Explorative exercises

I recommend to study the explorative problems before the first lecture of the week.

Problem 1. Let B be an arbitrary set.

a) Let $f : \mathbb{N} \to B$ be an injective function. (Think of a few concrete examples.) Can you use f to construct a surjective function $B \to \mathbb{N}$?
b) Let $g : \mathbb{N} \to B$ be a surjective function. (Think of a few concrete examples.) Can you use g to construct an injective function $B \to \mathbb{N}$?
c) Can you do the same if $\mathbb{N}$ is replaced by an arbitrary set A?

Solution: For a function $f : A \to B$ the set $f^{[-1]}(D)$ denotes the *pre-image* of the set $D \subseteq B$. It is the set of all elements x of A such that $f(x) \in D$, i.e.,

$$f^{[-1]}(D) = \{x \in A \mid f(x) \in D\}$$

Note that the pre-image is always defined for a function f, even if the inverse function f^{-1} does not exist.

a) Let us partition $B = B_1 \cup B_2$, where

$$B_1 = \{b \in B \mid \exists a \in A : f(a) = b\}$$
$$B_2 = B \setminus B_1$$

In other words, we split B into two *disjoint* (by construction) sets B_1 and B_2, where B_1 contains all the elements of B where some element of A gets mapped to and B_2 (a possibly empty set) contains the remaining elements of B. Since f was assumed to be injective, then $f^{[-1]}(\{b_1\})$ is a singleton (a set of size one) for every element $b_1 \in B_1$ (why?). Similarly, the pre-image $f^{[-1]}(\{b_2\}) = \emptyset$ for all $b_2 \in B_2$ (why?).

Let us define $g : B \to \mathbb{N}$ by setting

$$g(x) = \begin{cases} y \in f^{[-1]}(\{x\}), & x \in B_1 \\ 0, & x \in B_2 \end{cases}$$

Now, this function is a surjection as we can see from the following: because f is a function, it has to be defined for every element in its domain. On the other hand, because B_1 is precisely the image of the domain $\mathbb{N}$ under f, then g will actually work as a *left inverse* for f, since for any $n \in \mathbb{N}$ we have

$$g(f(n)) = g(b_1) = n, \quad b_1 \in B_1$$

Also because f was assumed to be injective, so we have that $f^{[-1]}(\{b_1\}) = \{n\}$.

As this holds for every $n \in \mathbb{N}$, we can see that g is indeed surjective. Note that the choice for where g maps the elements of B_2 does not matter: this is just to satisfy the fact that a function has to be defined for every element in its domain.

For a concrete example of the above: take $B = \mathbb{Z}$ and f to be the function $f(x) = |x|$. Then $B_1 = \mathbb{N}$ and $B_2 = \mathbb{Z} \setminus \mathbb{N} = \{\dots, -3, -2, -1\}$.

b) (Here we can use *axiom of choice*: if you are more interested in this topic, see for example Wikipedia for discussion.)

Because g is assumed to be surjective, then for every element b of B the preimage $f^{[-1]}(\{b\})$ is guaranteed to be non-empty (why?). However, we are not guaranteed that the pre-image is a singleton: therefore we will need to use axiom of choice to "choose" *some* element of the pre-image where we will map this element of b.

Define $f : B \to \mathbb{N}$ by setting

$$f(x) = n, \quad n \in g^{[-1]}(\{x\})$$

Now if $f(x) = f(y)$, then also $f(x), f(y) \in g^{[-1]}(\{x\})$. But this means that $g(f(x)) = x = g(f(y)) = y$, so $x = y$ and f is injective.

For a concrete example of the above: take $B = \{0, 1, 2\}$ and g to be the function

$$g(x) = \begin{cases} 0, & x = 0 \\ 1, & x \text{ is even and } x \text{ is not } 0 \\ 2, & x \text{ is odd} \end{cases}$$

Then the pre-images of each element b of B are given by the following:

$$g^{[-1]}(\{b\}) = \begin{cases} \{0\}, & b = 0 \\ \{2, 4, 6, \dots\}, & b = 1 \\ \{1, 3, 5, \dots\}, & b = 2 \end{cases}$$

so we could, for example, define f to be a function which always "chooses" the image of an element to be smallest element in the corresponding pre-image (with respect to the ordering $<$). Then

$$f(x) = \begin{cases} 0, & x = 0 \\ 2, & x = 1, \\ 1, & x = 2 \end{cases}$$

c) The answer is yes: neither of the two constructions we described were dependent or used facts about the set $\mathbb{N}$ (or its cardinality). However, note that in the example of part *b*) we assumed that the underlying set $\mathbb{N}$ has some *ordering* defined (i.e., "$<$") on it according to which we picked an element in the pre-image of g. If the domain of g is an arbitrary set A, then we have to make sure that the set A has some sort of ordering defined in it if we want to apply the example argument (or use axiom of choice). (Thanks for pointing this out Juho!)

Problem 2. The number of ways to select k elements out of a set of size n is denoted $\binom{n}{k}$

a) Argue that the number of ways to order a set of size n is

$$n! = n(n-1)(n-2)\cdots 2\cdot 1.$$

b) Argue that the number of ways to first select k elements out of a set of size n, then order these elements, and then order the remaining $n-k$ elements, is

$$\binom{n}{k} k!(n-k)!$$

c) Conclude that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

d) What does this formula say when $k = 0$, or when $k = n$? Do these formulas make sense?

Solution:

a) Let us consider an arbitrary ordering for a set X of size n. The first element in this ordering can be any of the elements in the set X: since there are n elements, we have a total of n options. Pick some element to be this the first respect to a given ordering: then for the second element in our ordering there are $n-1$ possible options. Since these choices can be done independently of each other, the total number of choices for the first two elements in an ordering is $n \cdot (n-1)$. Repeating this logic for the remaining $n-2$ elements of X gives a total of

$$n \cdot (n-1) \cdot \ldots 2 \cdot 1 = n!$$

options for ordering the elements of X.

b) The binomial coefficient $\binom{n}{k}$ describes the number of ways we can select a subset of size k from a set of size n. Therefore, there are $\binom{n}{k}$ possible choices to select k elements from a set of size n. For such given set, we can order the subset of size k in $k!$ possible ways and the remaining $(n-k)$ elements in $(n-k)!$ possible ways (part a).

c) Note that the above describes the number of ways to order a set of n elements but just does it in two parts: by writing a set X of size n in two disjoint sets $X = X_1 \cup X_2$ where $|X_1| = k$, ordering X_1 first and then ordering the remaining elements of X_2. But this is the same as ordering X directly. Therefore,

$$\binom{n}{k} k!(n-k)! = n! \iff \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

d) When $k = 0$ or $n = 0$, then the above term reduces to 1 (check!) when we use the fact that $0! = 1$. This makes sense: for a set of size n there are only 1 way to choose a subset of size $k = n$ (by taking the whole set) and $k = 0$ (by taking the empty set).

Problem 3. Prove that the *binomial coefficients* $\binom{n}{k}$ satisfy the identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

a) Using the interpretation of $\binom{n}{k}$ as the number of combinations of k out of n elements.

b) Using the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Solution:

a) As a combinatorial argument: let X be a set of size n and consider the number of ways to choose a subset Y of X that is of size k. Let us fix an element of X, say x. Then either x is contained in Y or it is not. If x is contained in Y, then there are $k-1$ more elements to be chosen in Y: this number is described by the binomial coefficient $\binom{n-1}{k-1}$. On the other hand, if x is not contained in Y, then there $|X \setminus \{x\}| = n-1$ elements to be considered that can be chosen in $\binom{n-1}{k}$ ways. Summing all these possibilities these gives the asked identity.

b) Using algebra: we get

$$\begin{aligned}
\binom{n}{k} &= \frac{n!}{k!(n-k)!} = \frac{n(n-1)! + k(n-1)! - k(n-1)!}{k!(n-k)!} \\
&= \frac{(n-1)!(n-k)}{k!(n-k)!} + \frac{k(n-1)!}{k!(n-k)!} \\
&= \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \binom{n-1}{k} + \binom{n-1}{k-1}
\end{aligned}$$

Problem 4. What is the sum $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n}$? If you do not see the answer immediately, first compute the sum for $n = 3$, $n = 4$, $n = 5$, etc.. Can you explain this phenomenon combinatorially?

Solution: A combinatorial interpretation: as the term $\binom{n}{k}$ describes the ways to choose a subset of size k from a set X of size n, then the sum $\sum_{k=0}^{n} \binom{n}{k}$ describes the total number of ways to choose a subset out of all the possible subsets of X.

An algebraic interpretation: as we might remember, the binomial theorem tells us that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Substituting $x = y = 1$ gives

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Note how this ties to the above combinatorial interpretation when we remember that the power set $\mathcal{P}(X)$ has cardinality $|\mathcal{P}(X)| = 2^{|X|} = 2^n$

Homework

The written solutions to the homework problems should be handed in on MyCourses by Monday 20.3 at 10:00. You are allowed and encouraged to discuss the exercises with your fellow students, but everyone should write down their own solutions.

Problem 1. (10pts)

(a) Let P be the set of all Finland's presidents, and let G be the set of all ordered pairs $(a, b) \in P \times P$ such that the president b succeeded president a in office. Is G the graph of a function? Explain your answer.
(b) Find the domain and range of the function which assigns to each nonnegative integer its last digit.

Solution:

(a) There are some different possible interpretations for this relation R: for example, G might be the relation $aRb \iff b$ succeeded a (but not necessarily immediately) or $aRb \iff b$ succeeded a immediately. In addition, it's possible that we might let a president be its own successor when it served more than one term (e.g., Kekkonen) or that we only consider a pair (x, y) to be in this relation when $y \neq x$. Depending on the interpretation, your answer to this question might differ.

If we take the latter interpretation literally (that is, P is the set of all presidents of Finland, including the current and the past presidents, where a successor is considered to be a distinct person from itself), then the relation is given by the set $G = \{(a, b) \in P \times P \mid a \neq b, b \text{ } was \text{ a successor of } a\}$. Remember that a relation R over a set X is a *function* when for all $x \in X$ there exists a unique $y \in X$ such that $(x, y) \in R$. However, since we include the current president p in the set P, then there does not exist an element p' such that $(p, p') \in P$, that is, G is not a graph of a function (with this interpretation).

(b) The described function f maps a natural number n to its last digit, i.e., it takes an natural number as an argument and outputs an element of the set $\{0, 1, \dots, 9\}$. Therefore the domain of f is $\mathbb{N}$ and the range is the set $\{0, 1, \dots, 9\}$.

Problem 2. (10pts) (Hint: this problem relates to exploratory problem 1). Let A and B be arbitrary sets and let $f : A \to B$ be a function. A function $g : B \to A$ such that $g(f(x) = x$ for all $x \in A$ is called a *left inverse* of f. A function $h : B \to A$ such that $f(h(y) = y$ for all $y \in B$ is called a *right.* inverse of f.

(a) Consider the example $p : [-1, 1] \to [0, 1]$ given by $p(x) = x^2$. Then $r(y) = \sqrt{y}$ is a right inverse. However, there is no left inverse. Why not?
(b) Prove that f is injective if and only f has a left inverse.
(c) Prove that f is surjective if and only if f has a right inverse.
(d) Prove that if f has both a left inverse and a right inverse, then the left and right inverse functions are equal. (This is why we can talk about "the" inverse of f).
(e) Prove that f is bijective if and only f has a two-sided inverse (that is, a single function that is both a left and right inverse).

Solution:

a) The problem arises from the fact that the pre-image of any element $y \in (0, 1)$ under p contains two distinct elements, since $(-\sqrt{y})^2 = (\sqrt{y})^2 = y$ (that is, p is not injective). If such left inverse $g : [0, 1] \to [-1, 1]$ existed, then g would have to satisfy the above definition. In particular, when $y \neq 0$, then $-\sqrt{y} \neq \sqrt{y}$, but

$$g(f(-\sqrt{y})) = g(y)$$
$$g(f(\sqrt{y})) = g(y)$$

so g would have to map y to two distinct elements. But this contradicts the definition of a function.

b) (See exploratory exercise 1 for details about the notation and axiom of choice).

To prove an "if and only if"-statement we have to prove two distinct implication claims:

" $\implies$ " : Assume that $f : A \to B$ is an injection. Then $f(x) = f(y) \implies x = y$ for all $x, y \in A$. Define $g : B \to A$ to be a function (as in exploratory problem 1) such that

$$g(x) = \begin{cases} y \in f^{[-1]}\{x\}, & x \in f(A) \\ a, & x \in B \setminus f(A) \end{cases}$$

where a denotes any element of the set A. Then $g(f(y)) = g(b) = y$ for all $y \in A$, since $b \in f(A)$ and because f was assumed to be injective, then we must have that $f^{[-1]}(\{b\}) = \{y\}$. So g is a left inverse of f.

" $\impliedby$ " : Assume $f : A \to B$ has a left inverse, say $g : B \to A$, and that $f(x) = f(y)$ for some $x, y \in A$. Then $x = g(f(x)) = g(f(y)) = y$. So f is injective.

c) " $\implies$ " : Assume that $f : A \to B$ is a surjection. Then for all $b \in B$ there exists $a \in A$ such that $f(a) = b$, so the pre-image of any element $b \in B$ is non-empty. Define h to be as follows:

$$h(b) = a, \quad a \in f^{[-1]}(\{b\})$$

that is, h maps an element b to *some* element of $f^{[-1]}(\{b\})$. Then $f(h(b)) = f(a) = b$ for all $b \in B$, so h is a right inverse of b.

" $\Longleftarrow$ " : Assume $f : A \to B$ has a right inverse, say $h : B \to A$. Let $b \in B$ be arbitrary. Then $f(g(b)) = b$ by definition of a right inverse. In particular, this means that there exists an element $g(b)$ such that $f(g(b)) = b$. Because b was arbitrary, this holds for all elements $b \in B$. So f is a surjection.

d) Assume that $f : B \to A$ has both a left and a right inverse. Note that two functions h and g are equal if i) the domain and range of h and g are the same ii) h and g agree for all inputs x, that is, $h(x) = g(x)$ for every element of the domain.
Let g denote the left inverse of f and h denote the right inverse of f. Clearly they have the same domain and codomain. Let b be any element of B. Then $g(f(b)) = b$ and $f(h(b)) = b$ so in particular $g(b) = g(f(h(b)) = h(b)$. Since b was arbitrary, this holds for every $b \in B$. So $h = g$.

e) " $\Longrightarrow$ " : Assume f is bijective. Then by definition f is injective and surjective, so by parts a) and b) we know that f has a left inverse and a right inverse. By part c) the left and right inverses have to be equal, so therefore f has a two-sided inverse.

" $\Longleftarrow$ " : Conversely, if f has a two-sided inverse g, then g is both the left inverse and the right inverse of f by part c). Therefore, by parts a) and b) we know that f is both injective and surjective and therefore bijective.

Problem 3. (10pts) Eight people are to be seated around a table; the chairs don't matter, only who is next to whom, but right and left are different. Two people, X and Y, cannot be seated next to each other. How many seating arrangements are possible?

Solution: There are multiple ways to approach the problem, we will use the fact that the set of all possible seatings can be separated into those where the two people X and Y are sitting next to each other and to those where X and Y are not sitting next to each other. That is, if we compute the number of all possible seatings and then remove the number of seatings where X and Y are sitting next to eachother, then we are left with those seatings that we are interested in.

Because the chairs do not matter, we first find the number of all possible seatings. Fix one person to a particular seat. Then the remaining 7 people can be seated in a total of 7! possible ways, so there are 7! possible seatigns in total. If X and Y are sitting next to eachother, then the remaining 6 seatings can be chosen in an arbitrary order. So there are a total of 6! possible ways (illustrate this to yourself with pen and paper!). However, since the X and Y can be sitting in two ways next to eachother (either X is on the left hand side of Y or Y is on the left hand side of X), then we have to multiply this number by two. Therefore, we get a total of

$$7! - 6! \cdot 2 = 3600$$

suitable seatings.

Problem 4. (10pts) Prove that for all $n \in \mathbb{N}$, $n \geq 9$, the following statement is true: for all $k \in \mathbb{N}$ with $0 \leq k \leq n$ we have

$$\binom{n}{k} < 2^{n-2}.$$

Hint: You can use induction in n with base case $n = 9$.

Solution: Base case: assume $n = 9$. We can find the maximum value of $\binom{n}{k}$ by considering the binomial coefficients as a sequence $(a_k)_{k=0}^n = \left(\binom{n}{k}\right)_{k=0}^n$. Then, the maxima of this with respect to k is found by studying the ratio of the terms $\frac{a_{k+1}}{a_k}$ as follows:

$$\begin{aligned}
\max_{k \in \{0,1,\ldots,n\}} \binom{n}{k} &= \max_{k \in \{0,1,\ldots,n\}} \left\{ \frac{n!}{(k+1)!(n-(k+1))!} \Big/ \frac{n!}{k!(n-k)!} \geq 1 \right\} \\
&= \max_{k \in \{0,1,\ldots,n\}} \left\{ \frac{n-k}{k+1} \geq 1 \right\} \\
&= \max_{k \in \{0,1,\ldots,n\}} \{ n \geq 2k+1 \}
\end{aligned}$$

So here we conclude that the maxima for $n = 9$ is found when $k = 4$ (you can end up in similar conclusions just using Pascal's triangle). Therefore,

$$\binom{9}{k} \leq \max_{k \in \{0,1\ldots,n\}} \binom{9}{k} \leq \binom{9}{4} = 126 < 2^7 = 128$$

Assume the claim is true for some $m \in \mathbb{N} \cap \{9, 10, \ldots\}$. In order the avoid the possibility that $k - 1 < 0$, we first consider the case when $k = 0$. Then clearly

$$\binom{m+1}{0} = 1 < 2^{9-2} < 2^{m+1-2}$$

Assume then that $k \geq 1$. By the identity in the exploratory exercise 3:

$$\begin{aligned}\binom{m+1}{k} &= \binom{m}{k} + \binom{m}{k-1} \\ &\overset{I.A}{<} 2 \cdot 2^{m-2} \\ &= 2^{(m+1)-2}\end{aligned}$$

Here we used the fact that the upper bound holds for all $1 \leq k \leq n$, so in particular it holds for $k-1$ too, which allowed us to apply the induction assumption for both of the terms.

Additional problems

These do not need to be returned for marking.

Problem 1. How many odd 5-digit numbers (in the decimal system) have all their digits different?

Solution: The set of odd 5-digit numbers are those where the last digit is in the set $\{1, 3, 5, 7, 9\}$, so there are 5 possible choices for the last digit. Fix the last digit to be one of these elements. Then, the remaining four digits can be chosen arbitrarily: the first digit can be any number in the set $\{1, 2, \ldots, 9\}$ except the last digit. Therefore, there are 8 possible choices. The second, third and fourth digits can be any numbers in the set $\{0, 1, 2, \ldots, 9\}$ except the previous ones. Thus, in total there are

$$8 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 13440$$

possible choices for such number.

Problem 2. How many bit strings contain exactly eight 0s and ten 1s if every 0 must be immediately followed by a 1?

Solution: Such binary strings have to contain exactly 18 bits. Because every 0 must be followed a 1 and the binary strings must contain eight 0s, then all the binary strings we consider must contain eight repetitions of the substring "01", and the remaining two bits are 1s. So we only have to consider in how many different way these remaining 1s can be placed: in total, this is

$$\binom{10}{2} = 45$$

Therefore there are 45 such binary strings.

Problem 3. Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

Solution: The only way we can form a committee of six members with even number of men and women is if the committee contains exactly 3 men and 3 women. Because the set of women and men committees candidates are disjoint, the total number is given by considering the product of the number of possible committees from each set, i.e.,

$$\binom{15}{3}\binom{10}{3} = 54600$$

Problem 4.

a) What is the coefficient of x^2y^3 in the expansion of $(x+y)^5$?
b) What is the coefficient of x^8y^9 in the expansion of $(x+y)^{17}$?
c) What is the coefficient of x^8y^9 in the expansion of $(2x+3y)^{17}$?

Solution: By the binomial theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

a) Let $n = 5$. When $k = 2$, then the coefficient of the corresponding term is given by $\binom{5}{2}$.
b) By similar observations as above: $\binom{17}{8}$
c) Let us substitute $x' = 2x, y' = 3y$. Then by part b), the coefficient is given by $\binom{17}{8}$. Taking in consideraration the coefficients of the original terms, the coefficient of x^8y^9 is then given by

$$\binom{17}{8} \cdot 2^8 \cdot 3^9$$

Problem 5.

a) How many relations on $\{1, 2, \ldots, n\}$ are reflexive?
b) How many relations on $\{1, 2, \ldots, n\}$ are symmetric?
c) How many relations on $\{1, 2, \ldots, n\}$ are antisymmetric?

Solution:

a) Let $X = \{1, 2, \ldots, n\}$.
Fact 1: there are a total over 2^{n^2} relations over X. This follows from the fact that a relation over X is any subset of the Cartesian product $X \times X$. By week 1 we remember that the power set $\mathcal{P}(X \times X)$ is of size $2^{|X \times X|} = 2^{n^2}$.
Fact 2: There are a total of n pairs of the form (a, a) in $X \times X$. This is hopefully clear, since $|X| = n$.

For a relation to be reflexive, it has to contain all these n pairs, since by definition a relation over X is symmetric if xRx for all $x \in X$. Therefore, $X \times X$ contains $n^2 - n$ pairs (x, y) that we can either include or not include in a relation in order for it to be reflexive. Since these choices can be done independently, the total number of reflexive relations is then 2^{n^2-n}.

b) Remember that a relation over a set X is symmetric, if $xRy \implies yRx$ for all $x, y \in X$. By a similar counting argument as above: for each element x of X we can either include or not include x in this relation. Since there are n such elements of x, there are 2^n total such options. For each of these n elements there are $n - 1$ other elements $y \neq x$ such that (x, y) can be added to a relation. Thus, we make a total of $n \cdot (n - 1)$ choices in for each of these elements. In order to avoid double counting (if we decide to include the pair (x, y) in the relation, then we also have to add the pair (y, x) by definition of a symmetric relation), we divide this number by two. Therefore there are a total of $2^n \cdot 2^{\frac{n^2-n}{2}} = 2^{\frac{n^2+n}{2}}$ relations to be considered.

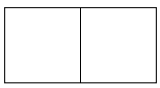
c) For a relation over X to be *antisymmetric* we require that if xRy and yRx, then $x = y$ for all $x, y \in X$. Therefore, the counting argument is similar as in part b): this time we only have three options for each possible pair (x, y), when $x \neq y$: either (x, y) is contained in the relation, (y, x) is contained the relation or neither of them is. This gives a total of $2^n \cdot 3^{\frac{n^2-n}{2}}$ possible options for such relations, when taking in consideration the double counting that happens.

Problem 6. The *pigeonhole principle* is the following very simple but surprisingly useful observation: If a set with n elements ("pigeons") is partitioned into m parts ("pigeonholes"), where $m < n$, then at least one of the pigeonholes contains at least two pigeons. Use this to show that, among 101 integers, there is a pair whose difference is divisible by 100.

Solution: Let us consider the *congruence* relation R_n defined by $xR_ny \iff x$ gives the same remainder as y when divided with number n (or in other words, $xR_ny \iff x \equiv y \bmod n$). This relation is also an equivalence relation (check!), so let us denote this relation by $x \sim_n y$. For example, $7 \sim_5 12$, because $7 = 1 \cdot 5 + 2$ and $12 = 2 \cdot 5 + 2$. Note that for such relation there are n different *equivalence classes*, since there are at most n different remainders $0, 1, \ldots, n - 1$ when a number x is divided by the number n.

Thus, by the above, let us consider this relation when $n = 100$, i.e., the relation $x \sim_{100} y$ over a set of 101 distinct integers (our "pigeons"). By the above, this relation partitions to 100 distinct equivalence classes (the "pigeonholes"). To place our 101 elements into 100 distinct classes, by the pigeonhole principle at least one these classes must contain at least two elements, say x and y. Therefore, there exists at least one pair (x, y) such that $x \sim_{100} y$, that is, x and y give the same remainder when divided by the number 100.

Problem 7. How many rectangles are bounded by the straight lines on a "chessboard" of size $n \times m$? (Example: On a board of size 1×2, like below, there are 3 rectangles: the left one, the right one, and the entire board.)



Solution: An $n \times m$ chess-board consists of $n + 1$ horizontal lines and $m + 1$ vertical lines. A rectangle is formed by considering two distinct vertical lines and two distinct horizontal lines. Since we can make such choices independently of each other for each distinct rectangle, the total number is then given by counting: there are a total of

$$\binom{m+1}{2}\binom{n+1}{2}$$

rectangles.

Problem 8. Prove the identity

$$\sum_{k=0}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

Solution: Note that

$$\sum_{k=0}^{n} k\binom{n}{k}^2 = \sum_{k=0}^{n} k\binom{n}{k}\binom{n}{n-k}$$

Suppose we have two groups of size n, for example a group of teachers and a group students. Now suppose we have to form a team of teachers and students of size n with a team leader who is a teacher. Say we choose k teachers, then we must have $n - k$ students. We have $\binom{n}{k}$ choices for the teachers and $\binom{n}{n-k}$ choices for the students, and k choices for the team leader from the chosen teachers. So for a given k, there are $k\binom{n}{k}\binom{n}{n-k}$ ways to form the team.

To get the number of all possible teams, we sum over k. On the other hand, the team formation can be done differently. If we pool the groups together, we have a total of $2n$ people. For choosing the team leader, who has to be a teacher, we have n choices. Now we have to choose $n-1$ members, who can be either teachers or students, from $2n-1$ people. Hence we get that the number of ways to do this is, $n\binom{2n-1}{n-1}$. Thus the claimed equality follows.

Problem 9. Prove that there is no bijection $\mathbb{N} \to P(\mathbb{N})$. Hint: Imitate the proof that there is no bijection $\mathbb{N} \to \mathbb{R}$.

Solution: These type of Diagonal arguments are attributed to Georg Cantor, whose ideas were not originally accepted (so this is an interesting topic from the perspective of history of mathematics!). If you are more interested in this, see, e.g., Section 22.2.4 in Katz or the following podcast.

Assume that there would exist a bijection $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$. Because f is bijective, we can enumerate the sets of $\mathcal{P}(\mathbb{N}_1)$ as, say $\{A_1, A_2, A_3..\}$. Let us *encode* (using bits) the information about where f maps the element $n \in \mathbb{N}$ in an array such that the i^{th} row and j^{th} column contains information about whether or not $f(i) = A_i$ contains the element j or not. For example, if $A_1 = \{1, 2, 3\} \subseteq \mathbb{N}$, then the first row of our array would contain the values 1 in the columns $1, 2$ and 3 and be 0 elsewhere. Consider flipping the bits for the diagonal elements of this array: since we assumed that this array contains all the possible encodings of f, then flipping the bits of the diagonal elements creates an encoding of a new value that could not exist in this array previously. Therefore, there must exist some element in $\mathcal{P}(\mathbb{N})$ which does not belong to the image of f, which contradicts the assumption that f is a bijection. (And therefore $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$.)