \phi_{2i-1}(x) = \frac{-(x - x_{i+1})^2(-h_i + 2(x_i - x))}{h_i^3} \quad \leftrightarrow \quad w_h(x_i), \quad h_i = x_{i+1} - x_i \]

\[ \phi_{2i}(x) = \frac{(x - x_i)(x - x_{i+1})^2}{h_i^2} \quad \leftrightarrow \quad w_h'(x_i) \]

\[ \phi_{2i+1}(x) = \frac{(x - x_i)^2(h_i + 2(x_{i+1} - x))}{h_i^3} \quad \leftrightarrow \quad w_h(x_{i+1}) \]

\[ \phi_{2(i+1)}(x) = \frac{(x - x_i)^2(x - x_{i+1})}{h_i^2} \quad \leftrightarrow \quad w_h'(x_{i+1}) \]
6.2 Finite element formulation for Euler–Bernoulli beams

Each element will give its contribution to the stiffness matrix and force vector as

\[ K_{pq}^{(e)} = a(\phi_p, \phi_q)_{K^{(e)}} = \int_{x_i}^{x_{i+1}} EI \phi_p'' \phi_q'' dx, \quad p, q = 1, \ldots, 2n \]

\[ F_p^{(e)} = l(\phi_p)_{K^{(e)}} = \int_{x_i}^{x_{i+1}} f \phi_p dx, \quad K^{(e)} = (x_i, x_{i+1}), \quad i = 0, \ldots, n. \]
6.2 Finite element formulation for Euler–Bernoulli beams

Each element will give its contribution to the stiffness matrix and force vector as

\[ K_{pq}^{(e)} = a(\phi_p, \phi_q) \bigg|_{K^{(e)}} = \int_{x_i}^{x_{i+1}} EI \phi_p'' \phi_q'' \, dx, \quad p, q = 1, \ldots, 2n \]

\[ F_p^{(e)} = l(\phi_p) \bigg|_{K^{(e)}} = \int_{x_i}^{x_{i+1}} f \phi_p \, dx, \quad K^{(e)} = (x_i, x_{i+1}), \quad i = 0, \ldots, n. \]

In practice, only four shape functions are nonzero in each element and hence

\[ K^{(e)} = \begin{bmatrix}
K_{w_1^{(e)}w_1^{(e)}}^{(e)} & K_{w_1^{(e)}w_1^{(e)}w_1^{(e)}}^{(e)} & K_{w_1^{(e)}w_1^{(e)}w_2^{(e)}}^{(e)} & K_{w_1^{(e)}w_1^{(e)}w_2^{(e)}}^{(e)} \\
K_{w_2^{(e)}w_1^{(e)}}^{(e)} & K_{w_2^{(e)}w_1^{(e)}w_1^{(e)}}^{(e)} & K_{w_2^{(e)}w_1^{(e)}w_2^{(e)}}^{(e)} & K_{w_2^{(e)}w_1^{(e)}w_2^{(e)}}^{(e)} \\
K_{w_2^{(e)}w_1^{(e)}w_1^{(e)}}^{(e)} & K_{w_2^{(e)}w_1^{(e)}w_1^{(e)}w_1^{(e)}}^{(e)} & K_{w_2^{(e)}w_1^{(e)}w_1^{(e)}w_2^{(e)}}^{(e)} & K_{w_2^{(e)}w_1^{(e)}w_2^{(e)}}^{(e)} \\
K_{w_2^{(e)}w_1^{(e)}w_2^{(e)}}^{(e)} & K_{w_2^{(e)}w_2^{(e)}w_2^{(e)}}^{(e)} & K_{w_2^{(e)}w_2^{(e)}w_2^{(e)}}^{(e)} & K_{w_2^{(e)}w_2^{(e)}w_2^{(e)}}^{(e)}
\end{bmatrix}, \quad F^{(e)} = \begin{bmatrix}
F_{w_1^{(e)}}^{(e)} \\
F_{w_1^{(e)}w_1^{(e)}}^{(e)} \\
F_{w_1^{(e)}w_2^{(e)}}^{(e)} \\
F_{w_2^{(e)}}^{(e)}
\end{bmatrix}, \quad d^{(e)} = \begin{bmatrix}
d_{w_1^{(e)}}^{(e)} \\
d_{w_1^{(e)}w_1^{(e)}}^{(e)} \\
d_{w_1^{(e)}w_2^{(e)}}^{(e)} \\
d_{w_2^{(e)}}^{(e)}
\end{bmatrix}^T. \]
The global stiffness matrix and force vector can now be assembled in a usual way.

**Remark.** Hermite type third order $C^1$ continuous deflection approximation $w_h$ implies, by taking (elementwise) derivatives, that
- rotation approximation $w_h'$ is piecewise quadratic and continuous,
- moment approximation is piecewise linear and discontinuous,
- shear force approximation is piecewise constant and discontinuous.
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Remark. In a similar manner, one can deduce that a typical continuous (Lagrange type) quadratic finite element approximation would lead to piecewise linear discontinuous rotation. Taking then the (global) derivative of the rotation would lead on element borders to Dirac delta functions which are not square-integrable. Hence, the bilinear form of the problem would not be defined for this type of trial functions and the problem would not be solvable. This argumentation gives a fairly intuitive justification for the $C^1$ continuity requirement implied by the conformity condition

$$W_h \subset W \subset H^2(\Omega).$$

Due to conformity, the error analysis can be carried out by standard techniques:
Error estimates. $H^2$ continuity and $H^2$ coercivity of the bilinear form, together with Galerkin orthogonality, imply an error estimate following the Cea’s lemma,

$$
\| w - w_h \|_2 \leq \frac{C}{\alpha} \| w - v \|_2 \quad \forall v \in W_h,
$$
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from which we get a more quantitative estimate (assuming a smooth solution)

$$\| w - w_h \|_2 \leq c_k h^{k-1} \| w \|_{k+1} = c_3 h^2 \| w \|_4, \quad \text{for } k = 3.$$
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$$\| w - w_h \|_2 \leq c_k \ h^{k-1} \ | w |_{k+1} = c_3 \ h^2 \ | w |_4, \quad \text{for } k = 3.$$

Above – as well as in earlier error estimates – we have used a result from *approximation theory* for the interpolation error of polynomials, stating that a polynomial $\tilde{w}$ of order $k$ interpolates a function $w$ with the following accuracy:

$$\| w - \tilde{w} \|_m \leq c_k \ h^{k+1-m} \ | w |_{k+1}.$$
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$$\| w - \tilde{w} \|_m \leq c_k h^{k+1-m} \| w \|_{k+1}.$$

**Remark.** It can be shown that the Hermite finite element approximation gives accurate nodal values for the deflection and its derivative (rotation) of the Euler–Bernoulli beam problem: $w(x_i) = w_h(x_i)$ and $w'(x_i) = w_h'(x_i)$ for all nodes $x_i$. 
Above, continuity was shown to imply an $H^1$ derivative under certain circumstances. With certain assumptions, the opposite implication holds as well (Sobolev embedding theorem):

$$H^k(\Omega) \subset C^m(\overline{\Omega}), \quad 0 \leq m < k - d / 2, \quad \text{with} \quad \Omega \subset \mathbb{R}^d.$$  

For instance, in 1D case an $H^1$ function is continuous, while in 2D case, instead, a function has to be $H^2$ regular in order to be continuous.

If a function is $H^1$ regular inside its domain $\Omega$, how regular it is on the boundary of the domain? If the domain $\Omega$ is bounded and its boundary is $C^1$ regular, then the trace $T_v$ of a function $v$ on the boundary $\partial\Omega$ is an $L^2$ function (Trace theorem):

$$\| T_v \|_{L^2(\partial\Omega)} \leq c \| v \|_{H^1(\Omega)},$$

where $T : H^1(\Omega) \rightarrow L^2(\partial\Omega)$ is linear.

If it further holds that $v \in C(\overline{\Omega})$, then $T_v = v|_{\partial\Omega}$. 
Coffee exercise 9

Find the derivative of the function

\[ v(x) = \begin{cases} 
  x^a \sin\left(\frac{1}{x}\right), & 0 < x \leq 1 \\
  0, & x = 0 
\end{cases} \]

with \( a > 1 \). Show that \( v' \not\in L^2(\Omega) \) whenever \( a \leq 3/2 \).
QUESTIONS?

ANSWERS"

LECTURE BREAK!