MISCELLANEOUS PAPERS

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WITH AN INTRODUCTION

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ON THE CONTACT OF ELASTIC SOLIDS

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IN the theory of elasticity the causes of the deformations are assumed to be partly forces acting throughout the volume of the body, partly pressures applied to its surface. For both classes of forces it may happen that they become infinitely great in one or more infinitely small portions of the body, but so that the integrals of the forces taken throughout these elements remain finite. If about the singular point we describe a closed surface of small dimensions compared to the whole body, but very large in comparison with the element in which the forces act, the deformations outside and inside this surface may be treated independently of each other. Outside, the deformations depend upon the shape of the whole body, the finite integrals of the force-components at the singular point, and the distribution of the remaining forces; inside, they depend only upon the distribution of the forces acting inside the element. The pressures and deformations inside the surface are infinitely great in comparison with those outside.

In what follows we shall treat of a case which is one of the class referred to above, and which is of practical interest,[1] namely, the case of two elastic isotropic bodies which touch each other over a very small part of their surface and exert upon each other a finite pressure, distributed over the common area of contact. The surfaces in contact are imagined as perfectly smooth, *i.e.* we assume that only a normal pressure

[1] *Cf.* Winkler, *Die Lehre von der Elasticität und Festigkeit*, vol. i. p. 43 (Prag. 1867); and Grashof, *Theorie der Elasticität und Festigkeit*, pp. 49-54 (Berlin, 1878).

acts between the parts in contact. The portion of the surface which during deformation is common to the two bodies we shall call the surface of pressure, its boundary the curve of pressure. The questions which from the nature of the case first demand an answer are these: What surface is it, of which the surface of pressure forms an infinitesimal part ?[1] What is the form and what is the absolute magnitude of the curve of pressure? How is the normal pressure distributed over the surface of pressure? It is of importance to determine the maximum pressure occurring in the bodies when they are pressed together, since this determines whether the bodies will be without permanent deformation; lastly, it is of interest to know how much the bodies approach each other under the influence of a given total pressure.

We are given the two elastic constants of each of the bodies which touch, the form and relative position of their surfaces near the point of contact, and the total pressure. We shall choose our units so that the surface of pressure may be finite. Our reasoning will then extend to all finite space; the full dimensions of the bodies in contact we must imagine as infinite.

In the first place we shall suppose that the two surfaces are brought into mathematical contact, so that the common normal is parallel to the direction of the pressure which one body is to exert on the other. The common tangent plane is taken as the plane xy, the normal as axis of z, in a rectangular rectilinear system of coordinates. The distance of any point of either surface from the common tangent plane will in the neighbourhood of the point of contact, *i.e.* throughout all finite space, be represented by a homogeneous quadratic function of x and y. Therefore the distance between two corresponding points of the two surfaces will also be represented by such a function. We shall turn the axes of x and y so that in the last-named function the term involving xy is absent.

[1] In general the radii of curvature of the surface of a body in a state of strain are only infinitesimally altered; but in our particular case they are altered by finite amounts, and in this lies the justification of the present question. For instance, when two equal spheres of the same material touch each other, the surface of pressure forms part of a plane, *i.e.* of a surface which is different in character from both of the surfaces in contact.

Then we may write the equations of the two surfaces

$$z_1 = A_1x^2 + Cxy + B_1y^2, \quad z_2 = A_2x^2 + Cxy + B_2y^2,$$

and we have for the distance between corresponding points of the two surfaces $z_1 - z_2 = Ax^2 + By^2$, where $A = A_1 - A_2$, $B = B_1 - B_2$, and A, B, C are all infinitesimal.[1] From the meaning of the quantity $z_1 - z_2$ it follows that A and B have the like sign, which we shall take positive. This is equivalent to choosing the positive z-axis to fall inside the body to which the index 1 refers.

Further, we imagine in each of the two bodies a rectangular rectilinear system of axes, rigidly connected at infinity with the corresponding body, which system of axes coincides with the previously chosen system of xyz during the mathematical contact of the two surfaces. When a pressure acts on the bodies these systems of coordinates will be shifted parallel to the axis of z relatively to one another; and their relative motion will be the same in amount as the distance by which those parts of the bodies approach each other which are at an infinite distance from the point of contact. The plane $z = 0$ in each of these systems is infinitely near to the part of the surface of the corresponding body which is at a finite distance, and therefore may itself be considered as the surface, and the direction of the z-axis as the direction of the normal to this surface.

Let ξ, η, ζ be the component displacements parallel to the axes of x, y, z; let Y_x denote the component parallel to Oy of the pressure on a plane element whose normal is parallel to Ox, exerted by the portion of the body for which x has smaller values on the portion for which x has larger values, and let a similar notation be used for the remaining com-

[1] Let ρ_{11}, ρ_{12} be the reciprocals of the principal radii of curvature of the surface of the first body, reckoned positive when the corresponding centres of curvature lie inside this body; similarly let ρ_{21}, ρ_{22} be the principal curvatures of the surface of the second body; lastly, let ω be the angle which the planes of the curvatures ρ_{11} and ρ_{21} make with each other. Then

$$2(A + B) = \rho_{11} + \rho_{12} + \rho_{21} + \rho_{22},$$

$$2(A - B) = \sqrt{(\rho_{11} - \rho_{12})^2 + 2(\rho_{11} - \rho_{12})(\rho_{21} - \rho_{22}) \cos 2\omega + (\rho_{21} - \rho_{22})^2}.$$

If we introduce an auxiliary angle τ by the equation $\cos \tau = (A - B)/(A + B)$, then

$$2A = (\rho_{11} + \rho_{12} + \rho_{21} + \rho_{22}) \cos^2\frac{\tau}{2}, \quad 2B = (\rho_{11} + \rho_{12} + \rho_{21} + \rho_{22}) \sin^2\frac{\tau}{2}.$$

ponents of pressure; lastly let $K_1\theta_1$ and $K_2\theta_2$[1] be the respective coefficients of elasticity of the bodies. Generally, where the quantities refer to either body, we shall omit the indices. We then have the following conditions for equilibrium:—

1. Inside each body we must have

$$0 = \nabla^2\xi + (1+2\theta)\frac{\partial\sigma}{\partial x}, \quad 0 = \nabla^2\eta + (1+2\theta)\frac{\partial\sigma}{\partial y},$$

$$0 = \nabla^2\zeta + (1+2\theta)\frac{\partial\sigma}{\partial z}, \quad \sigma = \frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial z};$$

and in 1 we have to put θ_1 for θ, in 2 θ_2 for θ.

2. At the boundaries the following conditions must hold:—

(*a*) At infinity ξ, η, ζ vanish, for our systems of coordinates are rigidly connected with the bodies there.

(*b*) For $z = 0$, *i.e.* at the surface of the bodies, the tangential stresses which are perpendicular to the z-axis must vanish, or

$$Y_z = -K\left(\frac{\partial\eta}{\partial z} + \frac{\partial\zeta}{\partial y}\right) = 0, \quad X_z = -K\left(\frac{\partial\zeta}{\partial x} + \frac{\partial\xi}{\partial z}\right) = 0.$$

(*c*) For $z = 0$, outside a certain portion of this plane, viz. outside the surface of pressure, the normal stress also must vanish, or

$$Z_z = 2K\left(\frac{\partial\zeta}{\partial z} + \theta\sigma\right) = 0.$$

Inside that part

$$Z_{z1} = Z_{z2}.$$

We do not know the distribution of pressure over that part, but instead we have a condition for the displacement ζ over it.

(*d*) For if a denote the relative displacement of the two systems of coordinates to which we refer the displacements, the distance between corresponding points of the two surfaces after deformation is $Ax^2 + By^2 + \zeta_1 - \zeta_2 - a$, and since this distance vanishes inside the surface of pressure we have

$$\zeta_1 - \zeta_2 = a - Ax^2 - By^2 = a - z_1 + z_2.$$

(*e*) To the conditions enumerated we must add the con-

[1] [Kirchhoff's notation, *Mechanik*, p. 121.—Tr.]

dition that inside the surface of pressure Z_z is everywhere positive, and the condition that outside the surface of pressure $\zeta_1 - \zeta_2 > a - Ax^2 - By^2$, otherwise the one body would overflow into the other.

(*f*). Lastly the integral $\int Z_z ds$, taken over the part of the surface which is bounded by the curve of pressure, must be equal to the given total pressure, which we shall call p.

The particular form of the surface of the two bodies only occurs in the boundary condition (2 *d*), apart from which each of the bodies acts as if it were an infinitely extended body occupying all space on one side of the plane $z = 0$, and as if only normal pressures acted on this plane. We therefore consider more closely the equilibrium of such a body. Let P be a function which inside the body satisfies the equation $\nabla^2 P = 0$; in particular, we shall regard P as the potential of a distribution of electricity on the finite part of the plane $z = 0$. Further let

$$\Pi = -\frac{zP}{K} + \frac{1}{K(1+2\theta)}\left\{\int_z^i P dz - J\right\},$$

where i is an infinitely great quantity, and J is a constant so chosen as to make Π finite. For this purpose J must be equal to the natural logarithm of i multiplied by the total charge of free electricity corresponding to the potential P.

From the definition of Π it follows that

$$\nabla^2 \Pi = -\frac{2}{K}\frac{\partial P}{\partial z}.$$

Introducing the contraction $\vartheta = \dfrac{2(1+\theta)}{K(1+2\theta)}$ we put

$$\xi = \frac{\partial \Pi}{\partial x}, \quad \eta = \frac{\partial \Pi}{\partial y}, \quad \zeta = \frac{\partial \Pi}{\partial z} + 2\vartheta P,$$

$$\sigma = \nabla^2 \Pi + 2\vartheta \frac{\partial P}{\partial z} = \frac{2}{K(1+2\theta)}\frac{\partial P}{\partial z}.$$

This system of displacements is easily seen to satisfy the differential equations given for ξ, η, ζ, and the displacements vanish at infinity. For the pressure components we find

$$X_x = -2K\left\{\frac{\partial^2 \Pi}{\partial x^2} + \frac{2\theta}{K(1+2\theta)}\frac{\partial P}{\partial z}\right\},$$

$$X_y = -2K\frac{\partial^2 \Pi}{\partial x \partial y},$$

$$Y_y = -2K\left\{\frac{\partial^2 \Pi}{\partial y^2} + \frac{2\theta}{K(1+2\theta)}\frac{\partial P}{\partial z}\right\},$$

$$X_z = -2K\left\{\frac{\partial^2 \Pi}{\partial x \partial z} + \vartheta\frac{\partial P}{\partial x}\right\} = 2z\frac{\partial^2 P}{\partial x \partial z},$$

$$Z_z = -2K\left\{\frac{\partial^2 \Pi}{\partial z^2} + \frac{2(2+3\theta)}{K(1+2\theta)}\frac{\partial P}{\partial z}\right\},$$

$$Y_z = -2K\left\{\frac{\partial^2 \Pi}{\partial y \partial z} + \vartheta\frac{\partial P}{\partial y}\right\} = 2z\frac{\partial^2 P}{\partial y \partial z}.$$

The last two formulæ show that for the given system the stress-components perpendicular to the z-axis vanish throughout the plane $z = 0$. We determine the displacement ζ and the normal pressure Z_z at the plane $z = 0$, and find

$$\zeta = \vartheta P, \quad Z_z = -2\frac{\partial P}{\partial z}.$$

The density of the electricity producing the potential P is $-(1/2\pi)(\partial P/\partial z)$, hence we have the following theorem. The displacement ζ in the surface, which corresponds to the normal pressure Z_z, is equal to $\vartheta/4\pi$ times the potential due to an electrical density numerically equal to the pressure Z_z.

We now consider again both bodies: we imagine the electricity whose potential is P to be distributed over a finite portion only of the plane $z = 0$; we make Π_1 and Π_2 equal to the expressions derived from the given expression for Π by giving to the symbols K and θ the index 1 and 2, and put

$$\xi_1 = \frac{\partial \Pi_1}{\partial x}, \quad \eta_1 = \frac{\partial \Pi_1}{\partial y}, \quad \zeta_1 = \frac{\partial \Pi_1}{\partial z} + 2\vartheta_1 P,$$

$$\xi_2 = -\frac{\partial \Pi_2}{\partial x}, \quad \eta_2 = -\frac{\partial \Pi_2}{\partial y}, \quad \zeta_2 = -\frac{\partial \Pi_2}{\partial z} - 2\vartheta_2 P;$$

whence we have for $z=0$

$$\zeta_1 = \vartheta_1 \mathrm{P}, \quad \zeta_2 = -\vartheta_2 \mathrm{P}, \quad Z_{z1} = -2\frac{\partial \mathrm{P}}{\partial z}, \quad Z_{z2} = 2\frac{\partial \mathrm{P}}{\partial z}$$

This assumption satisfies the conditions (1), (2 a), and (2 b) according to the explanations given. Since $\partial \mathrm{P}/\partial z$ has on the two sides of the plane $z=0$ values equal but of opposite sign, and since it vanishes outside the electrically charged surface whose potential is P, the conditions (2 c) also are fulfilled, provided the surface of pressure coincides with the electrically charged surface. From the fact that P is continuous across the plane $z=0$, it follows that for $z=0, \vartheta_2\zeta_1 + \vartheta_1\zeta_2 = 0$. But according to the condition (2 d) we have for the surface of pressure, $\zeta_1 - \zeta_2 = a - z_1 + z_2$; here therefore

$$\zeta_1 = \frac{\vartheta_1}{\vartheta_1 + \vartheta_2}(a - z_1 + z_2), \qquad \zeta_2 = -\frac{\vartheta_2}{\vartheta_1 + \vartheta_2}(a - z_1 + z_2).$$

Apart from a constant which depends on the choice of the system of coordinates, and need therefore not be considered, the equation of the surface of pressure is $z = z_1 + \zeta_1 = z_2 + \zeta_2$, or $(\vartheta_1 + \vartheta_2)z = \vartheta_2 z_1 + \vartheta_1 z_2$. Thus the surface of pressure is part of a quadric surface lying between the undeformed positions of the surfaces which touch each other; and is most like the boundary of the body having the greater coefficient of elasticity. If the bodies are composed of the same material it is the mean surface of the surfaces of the two bodies, since then $2z = z_1 + z_2$.

We now make a definite assumption as to the distribution of the electricity whose potential is P. Let it be distributed over an ellipse whose semi-axes a and b coincide with the axes of x and y, with a density $\dfrac{3p}{8\pi^2 ab}\sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}}$, so that it can be regarded as a charge which fills an infinitely flattened ellipsoid with uniform volume density. Then

$$\mathrm{P} = \frac{3p}{16\pi}\int_u^\infty \left(1 - \frac{x^2}{a^2+\lambda} - \frac{y^2}{b^2+\lambda} - \frac{z^2}{\lambda}\right)\frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)\lambda}},$$

where u, the inferior limit of integration, is the positive root of the cubic equation

$$\frac{x_2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{u}=1\ .$$

Inside the surface of pressure, which is bounded by the given ellipse, we have $u=0$, $\mathrm{P}=\mathrm{L}-\mathrm{M}x^2-\mathrm{N}y^2$; where L, M, N denote certain positive definite integrals. The condition $(2\ d)$ is satisfied by choosing a and b so that

$$(\vartheta_1+\vartheta_2)\mathrm{M}=\mathrm{A},\quad(\vartheta_1+\vartheta_2)\mathrm{N}=\mathrm{B},$$

which is always possible. The unknown α which occurs in the condition is then determined by the equation

$$(\vartheta_1+\vartheta_2)\mathrm{L}=\alpha.$$

It follows directly from the equation

$$\mathrm{Z}_z=\frac{3p}{2\pi ab}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

that the first of the conditions $(2\ e)$ is satisfied.

To show that the second also is satisfied is to prove that when $z=0$ and $x^2/a^2+y^2/b^2>1$, $(\vartheta_1+\vartheta_2)\mathrm{P}>\alpha-\mathrm{A}x^2-\mathrm{B}y^2$. For this purpose we observe that here

$$\mathrm{P}=\mathrm{L}-\mathrm{M}x^2-\mathrm{N}y^2$$

$$-\frac{3p}{16\pi}\int_0^u\left(1-\frac{x^2}{a^2+\lambda}-\frac{y^2}{b^2+\lambda}\right)\frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)\lambda}},$$

and hence $\mathrm{P}>\mathrm{L}-\mathrm{M}x^2-\mathrm{N}y^2$, for the numerator of the expression under the sign of integration is negative throughout the region considered. Multiplying by $\vartheta_1+\vartheta_2$ we get the inequality which was to be proved. Finally, a simple integration shows that the last condition $(2\ f)$ also is satisfied; therefore we have in the assumed expression for P and the corresponding system ξ, η, ζ a solution which satisfies all the conditions.

The equations for the axes of the ellipse of pressure written explicitly are

$$\int_0^\infty \frac{du}{\sqrt{(a^2+u)^3(b^2+u)u}} = \frac{A}{\vartheta_1+\vartheta_2}\,\frac{16\pi}{3p},$$

$$\int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)^3u}} = \frac{B}{\vartheta_1+\vartheta_2}\,\frac{16\pi}{3p},$$

or introducing the ratio $k=a/b$, and transforming,

$$\frac{1}{a^3}\int_0^\infty \frac{dz}{\sqrt{(1+k^2z^2)^3(1+z^2)}} = \frac{8\pi}{3p}\,\frac{A}{\vartheta_1+\vartheta_2},$$

$$\frac{1}{a^3}\int_0^\infty \frac{dz}{k^2\sqrt{(1+k^2z^2)(1+z^2)^3}} = \frac{8\pi}{3p}\,\frac{B}{\vartheta_1+\vartheta_2}.$$

By division we obtain a transcendental equation for the ratio k.[1] This depends only on the ratio A : B, and it follows at once, from the meaning we have attached to the forces and displacements, that the ellipse of pressure is always more elongated than the ellipses at which the distance between the bodies is constant. As regards the absolute magnitude of the surface of pressure for a given form of the surfaces it varies as

[1] The solution of this equation and the evaluation of the integrals required for the determination of a and b may be performed by the aid of Legendre's tables without necessitating any new quadratures. The calculation, usually somewhat laborious, may in most cases be avoided by the use of the following small table, of which the arrangement is as follows. If we express A and B in the equations for a and b in terms of the principal curvatures and the auxiliary angle τ introduced in a previous note, the solutions of these equations are expressible in the form

$$a=\mu\sqrt{\frac{3p(\vartheta_1+\vartheta)}{8(\rho_{11}+\rho_{12}+\rho_{21}+\rho_{22})}},\quad b=\nu\sqrt{\frac{3p(\vartheta_1+\vartheta_2)}{8(\rho_{11}+\rho_{12}+\rho_{21}+\rho_{22})}},$$

where μ, ν are transcendental functions of the angle τ. The table gives the values of these functions for ten values of the argument τ expressed in degrees.

τ	90	80	70	60	50	40	30	20	10	0
μ	1·0000	1·1278	1·2835	1·4858	1·7542	2·1357	2·7307	3·7779	6·6120	∞
ν	1·0000	0·8927	0·8017	0·7171	0·6407	0·5673	0·4930	0·4079	0·3186	0·0000

the cube root of the total pressure and as the cube root of the quantity $\vartheta_1 + \vartheta_2$. By the preceding the distance through which the bodies approach each other under the action of the given pressure is

$$a = \frac{3p}{8\pi} \cdot \frac{\vartheta_1 + \vartheta_2}{a} \int_0^\infty \frac{dz}{\sqrt{(1+k^2z^2)(1+z^2)}} .$$

If we perform the multiplication by $\vartheta_1 + \vartheta_2$, a splits up into two portions which have a special meaning. They denote the distances through which the origin approaches the infinitely distant portions of the respective bodies; we may call them the indentations which the respective bodies have undergone. With a given form of the touching surfaces the distance of approach varies as the pressure raised to the power $\frac{2}{3}$ and also as the same power of the quantity $\vartheta_1 + \vartheta_2$. When A and B alter in magnitude while their ratio remains unchanged, the dimensions of the surface of pressure vary inversely as the cube roots of the absolute values of A and B, and the distance of approach varies directly as these roots. When A and B become infinite, the distance of approach becomes infinite; bodies which touch each other at sharp points penetrate into each other.

In connection with this we shall determine what happens to the element at the origin of our system of coordinates by finding the three displacements $\frac{\partial \xi}{\partial x}$, $\frac{\partial \eta}{\partial y}$, $\frac{\partial \zeta}{\partial z}$. In the first place we have at the origin

$$\sigma = \frac{2}{K(1+2\theta)} \frac{\partial P}{\partial z} = -\frac{3p}{2K(1+2\theta)\pi} \frac{1}{ab} ,$$

$$\frac{\partial \zeta}{\partial z} = \frac{1}{K(1+2\theta)} \frac{\partial P}{\partial z} = -\frac{3p}{4K(1+2\theta)\pi} \frac{1}{ab} .$$

Further, at the plane $z = 0$

$$\xi = \frac{\partial \Pi}{\partial x} . \quad \eta = \frac{\partial \Pi}{\partial y} ,$$

$$\Pi = \frac{1}{K(1+2\theta)} \int_0^\infty P dz = \frac{1}{2K(1+2\theta)} \int_{-\infty}^\infty P dz .$$

We see that in the said plane ξ and η are proportional to the forces exerted by an infinitely long elliptic cylinder, which stands on the surface of pressure and whose density increases inwards, according to the law of increase of the pressure in the surface of pressure. In general then, ξ and η are given by complicated functions; but for points close to the axis they can be easily calculated. Surrounding the axis we describe a very thin cylindrical surface, similar to the whole cylinder; this [small] cylinder we may treat as homogeneous, and since the part outside it has no action at points inside it, the components of the forces in question, and therefore also ξ and η, must be equal to a constant multiplied respectively by x/a and by y/b. Hence

$$a\frac{\partial\xi}{\partial x} - b\frac{\partial\eta}{\partial y} = 0.$$

On the other hand we have

$$\frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} = \sigma - \frac{\partial\zeta}{\partial z} = -\frac{3p}{4\mathrm{K}(1+2\theta)\pi}\frac{1}{ab}.$$

From these equations we find for the three quantities which we sought

$$\frac{\partial\xi}{\partial x} = -\frac{3p}{4\mathrm{K}(1+2\theta)\pi}\frac{1}{a(a+b)},$$

$$\frac{\partial\eta}{\partial y} = -\frac{3p}{4\mathrm{K}(1+2\theta)\pi}\frac{1}{b(a+b)},$$

$$\frac{\partial\zeta}{\partial z} = -\frac{3p}{4\mathrm{K}(1+2\theta)\pi}\frac{1}{ab}.$$

The negative sign of these three quantities shows that the element in question is compressed in all three directions. The compressions vary as the cube root of the total pressure. It is easy to determine from them the pressures at the origin. These pressures are the most intense of all those occurring throughout the bodies pressed together; we may therefore say that the limit of elasticity will not be exceeded until these pressures become of the order of magnitude required for transgressing the elastic limit. In plastic bodies, *e.g.* in the softer metals, this transgression will at first consist in a lateral deformation accompanied by a permanent compression; so that it will not result in an infinitely increasing disturbance of equilibrium, but the surface of pressure will increase beyond the calculated dimensions until the pressure per unit area is sufficiently small to be sustained. It is more difficult to determine what happens in the case of brittle bodies, as hard steel, glass, crystals, in which a transgression of the elastic limit occurs only through the formation of a rent or crack, *i.e.* only under the influence of tensional forces. Such a crack cannot start in the element considered above, which is compressed in every direction; and with our present-day knowledge of the tenacity of brittle bodies it is indeed impossible exactly to determine in which element the conditions for the production of a crack first occur when the pressure is increased. However, a more detailed discussion shows this much, that in bodies which in their elastic behaviour resemble glass or hard steel, much the most intense tensions occur at the surface, and in fact at the boundary of the surface of pressure. Such a discussion shows it to be probable that the first crack starts at the ends of the smaller axis of the ellipse of pressure, and proceeds perpendicularly to this axis along that ellipse.

The formulæ found become especially simple when both the bodies which touch each other are spheres. In this case the surface of pressure is part of a sphere. If ρ is the reciprocal of its radius, and if ρ_1 and ρ_2 are the reciprocals of the radii of the touching spheres, then we have the relation $(\vartheta_1 + \vartheta_2)\rho = \vartheta_2\rho_1 + \vartheta_1\rho_2$; which for spheres of the same material takes the simpler form $2\rho = \rho_1 + \rho_2$. The curve of pressure is a circle whose radius we shall call a. If we put

$$x^2 + y^2 = r^2, \quad \frac{r^2}{a^2 + u} + \frac{z^2}{u} = 1,$$

then will

$$\mathrm{P} = \frac{3p}{16\pi}\int_u^\infty \left(1 - \frac{r^2}{a^2 + u} - \frac{z^2}{u}\right)\frac{du}{(a^2+u)\sqrt{u}},$$

which may also be expressed in a form free of integrals.

We easily find for a, the radius of the circle of pressure, and for α, the distance through which the spheres approach each other, and also for the displacement ζ over the part of the plane $z=0$ inside the circle of pressure:—

$$a = \sqrt[3]{\frac{3p(\vartheta_1+\vartheta_2)}{16(\rho_1+\rho_2)}}, \qquad \alpha = \frac{3p(\vartheta_1+\vartheta_2)}{16a},$$

$$\zeta = \frac{3p}{32}\vartheta\frac{2a^2-r^2}{a^3}.$$

Outside the circle of pressure ζ is represented by a somewhat more complicated expression, involving an inverse tangent. Very simple expressions may be got for ξ and η at the plane $z=0$. For the compression at the plane $z=0$ we find

$$\sigma = -\frac{3p}{2\mathrm{K}(1+2\theta)\pi}\frac{\sqrt{a^2-r^2}}{a^3}$$

inside the circle of pressure; outside it $\sigma=0$. For the pressure Z_z inside the circle of pressure we obtain

$$Z_z = \frac{3p}{2\pi}\frac{\sqrt{a^2-r^2}}{a^3};$$

at the centre we have

$$Z_z = \frac{3p}{2\pi a^2}, \qquad X_x = Y_y = \frac{1+4\theta}{4(1+2\theta)}\frac{3p}{\pi a^2}.$$

The formulæ obtained may be directly applied to particular cases. In most bodies θ may with a sufficient approximation be made equal to 1. Then K becomes $\frac{3}{8}$ of the modulus of elasticity; ϑ becomes equal to $\frac{32}{9}$ times the reciprocal of that modulus; in all bodies ϑ is between three and four times this reciprocal value. If, for instance, we press a glass lens of 100 metres radius with the weight of 1 kilogramme against a plane glass plate (in which case the first Newton's ring would have a radius of about 5·2 millimetres), we get a surface of pressure which is part of a sphere of radius equal to 200 metres. The radius of the circle of pressure is 2·67 millimetres; the distance of approach of the glass bodies amounts to only 71 millionths of a millimetre. The pressure Z_z at the centre of the surface of pressure is 0·0669 kilogrammes per square millimetre, and the perpendicular pressures X_x and Y_y have about $\frac{5}{6}$ that value. As a second example, consider a number of steel spheres pressed by their own weight against a rigid horizontal plane. We find that the radius of the circle of pressure in millimetres is very approximately $a = \frac{1}{1000}R^{\frac{4}{3}}$. Hence for spheres of radii

1 mm., 1 m., 1 km., 1000 km.,

a becomes about

$\frac{1}{1000}$ mm., 10 mm. 100 m., 1000 km.,

or $a = \frac{1}{1000}, \quad \frac{1}{100}, \quad \frac{1}{10}, \quad \frac{1}{1}$

of the radius. For spheres whose radius exceeds 1 km. the radius of the circle of pressure is more than $\frac{1}{10}$ of the radius of the sphere. Our calculations do not apply to such ratios, for we presupposed the ratio to be a small fraction. But the very fact that for such large spheres equilibrium is no longer possible with small deformations shows that equilibrium is altogether impossible. Consider further two steel spheres of equal radius touching one another and pressed together only by their mutual gravitational attraction. In millimetres we find[1] the radius of the circle of pressure to be $\rho = 0{\cdot}000000378R^{\frac{5}{3}}$. If the radius of the two spheres is 4·3 kilometres, then $\rho = \frac{1}{100}R$; if it is 136 kilometres, then $\rho = \frac{1}{10}R$. That value of R, for which the elastic forces cease to be able to equilibrate gravitational attraction, will lie between the above values and nearer to the greater. If steel spheres of greater radius be placed touching each other, they will break up into pieces whose dimensions are of the order of the values of R just mentioned.

Finally, we shall apply the formulæ we have obtained to the impact of elastic bodies. It follows, both from existing observations and from the results of the following considerations, that the time of impact, *i.e.* the time during which the impinging bodies remain in contact, is very small in absolute value; yet it is very large compared with the time taken by waves of elastic deformation in the bodies in question to traverse distances of the order of magnitude of that part of their surfaces which is common to the two bodies when in

[1] In these calculations the modulus of elasticity of steel is taken to be 20,000 kg/mm^2, its density 7·7, and the mean density of the earth 6.

closest contact, and which we shall call the surface of impact. It follows that the elastic state of the two bodies near the point of impact during the whole duration of impact is very nearly the same as the state of equilibrium which would be produced by the total pressure subsisting at any instant between the two bodies, supposing it to act for a long time. If then we determine the pressure between the two bodies by means of the relation which we previously found to hold between this pressure and the distance of approach along the common normal of two bodies at rest, and also throughout the volume of each body make use of the equations of motion of elastic solids, we can trace the progress of the phenomenon very exactly. We cannot in this way expect to obtain general laws; but we may obtain a number of such if we make the further assumption that the time of impact is also large compared with the time taken by elastic waves to traverse the impinging bodies from end to end. When this condition is fulfilled, all parts of the impinging bodies, except those infinitely close to the point of impact, will move as parts of rigid bodies; we shall show from our results that the condition in question may be realised in the case of actual bodies.

We retain our system of axes of xyz. Let α be the resolved part parallel to the axis of z of the distance of two points one in each body, which are chosen so that their distance from the surface of impact is small compared with the dimensions of the bodies as a whole, but large compared with the dimensions of the surface of impact; and let α' denote the differential coefficient of α with regard to the time. If dJ is the momentum lost in time dt by one body and gained by the other, then it follows from the theory of impact of rigid bodies that $d\alpha' = -k_1 dJ$, where k_1 is a quantity depending only upon the masses of the impinging bodies, their principal moments of inertia, and the situation of their principal axes of inertia relatively to the normal at the point of impact.[1] On

[1] See Poisson, *Traité de mécanique*, II. chap. vii. In the notation there employed we have for the constant k_1

$$k_1 = \frac{1}{M} + \frac{(b\cos\gamma - c\cos\beta)^2}{A} + \frac{(c\cos\alpha - a\cos\gamma)^2}{B} + \frac{(a\cos\beta - b\cos\alpha)^2}{C}$$

$$+ \frac{1}{M'} + \frac{(b'\cos\gamma' - c'\cos\beta')^2}{A'} + \frac{(c'\cos\alpha' - a'\cos\gamma')^2}{B'} + \frac{(a'\cos\beta' - b'\cos\alpha')^2}{C'}.$$

the other hand, dJ is equal to the element of time dt, multiplied by the pressure which during that time acts between the bodies. This is $k_2a^{\frac{3}{2}}$, where k_2 is a constant to be determined from what precedes, which constant depends only on the form of the surfaces and the elastic properties quite close to the point of impact. Hence $dJ = k_2a^{\frac{3}{2}}dt$ and $da' = -k_1k_2a^{\frac{3}{2}}dt$; integrating, and denoting by a'_0 the value of a' just before impact, we find

$$a'^2 - a'^2_0 + \tfrac{4}{5}k_1k_2a^{\frac{5}{2}} = 0,$$

which equation expresses the principle of the conservation of energy. When the bodies approach as closely as possible a' vanishes; if a_m denote the corresponding value of a, then $a_m = \left(\frac{5a'^2_0}{4k_1k_2}\right)^{\frac{2}{5}}$, and the simultaneous maximum pressure is $p_m = k_2a_m^{\frac{3}{2}}$. From this we at once obtain the dimensions of the surface of impact.

In order to deduce the variation of the phenomenon with the time, we integrate again and obtain

$$t = \int_a^{a_m} \frac{da}{\sqrt{a'^2_0 - \frac{4}{5}k_1k_2a^{\frac{5}{2}}}}.$$

The upper limit is so chosen that $t = 0$ at the instant of nearest approach. For each value of the lower limit a, the double sign of the radical gives two equal positive and negative values of t. Hence a is an even and a' an odd function of t; immediately after impact the points of impact separate along the normal with the same relative velocity with which they approached each other before impact. And the same transcendental function which represents the variation of a' between its initial and final values, also represents the variations of all the component velocities from their initial to their final values.

In the first place, the bodies touch when $a = 0$; they separate when a again $= 0$. Hence the duration of contact, that is the time of impact, is

$$\mathrm{T} = 2\int_0^{a_m} \frac{da}{\sqrt{a'^2_0 - \frac{4}{5}k_1k_2a^{\frac{5}{2}}}} = 2\eta\sqrt[5]{\frac{25}{16a'_0k_1^2k_2^2}} = 2\eta\frac{a_m}{a'_0},$$

$$\eta = \int_0^1 \frac{d\epsilon}{\sqrt{1-\epsilon^{\frac{5}{2}}}} = 1 \cdot 4716.$$

Thus the time of impact may become infinite in various ways without the time, with which it is to be compared, also becoming infinite. In particular the time of impact becomes infinite when the initial relative velocity of the impinging bodies is infinitely small; so that whatever be the other circumstances of any given impact, provided the velocities are chosen small enough, the given developments will have any accuracy desired. In every case this accuracy will be the same as that of the so-called laws of impact of perfectly elastic bodies for the given case. For the direct impact of two spheres of equal radius R and of the same material of density q the constants k_1 and k_2 are

$$k_1 = \frac{3}{2\mathrm{R}^3\pi q}, \quad k_2 = \frac{8}{3k_1}\sqrt{\frac{\mathrm{R}}{2}};$$

hence in the particular case of two equal steel spheres of radius R, taking the millimetre as unit of length, and the weight of one kilogramme as unit of force, we have

$$\log k_1 = 8 \cdot 78 - 3 \log \mathrm{R},$$
$$\log k_2 = 4 \cdot 03 + \tfrac{1}{2} \log \mathrm{R}.$$

Thus for two such spheres impinging with relative velocity v:

the radius of the surface of impact .	$a_m = 0 \cdot 0020 \mathrm{R} v^{\frac{2}{5}}$mm,
the time of impact .	$\mathrm{T} = 0 \cdot 000024 \mathrm{R} v^{-\frac{1}{5}}$sec,
the total pressure at the instant of nearest approach . . .	$p_m = 0 \cdot 00025 \mathrm{R}^2 v^{\frac{6}{5}}$kg,
the simultaneous maximum pressure at the centre of impact per unit area . . .	$p'_m = 29 \cdot 1 v^{\frac{2}{5}}$kg/mm^2.

For instance, when the radius of the spheres is 25 mm., the velocity 10 mm/sec, then $a_m = 0 \cdot 13$ mm., $\mathrm{T} = 0 \cdot 00038$ sec., $p_m = 2 \cdot 47$ kg., $p'_m = 73 \cdot 0$ kg/mm.^{2} For two steel spheres as large as the earth, impinging with an initial velocity of 10 mm/sec, the duration of contact would be nearly 27 hours.

VI

ON THE CONTACT OF RIGID ELASTIC SOLIDS AND ON HARDNESS

(*Verhandlungen des Vereins zur Beförderung des Gewerbefleisses*, November 1882.)

WHEN two elastic bodies are pressed together, they touch each other not merely in a mathematical point, but over a small but finite part of their surfaces, which part we shall call the surface of pressure. The form and size of this surface and the distribution of the stresses near it have been frequently considered (Winkler, *Lehre von der Elasticität und Festigkeit*, Prag. 1867, I. p. 43; Grashof, *Theorie der Elasticität und Festigkeit*, Berlin, 1878, pp. 49-54); but hitherto the results have either been approximate or have even involved unknown empirical constants. Yet the problem is capable of exact solution, and I have given the investigation of the problem in vol. xcii. of the *Journal für reine und angewandte Mathematik*, p. 156.[1] As some aspects of the subject are of considerable technical interest, I may here treat it more fully, with an addition concerning hardness. I shall first restate briefly the proof of the fundamental formulæ.

We first imagine the two bodies brought into mathematical contact; the common normal coincides with the line of action of the pressure which the one body exerts upon the other. In the common tangent plane we take rectangular rectilinear axes of xy, the origin of which coincides with the point of contact; the third perpendicular axis is that of z. We can confine our attention to that part of each body which is very close to the point of contact, since here the stresses are extremely great compared with those occurring elsewhere, and

[1] See V. p. 146.

consequently depend only to the very smallest extent on the forces applied to other parts of the bodies. Hence it is sufficient to know the form of the surfaces infinitely near the point of contact. To a first approximation, if we consider each body separately, we may even suppose their surfaces to coincide with the common tangent plane $z = 0$, and the common normal to coincide with the axis of z; to a second approximation, when we wish to consider the space between the bodies, it is sufficient to retain only the quadratic terms in xy in the development of the equations of the surfaces. The distance between opposite points of the two surfaces then becomes a homogeneous quadratic function of the x and y belonging to the two points; and we can turn our axes of x and y so that from this function the term in xy disappears. After completing this operation let the distance between the surfaces be given by the equation $e = Ax^2 + By^2$. A and B must of necessity have the same sign, since e cannot vanish; when we construct the curves for which e has the same value, we obtain a system of similar ellipses, whose centre is the origin. Our problem now is to assign such a form to the surface of pressure and such a system of displacements and stresses to its neighbourhood, that (1) these displacements and stresses may satisfy the differential equations of equilibrium of elastic bodies, and the stresses may vanish at a great distance from the surface of pressure; that (2) the tangential components of stress may vanish all over both surfaces; that (3) at the surface the normal pressure also may vanish outside the surface of pressure, but inside it pressure and counterpressure may be equal; the integral of this pressure, taken over the whole surface of pressure, must be equal to the total pressure p fixed beforehand; that, lastly (4) the distance between the surfaces, which is altered by the displacements, may vanish in the surface of pressure, and be greater than zero outside it. To express the last condition more exactly, let ξ_1, η_1, ζ_1 be the displacements parallel to the axes of x, y, z in the first body, ξ_2, η_2, ζ_2 those in the second. In each let them be estimated relatively to the undeformed parts of the bodies, which are at a distance from the surface of pressure; and let α denote the distance by which these parts are caused by the pressure to approach each other. Then any two points of the two bodies, which have the same coordinates x, y, have approached each other by a distance $\alpha - \zeta_1 + \zeta_2$ under the action of the pressure; this approach must in the surface of pressure neutralise the original distance $Ax^2 + By^2$ Hence here we must have $\zeta_1 - \zeta_2 = \alpha - Ax^2 - By^2$, whilst elsewhere over the surfaces $\zeta_1 - \zeta_2 > \alpha - Ax^2 - By^2$. All these conditions can be satisfied only by one single system of displacements; I shall give this system, and prove that it satisfies all requirements.

As surface of pressure we take an ellipse, whose axes coincide with those of the ellipses e = constant, but whose shape is more elongated than theirs. We reserve the determination of the lengths of its semi-axes a and b until later. First we define a function P by the equation

$$P = \frac{3p}{16\pi}\int_u^\infty \left(1 - \frac{x^2}{a^2+\lambda} - \frac{y^2}{b^2+\lambda} - \frac{z^2}{\lambda}\right)\frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)\lambda}},$$

where the lower limit of integration is the positive root of the cubic equation

$$0 = 1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{u}.$$

The quantity u is an elliptic coordinate of the point xyz; it is constant over certain ellipsoids, which are confocal with the ellipse of pressure, and vanishes at all points which are infinitely close to the surface of pressure. The function P has a simple meaning in the theory of potential. It is the potential of an infinitely flattened gravitating ellipsoid, which would just fill the surface of pressure; in that theory it is proved that P satisfies the differential equation

$$\nabla^2 P = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = 0.$$

Now from this P we deduce two functions Π, one of which refers to the one body, the second to the other, and we make

$$\Pi_1 = -\frac{1}{K_1}\left(zP - \frac{1}{1+2\Theta_1}\int_z^\infty P\,dz\right),$$

$$\Pi_2 = -\frac{1}{K_2}\left(zP - \frac{1}{1+2\Theta_2}\int_z^\infty P\,dz\right).$$

Here K, Θ denote the coefficients of elasticity in Kirchhoff's notation. Young's modulus of elasticity is expressed in terms of these coefficients by the equation

$$E = 2K\frac{1+3\Theta}{1+2\Theta}.$$

The ratio between lateral contraction and longitudinal extension is

$$\mu = \frac{\Theta}{1+2\Theta}.$$

For bodies like glass or steel, this ratio is nearly $\frac{1}{3}$, or Θ nearly 1, and K is nearly $\frac{3}{8}$ E. For slightly compressible bodies the ratio is nearly $\frac{1}{2}$; here then $\Theta = \infty$, $K = \frac{1}{3}E$. As a matter of fact a particular combination of K and Θ will play the principal part in our formulæ, for which we shall therefore introduce a special symbol. We put

$$\vartheta = \frac{2(1+\Theta)}{K(1+2\Theta)}.$$

In bodies like glass, $\vartheta = 4/3K = 32/9E$; in all bodies ϑ lies between $3/E$ and $4/E$, since Θ lies between 0 and ∞. In regard to the Π's we must note that calculated by the above formulæ they have infinite values; but their differential coefficients, which alone concern us, are finite. It would only be necessary to add to the Π's infinite constants of suitable magnitude to make them finite. By a simple differentiation, remembering that $\nabla^2 P = 0$, we find

$$\nabla^2\Pi_1 = -\frac{2}{K_1}\frac{\partial P}{\partial z}, \quad \nabla^2\Pi_2 = -\frac{2}{K_2}\frac{\partial P}{\partial z}.$$

We now assume the following expressions for the displacements in the two bodies:—

$$\xi_1 = \frac{\partial\Pi_1}{\partial x}, \quad \eta_1 = \frac{\partial\Pi_1}{dy}, \quad \zeta_1 = \frac{\partial\Pi_1}{dz} + 2\vartheta_1 P,$$

$$\xi_2 = -\frac{\partial\Pi_2}{\partial x}, \quad \eta_2 = -\frac{\partial\Pi_2}{dy}, \quad \zeta_2 = -\frac{\partial\Pi_2}{dz} - 2\vartheta_2 P,$$

whence follow

$$\sigma_1 = \frac{\partial\xi_1}{\partial x} + \frac{\partial\eta_1}{\partial y} + \frac{\partial\zeta_1}{\partial z} = \nabla^2\Pi_1 + 2\vartheta_1\frac{\partial P}{\partial z} = \frac{2}{K_1(1+2\Theta_1)}\frac{\partial P}{\partial z},$$

$$\sigma_2 = -\frac{2}{K_2(1+2\Theta_2)}\frac{\partial P}{\partial z}.$$

In the first place, this system satisfies the equations of equilibrium, for we have

$$\nabla^2\xi_1 + (1+2\Theta_1)\frac{\partial\sigma_1}{\partial x} = \frac{\partial\nabla^2\Pi_1}{\partial x} + \frac{2}{K_1}\frac{\partial^2 P}{\partial z\partial x} = 0,$$

and similar equations hold for ξ_2, η_1, η_2; for the ζs we get the same result, remembering that $\nabla^2 P = 0$. For the tangential stress components at the surface ($z=0$) we find, leaving out the indices:—

$$X_z = -K\left(\frac{\partial\zeta}{\partial x} + \frac{\partial\xi}{\partial z}\right) = -K\left(2\frac{\partial^2\Pi}{\partial z\partial x} + 2\vartheta\frac{\partial P}{\partial x}\right) = 2z\frac{\partial^2 P}{\partial x\partial z} = 0,$$

$$Y_z = -K\left(\frac{\partial\eta}{\partial z} + \frac{\partial\zeta}{\partial y}\right) = -K\left(2\frac{\partial^2\Pi}{\partial y\partial z} + 2\vartheta\frac{\partial P}{\partial x}\right) = 2z\frac{\partial^2 P}{\partial y\partial z} = 0,$$

as the second condition requires.

It is more troublesome to prove that the third condition is satisfied. We again omit indices, as the calculation applies equally to both bodies. We have generally

$$Z_z = -2K\left(\frac{\partial\zeta}{\partial z} + \Theta\sigma\right) = -2K\left\{\frac{\partial^2\Pi}{\partial z^2} + \frac{2(2+3\Theta)}{K(1+2\Theta)}\frac{\partial P}{\partial z}\right\}$$

$$= 2z\frac{\partial^2 P}{\partial z^2} - 2\frac{\partial P}{\partial z};$$

therefore at the surface $Z_z = -2\dfrac{\partial P}{\partial z}$. Now, using the equation for u, we have generally

$$\frac{\partial P}{\partial z} = -\frac{3p}{8\pi}z\int_u^\infty \frac{d\lambda}{\lambda\sqrt{(a^2+\lambda)(b^2+\lambda)\lambda}},$$

and therefore at the surface $\dfrac{\partial P}{\partial z}$ vanishes, as it must do, and with it Z_z, at any rate outside the surface of pressure. In the compressed surface, where $u=0$, the expression takes the form $0 . \infty$; the ordinary procedure for the evaluation of such an indeterminate form gives

$$\frac{\partial \mathrm{P}}{\partial z} = -\frac{3p}{8\pi ab}\frac{z^2\frac{\partial u}{\partial z}}{u\sqrt{u}},$$

that is, since for $u = 0$ we have

$$z^2 = u\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right),$$

$$\mathrm{Z}_z = -2\frac{\partial \mathrm{P}}{\partial z} = \frac{3p}{2\pi ab}\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}.$$

Here no quantity occurs which could be affected by an index. Hence in the surface of pressure Z_z is the same for both bodies; pressure and counter-pressure are equal. Lastly, the integral of Z_z over the surface of pressure is $3p/4\pi ab$ times the volume of an ellipsoid whose semi-axes are $1, a, b$; *i.e.* it equals p, and therefore the total pressure has the required value.

It remains to be shown that the fourth condition can be satisfied by a suitable choice of the semi-axes a and b. For this purpose we remark that

$$\zeta_1 = \frac{\partial \Pi_1}{\partial z} + 2\vartheta_1 \mathrm{P} = -\frac{z}{\mathrm{K}_1}\frac{\partial \mathrm{P}}{\partial z} + \vartheta_1 \mathrm{P},$$

so that at the surface $\zeta_1 = \vartheta_1 \mathrm{P}$ and $\zeta_2 = \vartheta_2 \mathrm{P}$. Since inside the surface of pressure the lower limit u of the integral is constantly zero, inside that surface P has the form $\mathrm{P} = \mathrm{L} - \mathrm{M}x^2 - \mathrm{N}y^2$; and therefore it is necessary so to determine a, b and α that $(\vartheta_1 + \vartheta_2)\mathrm{M} = \mathrm{A}$, $(\vartheta_1 + \vartheta_2)\mathrm{N} = \mathrm{B}$, $(\vartheta_1 + \vartheta_2)\mathrm{L} = \alpha$, so as to satisfy the equation $\zeta_1 - \zeta_2 = \alpha - \mathrm{A}x^2 - \mathrm{B}y^2$, and this determination is always possible. Written explicitly the equations for a and b are

$$\left.\begin{aligned} \int_0^\infty \frac{du}{\sqrt{(a^2+u)^3(b^2+u)u}} &= \frac{\mathrm{A}}{\vartheta_1+\vartheta_2}\frac{16\pi}{3p}, \\ \int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)^3u}} &= \frac{\mathrm{B}}{\vartheta_1+\vartheta_2}\frac{16\pi}{3p}. \end{aligned}\right\} \quad \text{(I)}$$

Finally, it is easily shown that the very essential inequality, which must be fulfilled outside the surface of pressure, is actually satisfied; but I omit the proof, since it requires the repetition of complicated integrals.

Thus our formulæ express the correct solution of the proposed problem, and we may use them to answer the chief questions which may be asked concerning the subject. It is necessary to carry the evaluation of the quantities a and b a step further; for the equations hitherto found for them cannot straightway be solved, and in general not even the quantities A and B are explicitly known. I assume that we are given the four principal curvatures (reciprocals of the principal radii of curvature) of the two surfaces, as well as the relative position of their planes; let the former be ρ_{11} and ρ_{12} for the one body, ρ_{21} and ρ_{22} for the other, and let ω be the angle between the planes of ρ_{11} and of ρ_{21}. Let the ρ's be reckoned positive when the corresponding centres of curvature lie inside the body considered. Let our axes of xy be placed so that the xz-plane makes with the plane of ρ_{11} the angle ω', so far unknown. Then the equations of the surfaces are

$$2z_1 = \rho_{11}(x\cos\omega' + y\sin\omega')^2 + \rho_{12}(y\cos\omega' - x\sin\omega')^2,$$

$$2z_2 = -\rho_{21}\{x\cos(\omega'-\omega) + y\sin(\omega'-\omega)\}^2 - \rho_{22}\{y\cos(\omega'-\omega) - x\sin(\omega'-\omega)\}^2.$$

The difference $z_1 - z_2$ gives the distance between the surfaces. Putting it $= Ax^2 + By^2$, and equating coefficients of x^2, xy, y^2 on both sides, we obtain three equations for ω', A and B; their solution gives for the angle ω', which evidently determines the position of the axes of the ellipse of pressure relatively to the surfaces, the equation

$$\tan 2\omega' = \frac{(\rho_{21} - \rho_{22})\sin 2\omega}{\rho_{11} - \rho_{12} + (\rho_{21} - \rho_{22})\cos 2\omega},$$

for A and B

$$2(A + B) = \rho_{11} + \rho_{12} + \rho_{21} + \rho_{22},$$

$$2(A - B) = -\sqrt{(\rho_{11} - \rho_{12})^2 + 2(\rho_{11} - \rho_{12})(\rho_{21} - \rho_{22})\cos 2\omega + (\rho_{21} - \rho_{22})^2}.$$

For the purpose of what follows it is convenient to introduce an auxiliary angle τ by the equation

$$\cos\tau = -\frac{A-B}{A+B},$$

and then

$$2A = (\rho_{11}+\rho_{12}+\rho_{21}+\rho_{22})\sin^2\frac{\tau}{2},$$

$$2B = (\rho_{11}+\rho_{12}+\rho_{21}+\rho_{22})\cos^2\frac{\tau}{2}.$$

We shall introduce these values into the equations for a and b, and at the same time transform the integrals occurring there by putting in the first $u=b^2z^2$, in the second $u=a^2z^2$. Denoting the ratio b/a by k we get

$$\frac{1}{a^3}\int_0^\infty \frac{dz}{\sqrt{(1+k^2z^2)^3(1+z^2)}} = \frac{4\pi}{3p}\,\frac{\rho_{11}+\rho_{12}+\rho_{21}+\rho_{22}}{\vartheta_1+\vartheta_2}\sin^2\frac{\tau}{2},$$

$$\frac{1}{b^3}\int_0^\infty \frac{dz}{\sqrt{\left(1+\frac{z^2}{k^2}\right)^3(1+z^2)}} = \frac{4\pi}{3p}\,\frac{\rho_{11}+\rho_{12}+\rho_{21}+\rho_{22}}{\vartheta_1+\vartheta_2}\cos^2\frac{\tau}{2}.$$

Dividing the one equation by the other we get a new one, involving only k and τ, so that k is a function of τ alone; and the same is true of the integrals occurring in the equations. If we solve them by writing

$$a = \mu\sqrt[3]{\frac{3p(\vartheta_1+\vartheta_2)}{8(\rho_{11}+\rho_{12}+\rho_{21}+\rho_{22})}},$$

$$b = \nu\sqrt[3]{\frac{3p(\vartheta_1+\vartheta_2)}{8(\rho_{11}+\rho_{12}+\rho_{21}+\rho_{22})}},$$

then μ and ν depend only on τ, that is on the ratio of the axes of the ellipse $e=$ constant. The integrals in question may all be reduced to complete elliptic integrals of the first species and their differential coefficients with respect to the modulus, and can therefore be found by means of Legendre's tables without further quadratures. But the calculations are wearisome, and I have therefore calculated the table given below,[1] in which are found the values of μ and ν for ten values of the argument τ; presumably interpolation between these values will always yield a sufficiently near approximation. We may sum up our results thus: The form of the ellipse of pressure is conditioned solely by the form of the ellipses $e =$ constant. With a given shape its linear dimensions vary as the cube root of the pressure, inversely as the cube root of the arithmetical mean of the curvatures, and also directly as the cube root of the mean value of the elastic coefficients ϑ; that is, very nearly as the cube root of the mean value of the reciprocals of the moduli of elasticity. It is to be noted that the area of the ellipse of pressure increases, other things being equal, the more elongated its form. If we imagine that of two bodies touching each other one be rotated about the common normal while the total pressure is kept the same, then the area of the surface of pressure will be a maximum and the mean pressure per unit area a minimum in that position in which the ratio of the axes of the ellipse of pressure differs most from 1.

Our next inquiry concerns the indentations experienced by the bodies and the distance by which they approach each other in consequence of the pressure; the latter we called α and found its value to be $(\vartheta_1 + \vartheta_2)\mathrm{L}$. Transforming the integral L a little, we get

$$\alpha = \frac{3p}{8\pi}\,\frac{\vartheta_1 + \vartheta_2}{a}\int_0^\infty \frac{dz}{\sqrt{(1+k^2z^2)(1+z^2)}}.$$

The distances by which the origin approaches the distant

1

τ	90	80	70	60	50	40	30	20	10	0
μ	1·000	1·128	1·284	1·486	1·754	2·136	2·731	3·778	6·612	∞
ν	1·000	0·893	0·802	0·717	0·641	0·567	0·493	0·408	0·319	0

parts of the bodies may be suitably denoted as indentations. Their values are easily found by multiplying by $\vartheta_1 + \vartheta_2$ and thus separating α into two portions. Substituting for α its value, we see that α involves a numerical factor which depends on the form of the ellipse of pressure; and that for a given value of this factor α varies as the $\frac{2}{3}$ power of the pressure, as the $\frac{2}{3}$ power of the mean value of the coefficients ϑ, and as the cube root of the mean value of the curvatures. If one or more of these curvatures become infinitely great, then distance of approach and indentations become infinitely great—a result sufficiently illustrated by the penetrating action of points and edges.

We assumed the surface of pressure to be so small that the deformed surfaces could be represented by quadric surfaces throughout a region large compared with the surface of pressure. Such an assumption can no longer be made after application of the pressure; in fact outside the surface of pressure the surface can only be represented by a complicated function. But we find that inside the surface of pressure the surface remains a quadric surface to the same approximation as before. Here we have $\zeta_1 - \zeta_2 = \alpha - \mathrm{A}x^2 - \mathrm{B}y^2 = \alpha - z_1 + z_2$, again $\zeta_1 = \vartheta_1 \mathrm{P}$, $\zeta_2 = \vartheta_2 \mathrm{P}$, or $\zeta_1 : \zeta_2 = \vartheta_1 : \vartheta_2$, and lastly, the equation of the deformed surface is $z = z_1 + \zeta_1 = z_2 + \zeta_2$; whence neglecting a constant, we easily deduce $(\vartheta_1 + \vartheta_2)z = \vartheta_2 z_1 + \vartheta_1 z_2$. This equation expresses what we wished to demonstrate; it also shows that the common surface after deformation lies between the two original surfaces, and most nearly resembles the body which has the greater modulus of elasticity. When spheres are in contact the surface of pressure also forms part of a sphere: when cylinders touch with axes crossed it forms part of a hyperbolic paraboloid.

So far we have spoken of the changes of form, now we will consider the stresses. We have already found for the normal pressure in the compressed surface

$$\mathrm{Z}_z = \frac{3p}{2ab\pi}\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}.$$

This increases from the periphery to the centre, as do the ordinates of an ellipsoid constructed on the ellipse of pressure; it vanishes at the edge, and at the centre is one and a half times as great as it would be if the total pressure were equally distributed over the surface of pressure. Besides Z_z the remaining two principal tensions at the origin can be expressed in a finite form. It may be sufficient to state that they are also pressures of the same order of magnitude as Z_z, and are of such intensity that, provided the material is at all compressible, it will suffer compression in all three directions. When the curve of pressure is a circle, these forces are to Z_z in the ratio of $(1+4\Theta)/2(1+2\Theta):1$; for glass about as $5/6:1$. The distribution of stress inside depends not only on the form of the ellipse of pressure, but also essentially on the elastic coefficient Θ; so that it may be entirely different in the two bodies which are in contact. When we compare the stresses in the same material for the same form but different sizes of the ellipse of pressure and different total pressures, we see that the stresses at points similarly situated with regard to the surface of pressure are proportional to each other. To get the pressures for one case at given points we must multiply the pressures at similarly situated points in the other case by the ratio of the total pressures, and divide by the ratio of the compressed areas. If we suppose two given bodies in contact and only the pressure between them to vary, the deformation of the material varies as the cube root of this total pressure.

It is desirable to obtain a clear view of the distribution

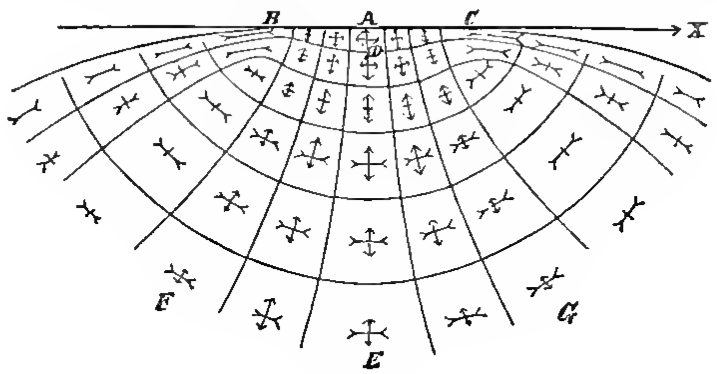

Fig. 19.

of stress in the interior; but the formulæ are far too complicated to allow of our doing this directly. But by considering the stresses near the z-axis and near the surface we can form a rough notion of this distribution. The result may be expressed by the following description and the accompanying diagram (Fig. 19), which represents a section through the axis of

z and an axis of the ellipse of pressure; arrow-heads pointing towards each other denote a tension, those pointing away from each other a pressure. The figure relates to the case in which $\Theta = 1$. The portion $ABDC$ of the body, which originally formed an elevation above the surface of pressure, is now pressed into the body like a wedge; hence the pressure is transmitted not only in the direct line AE, but also, though with less intensity, in the inclined directions AF and AG. The consequence is that the element is also powerfully compressed laterally; while the parts at F and G are pressed apart and the intervening portions stretched. Hence at A on the element of area perpendicular to the x-axis there is pressure, which diminishes inwards, and changes to a tension which rapidly attains a maximum, and then, with increasing distance, diminishes to zero. Since the part near A is also laterally compressed, all points of the surface must approach the origin, and must therefore give rise to stretching in a line with the origin. In fact the pressure which acts at A parallel to the axis of x already changes to a tension inside the surface of pressure as we proceed along the x-axis; it attains a maximum near its boundary and then diminishes to zero. Calculation shows that for $\Theta = 1$ this tension is much greater than that in the interior. As regards the third principal pressure which acts perpendicular to the plane of the diagram, it of course behaves like the one parallel to the x-axis; at the surface it is a pressure, since here all points approach the origin. If the material is incompressible the diagram is simplified, for since the parts near A do not approach each other, the tensions at the surface disappear.

We shall briefly mention the simplifications occurring in the formulæ, when the bodies in contact are spheres, or are cylinders which touch along a generating line. In the first case we have simply $k = \mu = \nu = 1$, $\rho_{11} = \rho_{12} = \rho_1$, $\rho_{21} = \rho_{22} = \rho_2$; hence

$$a = b = \sqrt{\frac{3p(\vartheta_1 + \vartheta_2)}{16(\rho_1 + \rho_2)}}, \quad \alpha = \frac{3p(\vartheta_1 + \vartheta_2)}{16a}.$$

The formulæ for the case of cylinders in contact are not got so directly. Here the major semi-axis a of the ellipse becomes infinitely great; we must also make the total pressure

p infinite, if the pressure per unit length of the cylinder is to be finite. We then have in the second of equations (I) $B = \frac{1}{2}(\rho_1 + \rho_2)$. Further, we may neglect u compared with a^2, take a outside the sign of integration, and put for the indeterminate quantity $p/a = \infty / \infty$ an arbitrary finite constant, say $\frac{4}{3}p'$; then, as we shall see directly, p' is the pressure per unit length of the cylinder. The integration of the equation can now be performed, and gives

$$b = \sqrt{\frac{p'(\vartheta_1 + \vartheta_2)}{\pi(\rho_1 + \rho_2)}}.$$

For the pressure Z_z we find

$$Z_z = \frac{2p'}{\pi b^2}\sqrt{b^2 - y^2},$$

and it is easy to see that p' has the meaning stated. The distance of approach a, according to our general formula, becomes logarithmically infinite. This means that it depends not merely on what happens at the place of contact, but also on the shape of the body as a whole; and thus its determination no longer forms part of the problem we are dealing with.

I shall now describe some experiments that I have performed with a view to comparing the formulæ obtained with experience; partly that I may give a proof of the reliability of the consequences deduced, and their applicability to actual circumstances, and partly to serve as an example of their application. The experiments were performed in such a way that the bodies used were pressed together by a horizontal one-armed lever. From its free end were suspended the weights which determined the pressure, and to it the one body was fastened close to the fulcrum. The other body, which formed the basis of support, was covered by the thinnest possible layer of lamp-black, which was intended to record the form of the surface of pressure. If the experiment succeeded, the lampblack was not rubbed away, but only squeezed flat; in transmitted light the places of action of the pressure could hardly be detected; but in reflected light they showed as small brilliant circles or ellipses, which could be measured fairly accurately by the microscope. The following numbers are the means of from 5 to 8 measurements.

I first examined whether the dimensions of the surface of pressure increased as the cube root of the pressure. To this end a glass lens of 28·0 mm. radius was fastened to the lever; the small arm of the lever measured 114·0 mm., the large one 930 mm. The basis of support was a plane glass plate; the Young's modulus was determined for a bar of the same glass and found to be 6201 kg/mm². According to Wertheim, Poisson's ratio for glass is 0·32, whence $\Theta = \frac{8}{9}$, K = 2349kg/mm,² and $\vartheta = 0005790$ mm²/kg. Hence our formula gives for the diameter of the circle of pressure in mm., $d = 0{\cdot}3650p^{\frac{1}{3}}$, where p is measured in kilogrammes weight. In the following table the first row gives in kilogrammes the weight suspended from the long arm of the lever, the second the measured diameter of the surface of pressure in turns of the micrometer screw of pitch 0·2737 mm. Lastly, the third row gives the quotient $d : \sqrt[3]{p}$, which should, according to the preceding, be a constant.

p	0·2	0·4	0·6	0·8	1·0	1·5	2·0	2·5	3·0	3·5
d	1·56	2·03	2·19	2·59	2·68	3·13	3·52	3·69	3·97	4·02
$d:\sqrt[3]{p}$	2·67	2·75	2·60	2·79	2·68	2·73	2·79	2·71	2·70	2·65

The ratio in question does indeed remain constant, apart from irregularities, though the weights vary up to fifteen times their initial value. To get the theoretical value of the ratio we must divide the factor ·3650 calculated above by the pitch in millimetres of the screw, and multiply by the cube root of the ratio of the long to the short arm of the lever; we thus obtain 2·685, a number almost exactly coincident with 2·707, the mean of the experimental numbers.

Secondly, I have tested the laws relating to the form of the curve of pressure by pressing together two glass cylinders, of equal diameter 7·37 mm., with their axes inclined at different angles to each other. If this angle be called ω, using former equations we get $\rho_{11} = \rho_{12} = \rho$, $\rho_{21} = \rho_{22} = 0$, $A + B = \rho$, $A - B = -\rho \cos \omega$, and therefore the auxiliary angle $\tau = \omega$ Hence if we determine the large and small axes of the ellipse of pressure for one and the same pressure but different inclinations, divide the major axes by the function μ belonging to the inclination used and the minor axes by the corresponding function ν, the quotient of all these divisions must be one and the same constant, namely, the quantity $2(3p\vartheta/8\rho)^{\frac{1}{3}}$. The following table gives in the first column the inclination ω in degrees, in the next two the values of $2\,a$ and $2\,b$ as measured in parts of the scale of the micrometer eye-piece, of which 96 equal one millimetre, and in the last two the quotients $2a/\mu$ and $2b/\nu$:—

ω	$2\,a$	$2\,b$	$\frac{2\,a}{\mu}$	$\frac{2\,b}{\nu}$
90	40·6	40·6	40·6	40·6
80	45·4	36·6	40·2	41·0
70	52·8	31·0	41·3	38·7
60	59·6	27·6	40·0	38·5
50	72·2	26·4	41·2	41·2
40	90·4	23·8	42·2	42·0
30	110·0	21·0	40·3	42·6
20	156·2	18·4	41·3	45·3
10	274·6	15·0	41·6	47·0

The quotients are fairly constant, excepting those for the minor axes at small inclinations. But at such an inclination it is extremely difficult to bring the cylinders together so as to make the common tangent plane exactly horizontal; and in any other position a slight slipping of one cylinder on the other occurs, which unduly magnifies the minor axis. In all these measurements the pressure was 12 kg. weight. Taking for ϑ the value ·0005790 already used, we get from the given values the value of the constant to be 40·80, which agrees almost exactly with 40·97, the mean resulting from the values for a; whilst it differs slightly, for the reasons explained, from 41·88, the mean resulting from the value for b.

Lastly, I have attempted to examine the effect of the moduli of elasticity by pressing a steel lens against planes of different metals. But here I encountered difficulties in the observation. In the first place, it is not so easy to obtain quite plane and smooth surfaces as for glass; secondly, the metallic surfaces cannot so easily be covered with lamp-black; thirdly, we have to confine ourselves to very small pressures so as not to exceed the elastic limits. All these causes together preclude our obtaining any but very imperfect curves of pressure, and in measuring these there is room for discretion. I obtained values which were always of the order of magnitude of those calculated, but were too uncertain to be of use in accurately testing the theory. However, the numbers given show conclusively that our formulæ are in no sense speculations, and so will justify the application now to be made of them. The object of this is to gain a clearer notion and an exact measure of that property of bodies which we call hardness.

The hardness of a body is usually defined as the resistance it opposes to the penetration of points and edges into it. Mineralogists are satisfied in recognising in it a merely comparative property; they call one body harder than another when it scratches the other. The condition that a series of bodies may be arranged in order of hardness according to this definition is that, if A scratches B and B scratches C, then A should scratch C and not *vice versâ*; further, if a point of A scratches a plane plate of B, then a point of B should not penetrate into a plane of A. The necessity of the concurrence of these presuppositions is not directly manifest. Although experience has justified them, the method cannot give a quantitative determination of hardness of any value. Several attempts have been made to find one. Muschenbroek measured hardness by the number of blows on a chisel which were necessary to cut through a small bar of given dimensions of the material to be examined. About the year 1850 Crace-Calvert and Johnson measured hardness by the weight which was necessary to drive a blunt steel cone with a plane end 1·25 mm. in diameter to a depth of 3·5 mm. into the given material in half an hour. According to a book published in 1865,[1] Hugueny measured the same property by the weight necessary to drive a perfectly determinate point 0·1 mm. deep into the material. More recent attempts at a definition I have not met with. To all these attempts we may urge the following objections: (1) The measure obtained is not only not absolute, since a harder body is essential for the determination, but it is also entirely dependent on a point selected at random. From the results obtained we can draw no conclusions at all

[1] F. Hugueny, *Recherches expérimentales sur la dureté des corps.*

as to the force necessary to drive in another point. (2) Since finite and permanent changes of form are employed, elastic after-effects, which have nothing to do with hardness, enter into the results of measurement to a degree quite beyond estimation. This is shown only too plainly by the introduction of the time into the definition of Crace-Calvert and Johnson, and it is therefore doubtful whether the hardness of bodies thus measured is always in the order of the ordinary scale. (3) We cannot maintain that hardness thus measured is a property of the bodies in their original state (although without doubt it is dependent upon that state). For in the position in the experiment the point already rests upon permanently stretched or compressed layers of the body.

I shall now try to substitute for these another definition, against which the same objections cannot be urged. In the first place I look upon the strength of a material as determined, not by forces producing certain permanent deformations, but by the greatest forces which can act without producing deviations from perfect elasticity, to a certain predetermined accuracy of measurement. Since the substance after the action and removal of such forces returns to its original state, the strength thus defined is a quantity really relating to the original substance, which we cannot say is true for any other definition. The most general problem of the strength of isotropic bodies would clearly consist in answering the question—Within what limits may the principal stresses X_x, Y_y, Z_z in any element lie so that the limit of elasticity may not be exceeded? If we represent X_x, Y_y, Z_z as rectangular rectilinear coordinates of a point, then in this system there will be for every material a certain surface, closed or in part extending to infinity, round the origin, which represents the limit of elasticity; those values of X_x, Y_y, Z_z which correspond to internal points can be borne, the others not so. In the first place it is clear that if we knew this surface or the corresponding function $\psi(X_x, Y_y, Z_z) = 0$ for the given material, we could answer all the questions to the solution of which hardness is to lead us. For suppose a point of given form and given material pressed against a second body. According to what precedes we know all the stresses occurring in the body; we need therefore only see whether amongst them there is one corresponding to a point outside the surface $\psi(X_x, Y_y, Z_z) = 0$, to be enabled to tell whether a permanent deformation will ensue and, if so, in which of the two bodies. But so far there has not even been an attempt made to determine that surface. We only know isolated points of it: thus the points of section by the positive axes correspond to resistance to compression; those by the negative axes to tenacity; other points to resistance to torsion. In general we may say that to each point of the surface of strength corresponds a particular kind of strength of material. As long as the whole of the surface is not known to us, we shall let a definite discoverable point of the surface correspond to hardness, and be satisfied with finding out its position. This object we attain by the following definition,—*Hardness is the strength of a body relative to the kind of deformation which corresponds to contact with a circular surface of pressure.* And we get an absolute measure of the hardness if we decide that—*The hardness of a body is to be measured by the normal pressure per unit area which must act at the centre of a circular surface of pressure in order that in some point of the body the stress may just reach the limit consistent with perfect elasticity.* To justify this definition we must show (1) that the neglected circumstances are without effect; (2) that the order into which it brings bodies according to hardness coincides with the common scale of hardness. To prove the first point, suppose a body of material A in contact with one of material B, and a second body made of A in contact with one made of C. The form of the surfaces may be arbitrary near the point of contact, but we assume that the surface of pressure is circular, and that B and C are harder or as hard as A. Then we may simultaneously allow the total pressures at both contacts to increase from zero, so that the normal pressure at the centre of the circle of pressure may be the same in both cases. We know that then the same system of stresses occurs in both cases, therefore the elastic limit will first be exceeded at the same time and at points similarly situated with respect to the surface of pressure. We should from both cases get the same value for the hardness, and this hardness would correspond to the same point of the surface of strength. It is obvious that the elements in which the elastic limit is first exceeded may have very different positions relatively to the surface of pressure in different materials, and that the positions of the points of hardness in the surface of strength may be very dissimilar. We have to remark that the second body which was used to determine the hardness of A might have been of the same material A; we therefore do not require a second material at all to determine the hardness of a given one. This circumstance justifies us in designating the above as an absolute measurement. To prove the second point, suppose two bodies of different materials pressed together; let the surface of pressure be circular; let the hardness, defined as above, be for one body H, for the second softer one h. If now we increase the pressure between them until the normal pressure at the origin just exceeds h, the body of hardness h will experience a permanent indentation, whilst the other one is nowhere strained beyond its elastic limit; by moving one body over the other with a suitable pressure we can in the former produce a series of permanent indentations, whilst the latter remains intact. If the latter body have a sharp point we can describe the process as a scratching of the softer by the harder body, and thus our scale of hardness agrees with the mineralogical one. It is true that our theory does not say whether the same holds good for all contacts, for which the compressed surface is elliptical; but this silence is justifiable. It is easy to see that just as hardness has been defined by reference to a circular surface of pressure, so it could have been defined by assuming for it any definite ellipticity. The hardnesses thus diversely defined will show slight numerical variations. Now the order of the bodies in the different scales of hardness is either the same, or it is not. In the first case, our definition agrees generally with the mineralogical one. In the second case, the fault lies with the mineralogical definition, since it cannot then give a definite scale of hardness at all. It is indeed probable that the deviations from one another of the variously defined hardnesses would be found only very small; so that with a slight sacrifice of accuracy we might omit the limitation to a circular surface of pressure both in the above and in what follows. Experiments alone can decide with certainty.

Now let H be the hardness of a body which is in contact with another of hardness greater than H. Then by help of this value we can make this assertion, that all contacts with a circular surface of pressure for which

$$Z_z = \frac{3p}{2\pi a^2} = \frac{2}{\pi}\sqrt[3]{\frac{3p(\rho_{11}+\rho_{12}+\rho_{21}+\rho_{22})^2}{(\vartheta_1+\vartheta_2)^2}} \leqq \mathrm{H},$$

or for which

$$\frac{p(\rho_{11}+\rho_{12}+\rho_{21}+{}_{22})^2}{(\vartheta_1+\vartheta_2)^2} \leqq \frac{\pi^3\mathrm{H}^3}{24}$$

can be borne, and only these.

The force which is just sufficient to drive a point with spherical end into the plane surface of a softer body, is proportional to the cube of the hardness of this latter body, to the square of the radius of curvature of the end of the point, and also to the square of the mean of the coefficients ϑ for the two bodies. To bring this assertion into better accord with the usual determinations of hardness we might be tempted to measure the latter not by the normal pressure itself, but rather by its cube. Apart from the fact that the analogy thus produced would be fictitious (for the force necessary to drive one and the same point into different bodies would not even then be proportionate to the hardness of the bodies), this proceeding would be irrational, since it would remove hardness from its place in the series of strengths of material.

Though our deductions rest on results which are satisfactorily verified by experience, still they themselves stand much in need of experimental verification. For it might be that actual bodies correspond very slightly with the assumptions of homogeneity which we have made our basis. Indeed, it is sufficiently well known that the conditions as to strength near the surface, with which we are here concerned, are quite different from those inside the bodies. I have made only a few experiments on glass. In glass and all similar bodies the first transgression beyond the elastic limit shows itself as a circular crack which arises in the surface at the edge of the compressed surface, and is propagated inwards along a surface conical outwards when the pressure increases. When the pressure increases still further, a second crack encircles the first and similarly propagates itself inwards; then a third appears, and so on, the phenomenon naturally becoming more and more irregular.

From the pressures necessary to produce the first crack under given circumstances, as well as from the size of this crack, we get the hardness of the glass. Thus experiments in which I pressed a hard steel lens against mirror glass gave the value 130 to 140 kg/mm^2 for the hardness of the latter. From the phenomena accompanying the impact of two glass spheres, I estimated the hardness at 150; whilst a much larger value, 180 to 200, was deduced from the cracks produced in pressing together two thin glass bars with natural surfaces. These differences may in part be due to the deficiencies of the methods of experimenting (since the same method gave rise to considerable variations in the various results); but in part they are undoubtedly caused by want of homogeneity and by differences in the value of the surface-strength. If variations as large as the above are found to be the rule, then of course the numerical results drawn from our theory lose their meaning; even then the considerations advanced above afford us an estimate of the value which is to be attributed to exact measurements of hardness.