1 Materials used in the Lab

1. Rotary Pendulum Workbook (Student).pdf
2. Rotary Inverted Pendulum - User Manual.pdf

Supplementary material is provided in this handout owing to errors in the workbook and also due to the fact that the theory is badly written. You can find that in section 1 of the handout. These files are provided in the Materials folder on the MyCourses page for this lab. In this lab, we will be covering chapters 1-4. It is expected that you go through these materials and submit the solutions to the questions to be solved before the pre-hearing session. Only one copy per group has to be submitted.

2 Supplementary Material: Theory of State Feedback Design

Let the single-input single-output process be given in state-space form as:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$
$$y(t) = Cx(t) \tag{2}$$

Where A, x, B, u, y and C are $n \times n$, $n \times 1$, $n \times 1$, 1×1, 1×1, and $1 \times n$ – dimensional matrices, respectively. This is a *representation* of the process, describing its input-output dynamics. Consider the state variable change:

$$x(t) = W\widetilde{x}(t) \Rightarrow \widetilde{x}(t) = W^{-1}x(t) \tag{3}$$

Where W is a non-singular $n \times n$ matrix (essentially one with an inverse). From this, it can be made clear that (time t is dropped from equations when convenient and when no confusion is possible):

$$\dot{\widetilde{x}} = W^{-1}\dot{x} = W^{-1}Ax + W^{-1}Bu = W^{-1}AW\widetilde{x} + W^{-1}Bu \tag{4}$$
$$y = Cx = CW\widetilde{x} \tag{5}$$

By defining $\widetilde{A} = W^{-1}AW$, $\widetilde{B} = W^{-1}B$, and $\widetilde{C} = CW$, we now have:

$$\dot{\widetilde{x}} = \widetilde{A}\widetilde{x} + \widetilde{B}u \tag{6}$$
$$y = \widetilde{C}\widetilde{x} \tag{7}$$

So the representations given by (1) and (2) are *equivalent* to (6) and (7) through the *similarity transformation* given by (3). They represent the same process, however, demonstrating the fact that the state-space representation of a process is not unique. In fact, by the above transformation an infinite number of state-space representations for the same process exist, by the change of the state variable as given by (3).

Let us now check that (1) and (6) give the same transfer function (input-output behavior) and the same characteristic polynomial. To that end, first look at two results of matrix algebra, which are used:

1. It is well-known (not proved here) that for any two square matrices of the same dimension, $\det(XY) = \det(X)\det(Y)$.
2. For any invertible square matrix, $\det(X^{-1}) = 1/\det(X)$. This can be proved as follows: $\det(X)\det(X^{-1}) = \det(XX^{-1}) = \det(I) = 1 \Rightarrow \det(X^{-1}) = 1/\det(X)$.

The transfer function corresponding to (1) is $G_1(s) = \det(C(sI - A)^{-1}B) = Y(s)/U(s)$. The transfer function corresponding to (6) is $G_2(s) = \det(\widetilde{C}(sI - \widetilde{A})^{-1}\widetilde{B}) = Y(s)/U(s) = \det(CW(sI - W^{-1}AW)^{-1}W^{-1}B) = \det(CW(sW^{-1}W - W^{-1}AW)^{-1}W^{-1}B) = \det(CW(sW^{-1}IW - W^{-1}AW)^{-1}W^{-1}B) = \det(CW(sIW - AW)^{-1}WW^{-1}B) = \det(CWW^{-1}(sI - A)^{-1}B) = \det(C(sI - A)^{-1}B)$. Observe that here, we have used $X_1^{-1}X_2^{-1} = (X_2X_1)^{-1}$.

The characteristic polynomial is of course the same thanks to the equality in the transfer functions, but let us still look at it separately. The characteristic polynomial corresponding to (6) is given by $\det(sI - \widetilde{A}) = \det(sI - W^{-1}AW) = \det(sW^{-1}W - W^{-1}AW) = \det(W^{-1}(sI - A)W) = \det(W^{-1})\det(sI - A)\det(W) = \det(sI - A)$. This is the same as the characteristic polynomial corresponding to (1). In addition to using the results given by points 1 and 2, we have utilized the fact that determinants are scalars which is why the order of multiplication can be interchanged freely.

It is now clear that for a system given by (1) and any invertible transformation (2), the equivalent system (6) can be calculated. But this is not the real target. The main question is: Given (1), what possible target representations (6) can we choose so that the transformation (3) exists. This is not so trivial question. Of course, the equivalent representations (1) and (6) must have the same transfer function. But that information does not help in calculating the matrix W.

Consider the representation of the system given by (1). Denote its transfer function as:

$$G_1(s) = \frac{b_n s^{n-1} + b_{n-1}s^{n-2} + ... + b_1}{s^n + a_n s^{n-1} + a_{n-1}s^{n-2} + ... + a_1} \tag{8}$$

It has been assumed that all pole-zero-cancellations have been done. The system is *reachable* (the student Workbook and many other sources use the term *controllable*; the two concepts are not exactly the same but are equivalent in time continuous linear time-invariant systems). The controllability matrix of the system is given by:

$$T = [B|AB|A^2B|...|A^{n-1}B] \tag{9}$$

and it is of full rank (it is invertible) because the system is reachable (controllable). The controllability matrix for the system characterized by (6) is given by:

$$\widetilde{T} = [\widetilde{B}|\widetilde{A}\widetilde{B}|\widetilde{A}^2\widetilde{B}|...|\widetilde{A}^{n-1}\widetilde{B}] \tag{10}$$

Note that in the Student Workbook there is an error in this formula. Easy to observe that $W\widetilde{T} = T$. This can be proved using the fact that $W\widetilde{B} = WW^{-1}B = B$ and $W\widetilde{A}^k\widetilde{B} = W \cdot (W^{-1}AW) \cdot (W^{-1}AW) \cdot ...k-times... \cdot (W^{-1}AW) \cdot (W^{-1}B) = A^kB$. We are almost there. For our final act, we shall define our matrices in the following manner:

$$\widetilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & ... & 0 \\ 0 & 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & -a_2 & -a_3 & -a_4 & ... & -a_n \end{bmatrix} \tag{11}$$

$$\widetilde{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{12}$$

$$\widetilde{C} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & ... & b_n \end{bmatrix} \tag{13}$$

This is called the control canonical form of the process given by (1) or by the transfer function in (8). (There exist other canonical forms, e.g. observable canonical form, diagonal form, Jordan form). It is easy to check that this form gives the transfer function in (8), which means that the system given by (6) is a realization (state-space representation) of the original process given by (1). Moreover, it is easy to calculate the controllability matrix using (9). Note that it has full rank (it is invertible), so it is reachable. From (10) we obtain the transformation W which transforms the system represented by (1) to the system represented by (6):

$$W = T\widetilde{T}^{-1} \tag{14}$$

The representations given by (1) and (6) are then equivalent, e.g. their poles are the same. Making a controller design for the process can be easier by using the controller canonical form because of its special matrix structure. That has been explained in **Section 3.2.4** of the Student Workbook. The state feedback control law for the system represented by (6) is given by:

$$u(t) = -\widetilde{K}\widetilde{x}(t) \tag{15}$$

(15) is used to get the desired closed-loop poles. Finally, to control the real process, we make use of (3). The final control law to be implemented by the system is given by:

$$u(t) = -\widetilde{K}W^{-1}x(t) \tag{16}$$

3 Pre-Hearing Questions

1. The nonlinear equations of motion for the SRV02 rotary inverted pendulum is given in equations **2.2** and **2.3** in the workbook. Linearize these equations by the classical linearization method. The initial conditions for all variables are set to zero.
   Hints:
   - Define a vector z such that $z = [\theta, \alpha, \dot{\theta}, \dot{\alpha}, \ddot{\theta}, \ddot{\alpha}]^T$ and $z_0 = [0, 0, 0, 0, 0, 0]^T$
   - Consider the left-hand side of equations **2.2** and **2.3** in the workbook as functions of z. Let's denote them as $f_1(z)$ and $f_2(z)$. Now linearize $f_1(z)$ and $f_2(z)$ about $z = z_0$

2. Find the linear state-space representation of the system when the input u is the torque applied at the servo load gear. Matrices C and D have been given.

3. What is the characteristic equation of this system?

4. What are the open-loop poles of the system? What can be said of the system based on the poles you just found?

5. Recall that the input of the state-space model you just found is the torque applied at the servo load gear. However, in the lab session, we control the servo input voltage instead. By using the voltage-torque relationship in equation **2.4** in the workbook, derive a new state-space model with the voltage to be the input. **Important**: We assume that the gears' and the motor's efficiencies (η_g, η_m) are 1.
   In MATLAB, write a script to find the open-loop poles of the new state-space model. **Hint**: Make sure your final state-space model is similar to the following one, otherwise your later results will be meaningless.

State-Space Matrix	A	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 80.3 & -45.8 & -0.930 \\ 0 & 122 & -44.1 & -1.40 \end{bmatrix}$
State-Space Matrix	B	$\begin{bmatrix} 0 \\ 0 \\ 83.4 \\ 80.3 \end{bmatrix}$
State-Space Matrix	C	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$
State-Space Matrix	D	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

6. From this point, **only the state-space model in terms of servo input voltage** is considered.
   - What is the characteristic polynomial of A?
   - Compute the corresponding companion matrices ($\widetilde{A}$, $\widetilde{B}$) of A and B.

7. In Figure 3.1 of the workbook, this system has four closed-loop poles and two of them (p_1 and p_2) are complex conjugate dominant poles. Find the location of these two dominant poles so that they satisfy the specifications given in **Section 3.1** of the workbook. Assume that $p_3 = -30$ and $p_4 = -40$, find the desired characteristic equation.

8. Calculate companion gain $\widetilde{A}$ that assigns the poles to the locations found in question 7. Convert the companion gain $\widetilde{A}$ into controller gain K (**Hint**: read **Section 3.2.4** of the workbook).
   Note:

- Typo in equation **3.1** of the workbook: $A^n \rightarrow A^{n-1}$
- Typo in the formula of $\widetilde{T}$: $\widetilde{A}^n \rightarrow \widetilde{A}^{n-1}$

The questions have to be solved and submitted before your pre-hearing slot. Upon your progress in answering the pre-hearing questions (those given in this handout as well as the questions asked by the instructor), you will be allowed to book the slots for the In-Lab Tasks.

4 In-Lab Tasks

1. The first task of this lab is to validate the state-space model of the inverted pendulum against the state-space model calculated in the pre-hearing questions.
2. The second task of this lab is to validate the controller gain K from question 8 which will be used to test balance control.
3. The third task of this lab will be a demonstration of swing-up and balance control. The instructor will give you a brief idea about what swing-up control looks like before the actual demonstration where you can sit back and enjoy the show!

Should you have any questions, please ask the instructor!