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STUDENT WORKBOOK
Inverted Pendulum Experiment for LabVIEW™ Users
Standardized for ABET* Evaluation Criteria

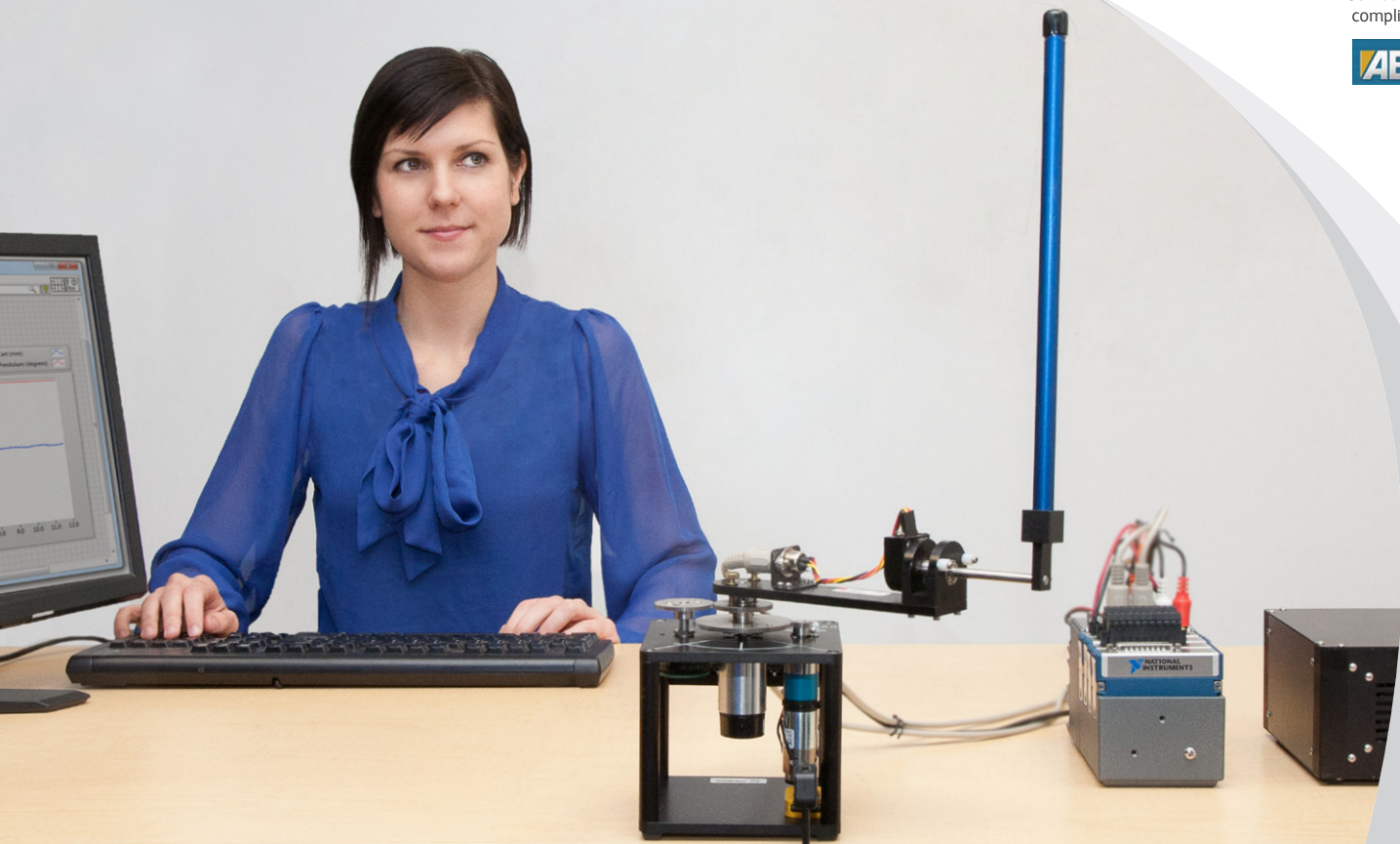Developed by:
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ACKNOWLEDGEMENTS

CONTENTS

1 INTRODUCTION

The objective of this laboratory is to design and implement a state-feedback control system that will balance the pendulum in the upright, vertical position.

Topics Covered

- Linearizing nonlinear equations of motion.

- Obtaining the linear state-space representation of the rotary pendulum plant.

- Designing a state-feedback control system that balances the pendulum in its upright vertical position using Pole Placement.

- Simulating the closed-loop system to ensure the specifications are met.

- Introduction to a nonlinear, energy-based swing up control.

- Implementing the controllers on the Quanser SRV02 Rotary Pendulum plant and evaluating its performance.

Prerequisites

- Know the basics of LabVIEW™ .

- Understand state-space modeling fundamentals.

- Some knowledge of state-feedback.

2 MODELING

2.1 Background

2.1.1 Model Convention

The rotary inverted pendulum model is shown in Figure 2.1. The rotary arm pivot is attached to the SRV02 system and is actuated. The arm has a length of L_r, a moment of inertia of J_r, and its angle, θ, increases positively when it rotates counter-clockwise (CCW). The servo (and thus the arm) should turn in the CCW direction when the control voltage is positive, i.e., $V_m > 0$.

The pendulum link is connected to the end of the rotary arm. It has a total length of L_p and it center of mass is $\frac{L_p}{2}$. The moment of inertia about its center of mass is J_p. The inverted pendulum angle, α, is zero when it is perfectly upright in the vertical position and increases positively when rotated CCW.
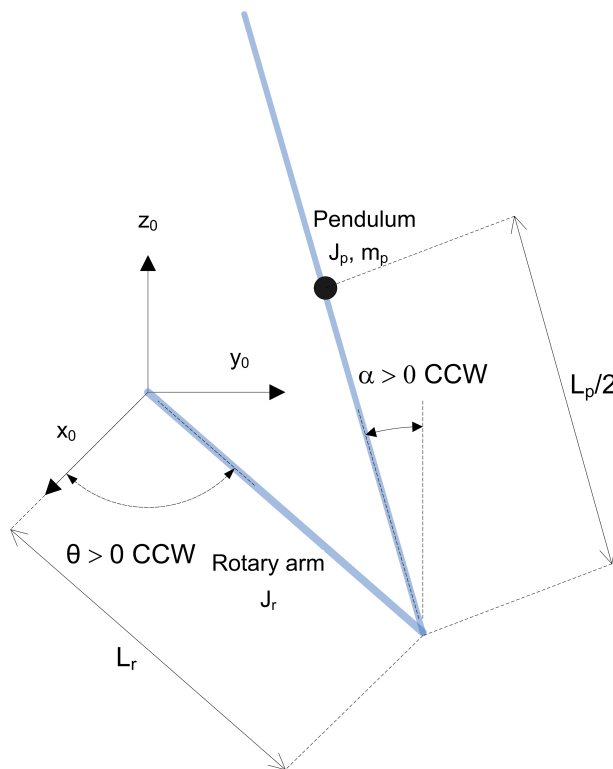
Figure 2.1: Rotary inverted pendulum conventions

2.1.2 Nonlinear Equations of Motion

Instead of using classical mechanics, the Lagrange method is used to find the equations of motion of the system. This systematic method is often used for more complicated systems such as robot manipulators with multiple joints.

More specifically, the equations that describe the motions of the rotary arm and the pendulum with respect to the servo motor voltage, i.e. the dynamics, will be obtained using the Euler-Lagrange equation:

$$\frac{\partial^2 L}{\partial t \partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i$$

The variables q_i are called *generalized coordinates*. For this system let

$$q(t)^\top = [\theta(t)\ \alpha(t)] \tag{2.1}$$

where, as shown in Figure 2.1, $\theta(t)$ is the rotary arm angle and $\alpha(t)$ is the inverted pendulum angle. The corresponding velocities are

$$\dot{q}(t)^\top = \left[\frac{\partial\theta(t)}{\partial t}\ \frac{\partial\alpha(t)}{\partial t}\right]$$

Note: The dot convention for the time derivative will be used throughout this document, i.e., $\dot{\theta} = \frac{d\theta}{dt}$. The time variable t will also be dropped from θ and α, i.e., $\theta = \theta(t)$ and $\alpha = \alpha(t)$.

With the generalized coordinates defined, the Euler-Lagrange equations for the rotary pendulum system are

$$\frac{\partial^2 L}{\partial t\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = Q_1$$

$$\frac{\partial^2 L}{\partial t\partial\dot{\alpha}} - \frac{\partial L}{\partial\alpha} = Q_2$$

The Lagrangian of the system is described

$$L = T - V$$

where T is the total kinetic energy of the system and V is the total potential energy of the system. Thus the Lagrangian is the difference between a system's kinetic and potential energies.

The generalized forces Q_i are used to describe the non-conservative forces (e.g., friction) applied to a system with respect to the generalized coordinates. In this case, the generalized force acting on the rotary arm is

$$Q_1 = \tau - B_r\dot{\theta}$$

and acting on the pendulum is

$$Q_2 = -B_p\dot{\alpha}.$$

See [2] for a description of the corresponding SRV02 parameters (e.g. such as the back-emf constant, k_m). Our control variable is the input servo motor voltage, V_m. Opposing the applied torque is the viscous friction torque, or viscous damping, corresponding to the term B_r. Since the pendulum is not actuated, the only force acting on the link is the damping. The viscous damping coefficient of the pendulum is denoted by B_p.

The Euler-Lagrange equations is a systematic method of finding the equations of motion, i.e., EOMs, of a system. Once the kinetic and potential energy are obtained and the Lagrangian is found, then the task is to compute various derivatives to get the EOMs. After going through this process, the nonlinear equations of motion for the SRV02 rotary inverted pendulum are:

$$\left(m_pL_r^2 + \frac{1}{4}m_pL_p^2 - \frac{1}{4}m_pL_p^2\cos(\alpha)^2 + J_r\right)\ddot{\theta} - \left(\frac{1}{2}m_pL_pL_r\cos(\alpha)\right)\ddot{\alpha}$$

$$+ \left(\frac{1}{2}m_pL_p^2\sin(\alpha)\cos(\alpha)\right)\dot{\theta}\dot{\alpha} + \left(\frac{1}{2}m_pL_pL_r\sin(\alpha)\right)\dot{\alpha}^2 = \tau - B_r\dot{\theta} \tag{2.2}$$

$$-\frac{1}{2}m_pL_pL_r\cos(\alpha)\ddot{\theta} + \left(J_p + \frac{1}{4}m_pL_p^2\right)\ddot{\alpha} - \frac{1}{4}m_pL_p^2\cos(\alpha)\sin(\alpha)\dot{\theta}^2$$

$$-\frac{1}{2}m_pL_pg\sin(\alpha) = -B_p\dot{\alpha}. \tag{2.3}$$

The torque applied at the base of the rotary arm (i.e., at the load gear) is generated by the servo motor as described by the equation

$$\tau = \frac{\eta_g K_g \eta_m k_t(V_m - K_g k_m\dot{\theta})}{R_m}. \tag{2.4}$$

See [2] for a description of the corresponding SRV02 parameters (e.g. such as the back-emf constant, k_m).

Both the equations match the typical form of an EOM for a single body:

$$J\ddot{x} + b\dot{x} + g(x) = \tau_1$$

where x is an angular position, J is the moment of inertia, b is the damping, $g(x)$ is the gravitational function, and τ_1 is the applied torque (scalar value).

For a generalized coordinate vector q, this can be generalized into the matrix form

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau \tag{2.5}$$

where D is the inertial matrix, C is the damping matrix, $g(q)$ is the gravitational vector, and τ is the applied torque vector.

The nonlinear equations of motion given in 2.2 and 2.3 can be placed into this matrix format.

2.1.3 Linearizing

Here is an example of how to linearize a two-variable nonlinear function called $f(z)$. Variable z is defined

$$z^\top = [z_1 \; z_2]$$

and $f(z)$ is to be linearized about the operating point

$$z_0{}^\top = [a \; b]$$

The linearized function is

$$f_{lin} = f(z_0) + \left(\frac{\partial f(z)}{\partial z_1}\right)\bigg|_{z=z_0} (z_1 - a) + \left(\frac{\partial f(z)}{\partial z_2}\right)\bigg|_{z=z_0} (z_2 - b)$$

2.1.4 Linear State-Space Model

The linear state-space equations are

$$\dot{x} = Ax + Bu \tag{2.6}$$

and

$$y = Cx + Du \tag{2.7}$$

where x is the state, u is the control input, A, B, C, and D are state-space matrices. For the rotary pendulum system, the state and output are defined

$$x^\top = [\theta \; \alpha \; \dot{\theta} \; \dot{\alpha}] \tag{2.8}$$

and

$$y^\top = [x_1 \; x_2]. \tag{2.9}$$

In the output equation, only the position of the servo and link angles are being measured. Based on this, the C and D matrices in the output equation are

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{2.10}$$

and

$$D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.11}$$

The velocities of the servo and pendulum angles can be computed in the digital controller, e.g., by taking the derivative and filtering the result though a high-pass filter.

2.2 Pre-Lab Questions

1. Linearize the first nonlinear inverted rotary pendulum equation, Equation 2.2. The initial conditions for all the variables are zero, i.e., $\theta_0 = 0$, $\alpha_0 = 0$, $\dot{\theta}_0 = 0$, $\dot{\alpha}_0 = 0$.

2. Linearize the second nonlinear inverted rotary pendulum equation, Equation 2.3, with initial conditions $\theta_0 = 0$, $\alpha_0 = 0$, $\dot{\theta}_0 = 0$, $\dot{\alpha}_0 = 0$.

3. Fit the two linear equations of motion found in the above exercises into the matrix form shown in Equation 2.5. Make sure the equation is in terms of θ and α (and its derivatives).

4. Solve for the acceleration terms in the equations of motion. You can either solve this using the two linear equations or using the matrix form. If you're doing it in the matrix form, recall that the inverse of a 2x2 matrix is

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \tag{2.12}$$

where $\det(A) = ad - bc$.

In any case, you'll have two equations of the form: $\ddot{\theta} = g_1(\theta, \alpha, \dot{\theta}, \dot{\alpha})$ and $\ddot{\alpha} = g_2(\theta, \alpha, \dot{\theta}, \dot{\alpha})$. Make sure you collect the terms with respect to the θ, α, $\dot{\theta}$, and $\dot{\alpha}$ variables.

5. Find the linear state-space of the rotary inverted pendulum system. Make sure you give the A and B matrices (C and D have already been given in Section 2.1).

2.3 In-Lab Exercises

Before going through these exercises, make sure to open the LabVIEW project called *Quanser SRV02 Rotary Pendulum (Student)*, pictured in Figure 2.2.
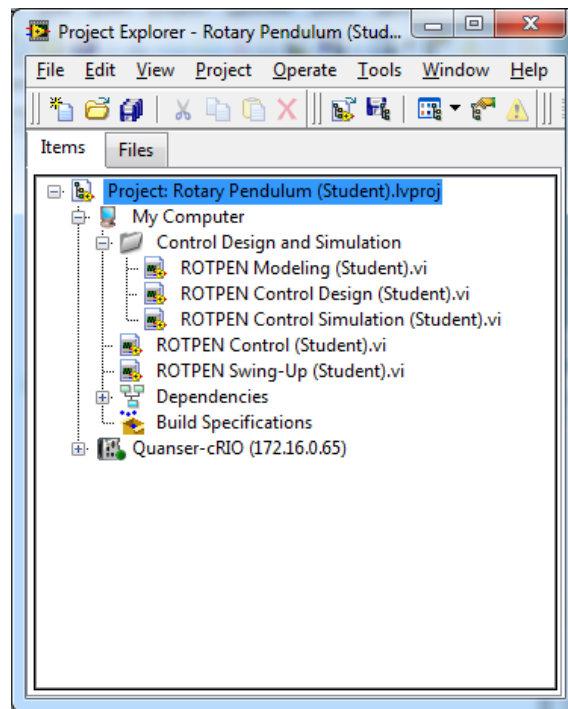

Figure 2.2: Rotary Pendulum LabVIEW project

2.3.1 Simulation: Model Analysis

1. Under the *Control Design and Simulation* folder, open *ROTPEN Modeling (Student)* VI shown in Figure 2.3.

2. Go the VI block diagram. As shown in Figure 2.4, it is not complete. The MathScript node contains the following code:

```
% State Space Representation
A = eye(4,4);
B = [0;0;0;1];
C = eye(2,4);
D = zeros(2,1);

% Add actuator dynamics
A(3,3) = A(3,3) - Kg^2*kt*km/Rm*B(3);
A(4,3) = A(4,3) - Kg^2*kt*km/Rm*B(4);
B = Kg * kt * B / Rm;

system = ss(A,B,C,D);
```

The representative C and D matrices have already been included. You need to enter the state-space matrices A and B that you found in Section 2.2. Notice the actuator dynamics have been added to convert your state-space matrices to be in terms of voltage. Recall that the input of the state-space model you found in Section 2.2 is the torque acting at the servo load gear (or the pivot of the pendulum). However, *we do not control torque directly - we control the servo input voltage*. The above code uses the voltage-torque relationship given in Equation 2.4 in Section 2.1.2 to tranform torque to voltage.
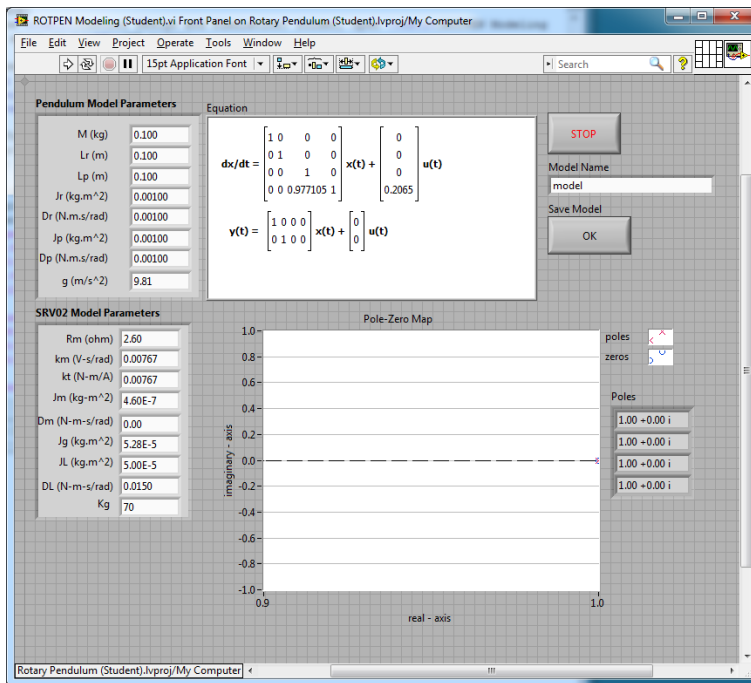
Figure 2.3: Front panel of ROTPEN Modeling.vi
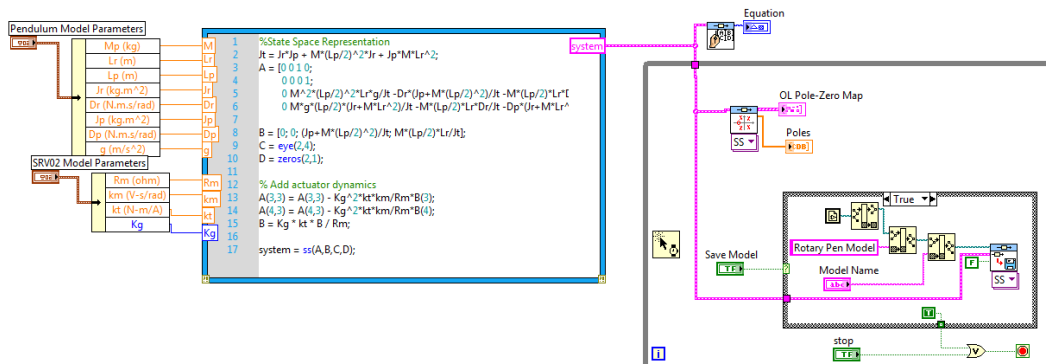
Figure 2.4: Block diagram of ROTPEN Modeling.vi

3. Enter the correct values in the *Rotary Model Parameters* control and run the VI. You can find these values in [3]. The VI is shown running with the *default* parameter values in Figure 2.5. Attach to your report a screen capture of the model when the VI is running and enter the numeric state-space matrices in Table 2.1.

4. Record the open-loop poles of the system in Table 2.1.

5. Enter a name in *Model Name* and click on the OK button to save the model. This model will later be used with other VIs.

Before ending this lab... To do the pre-lab questions in Section 3.3, you need the A and B matrices (numerical representation) and the open-loop poles. Make sure you record these.

2.3.2 Implementation: Calibration

1. In the *Quanser SRV02 Rotary Pendulum* LabVIEW project, open the *ROTPEN Control (Student)* VI. Make sure the HIL Initialize is configured for your data acquisition device, as explained in Section 5.3.
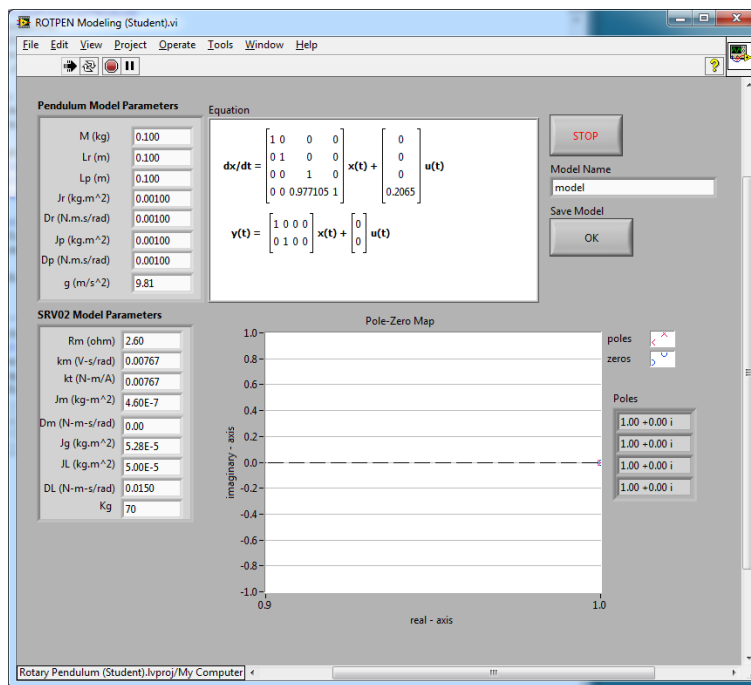
Figure 2.5: Running ROTPEN Modeling.vi with default values

2. Turn ON the power amplifier.

3. Run the VI. It should look similarly as shown in Figure 2.6.

4. Rotate the arm and the pendulum in the counter-clockwise direction and examine the direction of their response. Does the direction of these measurements agree with the modeling conventions given in Section 2.1.1? Explain why or why not.

5. Click on the *STOP* button to stop the VI.

6. Go to the VI block diagram. As shown in Figure 2.7, the block diagram is not completed.

7. Connect a control called *u (V)*, which denotes the control input, to the analog output/write VI (remove the constant 0). When you enter 1 V (as the VI is running), ensure the rotary arm is moving according to the model conventions that were defined in Section 2.1.1. Attach the response you obtain to your report. Use the Gain or Multiply VI if changes are necessary.

8. Make sure the VI is stopped by clicking on the STOP button

9. Close the VI when done.

10. Shut off the power amplifier.
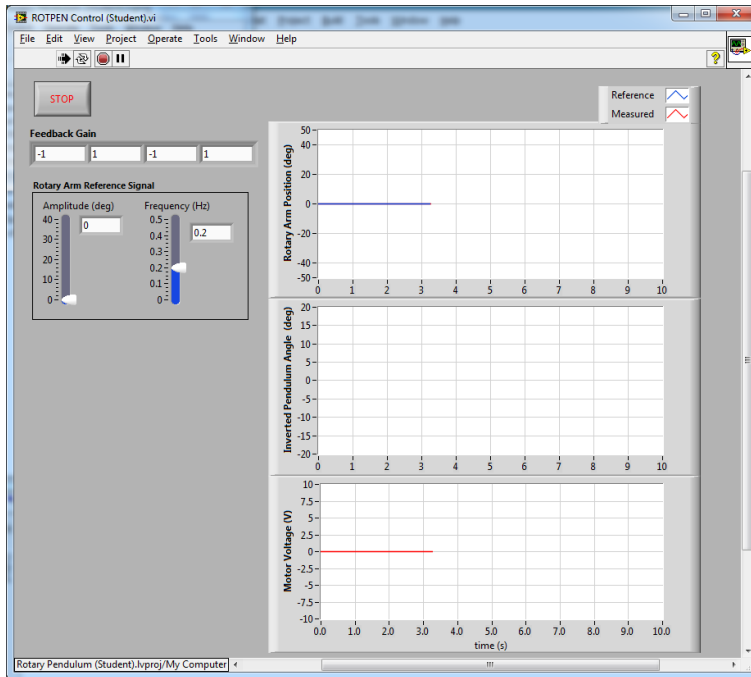
Figure 2.6: Running the ROTPEN Control.vi

Figure 2.7: Incomplete ROTPEN Control.vi block diagram

2.4 Results

Fill out Table 2.1 with your answers from your modeling lab results - both simulation and implementation.
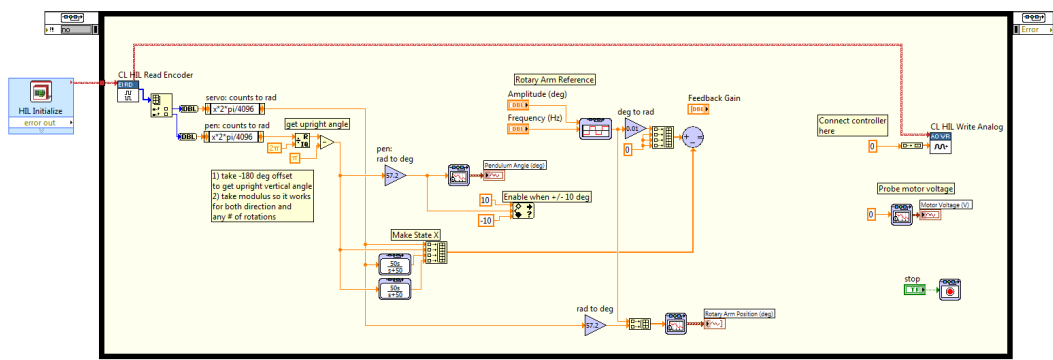
Description	Symbol	Value	Units
State-Space Matrix	A		
State-Space Matrix	B		
State-Space Matrix	C		
State-Space Matrix	D		
Open-loop poles	OL		

Table 2.1: Results

3 BALANCE CONTROL

3.1 Specifications

The control design and time-response requirements are:

Specification 1: Damping ratio: $\zeta = 0.7$.

Specification 2: Natural frequency: $\omega_n = 4$ rad/s.

Specification 3: Maximum pendulum angle deflection: $|\alpha| < 15$ deg.

Specification 4: Maximum control effort / voltage: $|V_m| < 10$ V.

The necessary closed-loop poles are found from specifications 1 and 2. The pendulum deflection and control effort requirements (i.e., specifications 3 and 4) are to be satisfied when the rotary arm is tracking a ± 20 degree angle square wave.

3.2 Background

In Section 2, we found a linear state-state space model that represents the inverted rotary pendulum system. This model is used to investigate the inverted pendulum stability properties in Section 3.2.1. In Section 3.2.2, the notion of controllabitliy is introduced. The procedure to transform matrices to their companion form is described in Section 3.2.3. Once in their companion form, it is easier to design a gain according to the pole-placement principles, which is discussed in Section 3.2.4. Lastly, Section 3.2.6 describes the state-feedback control used to balance the pendulum.

3.2.1 Stability

The stability of a system can be determined from its poles ([6]):

- Stable systems have poles only in the left-hand plane.

- Unstable systems have at least one pole in the right-hand plane and/or poles of multiplicity greater than 1 on the imaginary axis.

- Marginally stable systems have one pole on the imaginary axis and the other poles in the left-hand plane.

The poles are the roots of the system's characteristic equation. From the state-space, the characteristic equation of the system can be found using

$$\det\left(sI - A\right) = 0$$

where *det()* is the determinant function, s is the Laplace operator, and I the identity matrix. These are the *eigenvalues* of the state-space matrix A.

3.2.2 Controllability

If the control input u of a system can take each state variable, x_i where $i = 1 \ldots n$, from an initial state to a final state then the system is controllable, otherwise it is uncontrollable ([6]).

Rank Test The system is controllable if the rank of its controllability matrix

$$T = \begin{bmatrix} B & AB & A^2B & \ldots & A^nB \end{bmatrix} \tag{3.1}$$

equals the number of states in the system,

$$\text{rank}(T) = n.$$

3.2.3 Companion Matrix

If (A, B) are controllable and B is $n \times 1$, then A is similar to a companion matrix ([1]). Let the characteristic equation of A be

$$s^n + a_n s^{n-1} + \ldots + a_1.$$

Then the companion matrices of A and B are

$$\tilde{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \end{bmatrix} \qquad (3.2)$$

and

$$\tilde{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad (3.3)$$

Define

$$W = T\tilde{T}^{-1}$$

where T is the controllability matrix defined in Equation 3.1 and

$$\tilde{T} = [\tilde{B} \ \tilde{B}\tilde{A} \ldots \tilde{B}\tilde{A}^n].$$

Then

$$W^{-1}AW = \tilde{A}$$

and

$$W^{-1}B = \tilde{B}.$$

3.2.4 Pole Placement

If (A,B) are controllable, then pole placement can be used to design the controller. Given the control law $u = -Kx$, the state-space in Equation 2.6 becomes

$$\begin{aligned} \dot{x} &= Ax + B(-Kx) \\ &= (A - BK)x \end{aligned}$$

To illustate how to design gain K, consider the following system

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -1 & -5 \end{bmatrix} \qquad (3.4)$$

and

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad (3.5)$$

Note that A and B are already in the companion form. We want the closed-loop poles to be at $[-1 - 2 - 3]$. The *desired* characteristic equation is therefore

$$(s + 1)(s + 2)(s + 3) = s^3 + 6s^2 + 11s + 6 \qquad (3.6)$$

For the gain $K = [k_1 \ k_2 \ k_3]$, apply control $u = -Kx$ and get

$$A - KB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 - k_1 & -1 - k_2 & -5 - k_3 \end{bmatrix}.$$

The characteristic equation of $A - KB$ is

$$s^3 + (k_3 + 5)s^2 + (k_2 + 1)s + (k_1 - 3) \tag{3.7}$$

Equating the coefficients between Equation 3.7 and the desired polynomial in Equation 3.6

$$\begin{aligned} k_1 - 3 &= 6 \\ k_2 + 1 &= 11 \\ k_3 + 5 &= 6 \end{aligned}$$

Solving for the gains, we find that a gain of $K = [9 \ 10 \ 1]$ is required to move the poles to their desired location.

We can generalize the procedure to design a gain K for a controllable (A,B) system as follows:

Step 1 Find the companion matrices $\tilde{A}$ and $\tilde{B}$. Compute $W = T\tilde{T}^{-1}$.

Step 2 Compute $\tilde{K}$ to assign the poles of $\tilde{A} - \tilde{B}\tilde{K}$ to the desired locations. Applying the control law $u = -Kx$ to the general system given in Equation 3.2,

$$\tilde{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_1 - k_1 & -a_2 - k_2 & \cdots & -a_{n-1} - k_{n-1} & -a_n - k_n \end{bmatrix} \tag{3.8}$$

Step 3 Find $K = \tilde{K}W^{-1}$ to get the feedback gain for the original system (A,B).

Remark: It is important to do the $\tilde{K} \to K$ conversion. Remember that (A,B) represents the actual system while the companion matrices $\tilde{A}$ and $\tilde{B}$ do not.

3.2.5 Desired Poles

The rotary inverted pendulum system has four poles. As depicted in Figure 3.1, poles p_1 and p_2 are the complex conjugate *dominant* poles and are chosen to satisfy the natural frequency, ω_n, and damping ratio, ζ, specifications given in Section 3.1. Let the conjugate poles be

$$p_1 = -\sigma + j\omega_d \tag{3.9}$$

and

$$p_2 = -\sigma - j\omega_d \tag{3.10}$$

where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1 - \zeta^2}$ is the *damped* natural frequency. The remaining closed-loop poles, p_3 and p_4, are placed along the real-axis to the left of the dominant poles, as shown in Figure 3.1.

3.2.6 Feedback Control

The feedback control loop that balances the rotary pendulum is illustrated in Figure 3.2. The reference state is defined
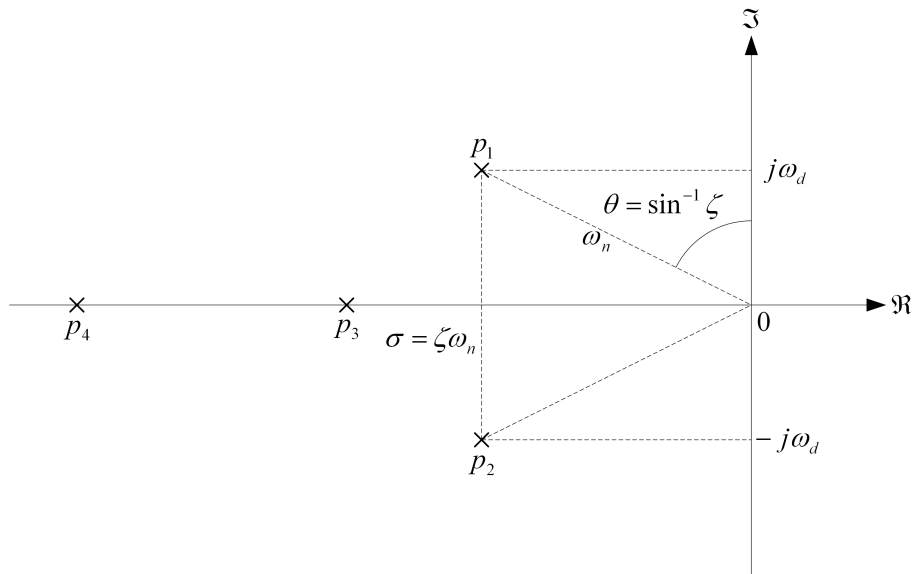
$$x_d = [\theta_d \ 0 \ 0 \ 0]$$

Figure 3.1: Desired closed-loop pole locations

where θ_d is the desired rotary arm angle. The controller is

$$u = K(x_d - x). \tag{3.11}$$

Note that if $x_d = 0$ then $u = -Kx$, which is the control used in the pole-placement algorithm.
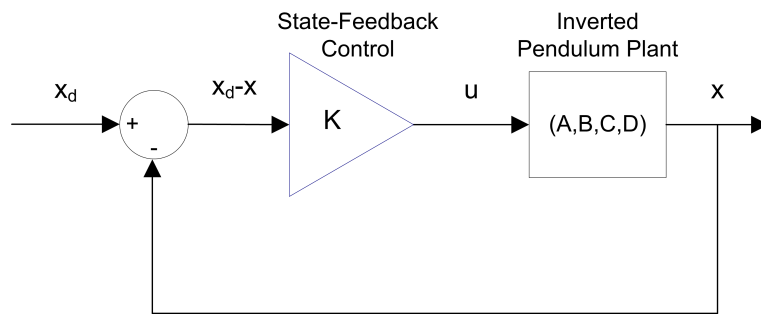
Figure 3.2: State-feedback control loop

When running this on the actual system, the pendulum begins in the hanging, downward position. We only want the balance control to be enabled when the pendulum is brought up around its upright vertical position. The controller is therefore

$$u = \begin{cases} K(x_d - x) & |x_2| < \epsilon \\ 0 & \text{otherwise} \end{cases}$$

where ϵ is the angle about which the controller should engage. For example if $\epsilon = 10$ degrees, then the control will begin when the pendulum is within ± 10 degrees of its upright position, i.e., when $|x_2| < 10$ degrees.

3.3 Pre-Lab Questions

1. Based on your analysis in Section 2.3, is the system stable, marginally stable, or unstable? Did you expect the stability of the inverted pendulum to be as what was determined?

2. Using the open-loop poles, find the characteristic equation of A.

3. Give the corresponding companion matrices $\tilde{A}$ and $\tilde{B}$. Do not compute the transformation matrix W (this will be done in the lab using LabVIEW™).

4. Find the location of the two dominant poles, p_1 and p_2, based on the specifications given in Section 3.1. Place the other poles at $p_3 = -30$ and $p_4 = -40$. Finally, give the desired characteristic equation.

5. When applying the control $u = -\tilde{K}x$ to the companion form, it changes $(\tilde{A}, \tilde{B})$ to $(\tilde{A} - \tilde{B}\tilde{K}, \tilde{B})$. Find the gain $\tilde{K}$ that assigns the poles to their new desired location.

3.4 In-Lab Exercises

Before going through these exercises, make sure you open the LabVIEW project called *Rotary Pendulum (Student)*, pictured in Figure 2.2.

3.4.1 Control Design

1. Under the *Control Design and Simulation* folder, open *ROTPEN Control Design (Student)* VI shown in Figure 3.3.
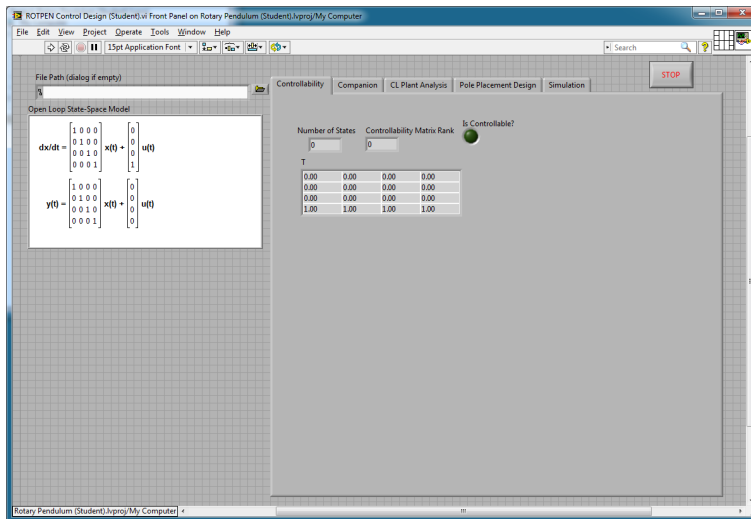
Figure 3.3: Front panel of ROTPEN Control Design.vi

2. Run the VI. It should look similarly as shown in Figure 3.4 (except using your model). The VI will prompt you to find a model file (unless you selected a model in *File Path* aleady). Find the model you saved from the previous modeling lab.
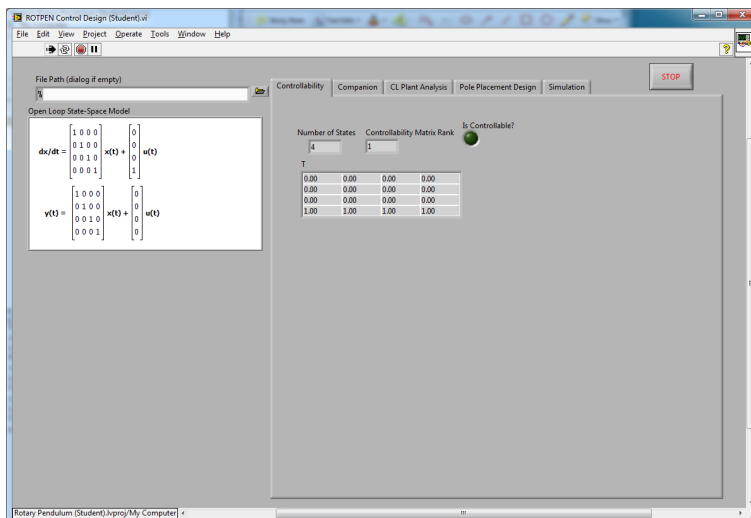
Figure 3.4: ROTPEN Control Design (Student).vi when running

3. Go to the *Controllability Test* tab. Is the system controllable? Explain why.

4. Click on the *Companion* tab. The *Companion* matrices $\tilde{A}$ and $\tilde{B}$ for the model loaded are automatically found (denoted as Ac and Bc in LabVIEW). In order to find the gain K, we need to find the transformation matrix

$W = T\tilde{T}^{-1}$ (note: $\tilde{T}$ is denoted as T_c in the VI). Stop the VI. In the VI block diagram, go to the *Companion* case. As shown in Figure 3.5, the block diagram is incomplete. Using the VIs from the *Diagram Disabled Structure*, calculate the controllability matrix T, the companion controllabilty matrix Tc, and the inverse of Tc. Show your completed block diagram and the resulting T, Tc, *Inverse Tc*, and W matrices.
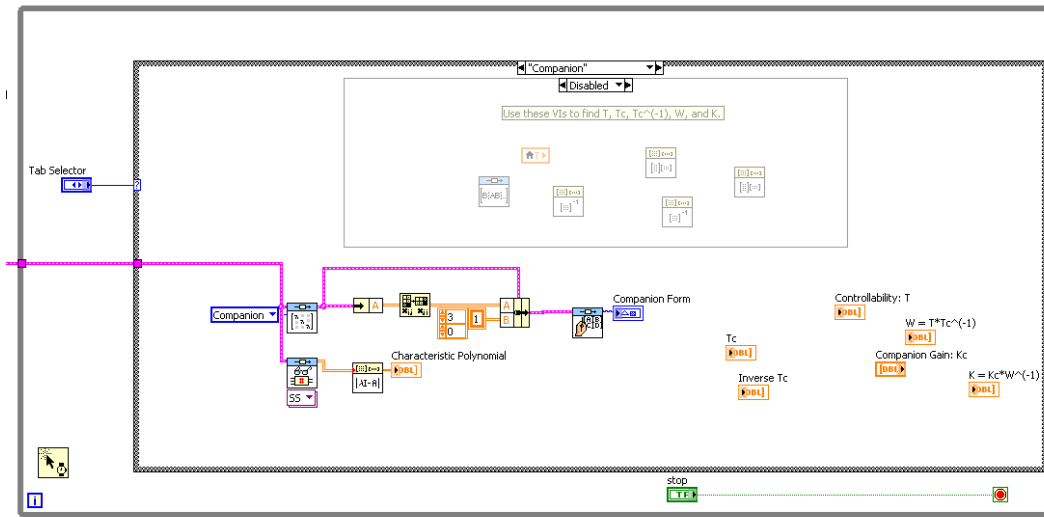
Figure 3.5: Incomplete block diagram of *Companion* tab

5. Enter the companion gain, $\tilde{K}$, you found in the pre-lab in the *Companion Gain: Kc* input box on the VI front panel. Run the VI again to calculate the feedback gain K and record its value in Table 3.1.

6. Click on the *CL Plant Analysis*. This shows the closed-loop poles of the system, i.e., the eigenvalues of $A - BK$. Record the closed-loop poles of the system when using the gain K calculated above. Have the poles been placed to their desired locations? If not, then go back and re-investigate your control design until you find a gain that positions the poles to the required location.

7. Select the *Pole Placement Design*. No gain is generated because, as shown in Figure 3.6, the block diagram is not complete. Back in the *Companion* tab, the necessary matrices to find gain K were found manually through matrix operations. All that work can instead be done using the *Pole Placement* VI. Add the *Pole Placement* VI to the block diagram so the gain is automatically generated from the state-space model and the desired poles. The *Pole Placement* VI is found in the *Control Design* palette. Run the VI again and verify that the gain is the same as generated before.

Figure 3.6: Incomplete block diagram of *Pole Placement* tab

3.4.2 Simulating the Balance Control

1. Under the *ROTPEN Control Design and Simulation* folder, open *ROTPEN Control Simulation (Student)* VI shown in Figure 3.7.
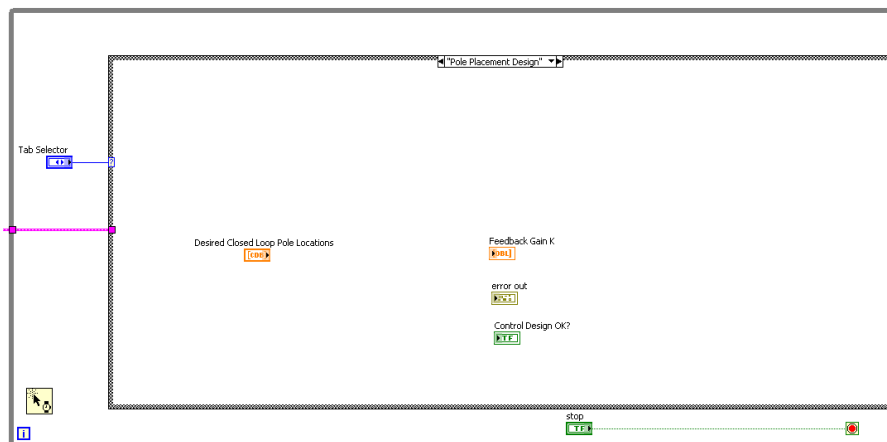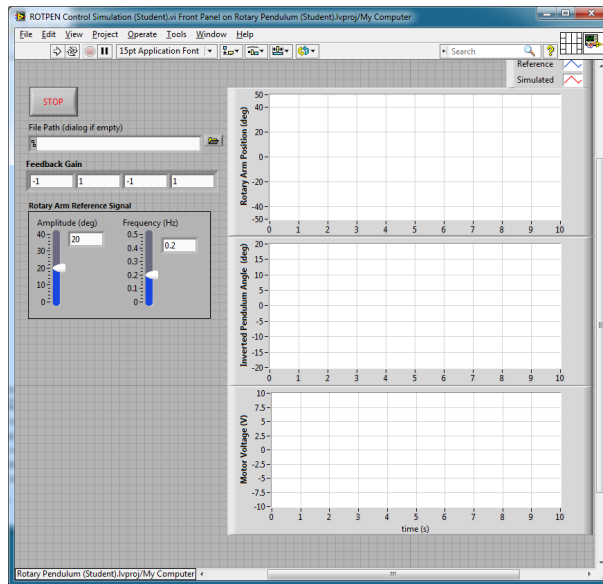

Figure 3.7: Front panel of ROTPEN Control Simulation (Student).vi

2. Go to the block diagram. As depicted in Figure 3.8, the block diagram is not finished. Add the necessary Gain and State-Space VIs to complete the feedback loop (remove the dummy state and control input constant arrays). In order to use your inverted pendulum model, make sure you use the model that is loaded from *Read Model from File* VI.

   Hint: Because you are using a vector gain, make sure you set the *Polymorphic instance* of the *Gain* VI to *Vector,MatrixGain*.
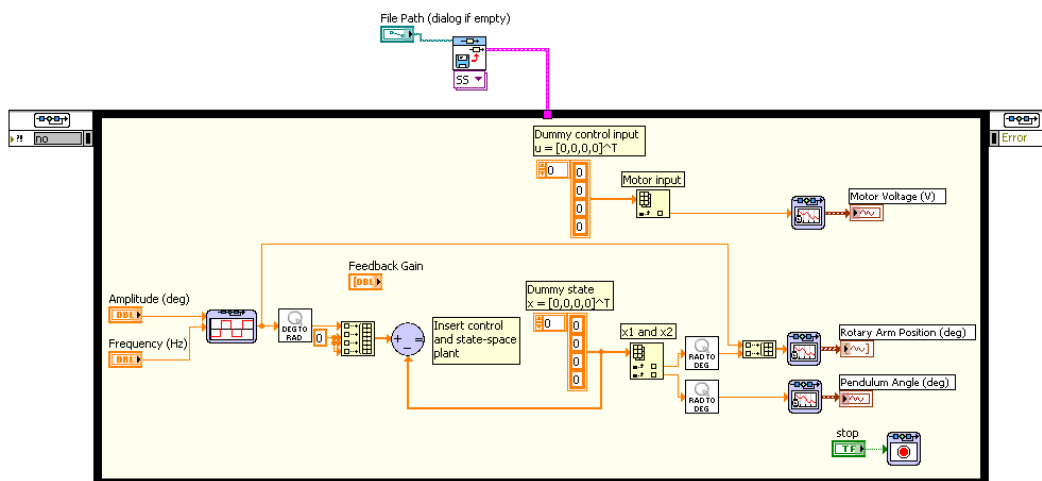
Figure 3.8: Incomplete block diagram of ROTPEN Control Simulation (Student).vi

3. Go to the front panel of the VI and enter the gain K you found in Section 3.4.1.

4. Run the VI. The VI will prompt you to find a model file (unless you selected a model in *File Path* aleady). Find the model you saved in the Modeling Laboratory in Section 2.3. The VI is shown running with an arbitrary gain in Figure 3.9. Include the response of the VI running with your designed gain and attach it to your report.
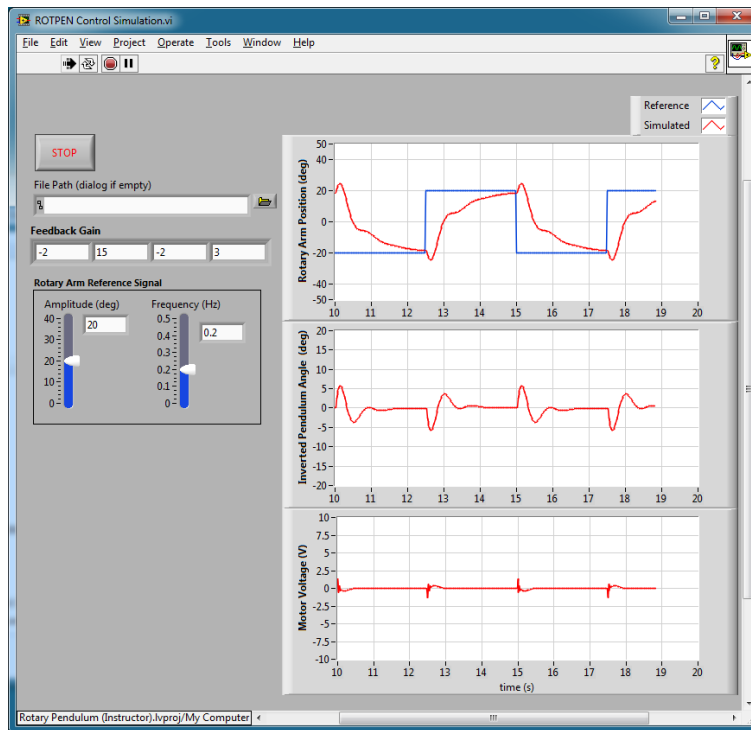
Figure 3.9: ROTPEN Control Simulation.vi when running

5. Measure the pendulum deflection and voltage used. Are the specifications given in Section 3.1 satisfied?

6. Press on the STOP button and close the VI when you are done.

3.4.3 Implementing the Balance Controller

In this section, ths state-feedback control that was designed and simulated in the previous sections is run on the actual SRV02 Rotary Pendulum device.

1. In the *Rotary Pendulum (Student)* LabVIEW project, open the *ROTPEN Control* VI. Make sure the HIL Initialize block has been confirued for your data acquisition device, as explained in Section 5.3.

2. Turn ON the power amplifier.

   Ensure the modifications you made to the VI in the Modeling Laboratory (Section 2.3) have been applied. Run the VI and verify that the model conventions still hold (e.g., motor turns in expected way when a positive voltage is supplied).

3. As shown in Figure 3.10, the VI block diagram is incomplete. Similarly as done in Section 3.4.2, add the necessary VIs to implement the balance control. When implementing the control, keep in mind the following:

   • Unlike in the simulation, where the pendulum is already upright, the pendulum begins in the hanging down position. Thus when the VI starts, the inverted pendulum angle reads ± 180 and it goes up to zero when brought to the upright position. You will need to add a switch logic to implement the Equation 3.2.6.

   • Recall in the simulation, the *Gain* VI outputs a vector. Use the first element of the vector to drive the analog output channel. Make sure you use the proper conventions as dictated in Section 2.3.

   • Saturate your control voltage to ± 10 V. On initial tests, you may want to set it to a lower range (e.g., ± 5V).

4. Go to the front panel of the VI and enter the gain K you found in Section 3.4.1.
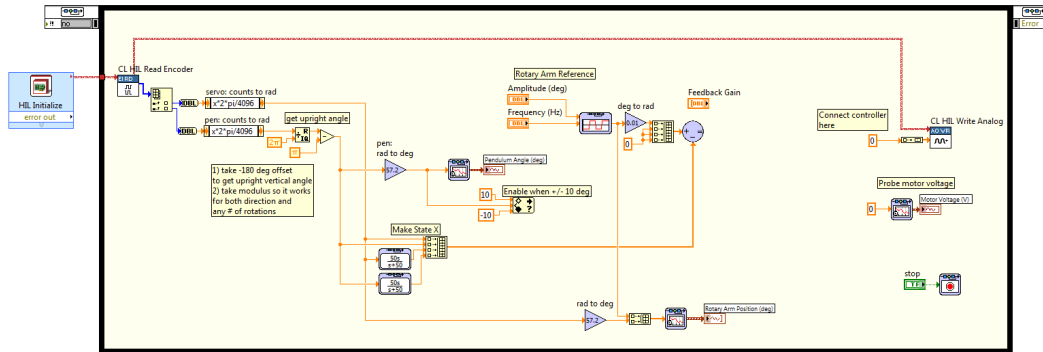
Figure 3.10: Incomplete block diagram of ROTPEN Control Simulation (Student).vi

5. Ensure the pendulum is in the hanging down position and is motionless. Run the VI and manually bring up the pendulum to its upright, vertical position. You should feel the voltage kick-in when it is within the range where the balance control engages. Once it is balanced, introduce the ± 20 degree rotary arm command by setting *Amplitude (deg)* to 20 in the VI. The response should look similar to your simulation. Attach the measured rotary pendulum responses.

6. Measure the pendulum deflection and voltage used. Are the specifications given in Section 3.1 satisfied for the implementation?

7. Click on the STOP button and close the VI when done. Be careful, as the pendulum will fall down when the VI is stopped.

8. Shut off the power amplifier.

3.5 Results

Fill out Table 3.1 with your answers from your control lab results - both simulation and implementation.

Description	Symbol	Value	Units
Pre Lab Questions			
Desired poles	DP		
Companion Gain	$\tilde{K}$		
Simulation: Control Design			
Transformation Matrix	W		
Control Gain	K		
Closed-loop poles	CLP		
Simulation: Closed-Loop System			
Maximum deflection	$|\alpha|_{max}$		deg
Maximum voltage	$|V_m|_{max}$		V
Implementation			
Control Gain	K		
Maximum deflection	$|\alpha|_{max}$		deg
Maximum voltage	$|V_m|_{max}$		V

Table 3.1: Results

4 SWING-UP CONTROL

4.1 Background

In this section a nonlinear, energy-based control scheme is developed to swing the pendulum up from its hanging, downward position. The swing-up control described herein is based on the strategy outlined in [7]. Once upright, the control developed in Section 3 can be used to balance the pendulum in the upright vertical position.

4.1.1 Pendulum Dynamics

The dynamics of the pendulum can be redefined in terms of pivot acceleration as

$$J_p\ddot{\alpha} + \frac{1}{2}m_pgL_p\sin(\alpha) = \frac{1}{2}m_pL_pu\cos(\alpha). \tag{4.1}$$

The pivot accleration, u, is the linear acceleration of the pendulum link base. The acceleration is proportional to the torque of the rotary arm and is expressed as

$$\tau = m_rL_ru \tag{4.2}$$

where m_r is the mass of the rotary arm and L_r is its length, as shown in Section 2. The voltage-torque relationship is given in Equation 2.4.

4.1.2 Energy Control

If the arm angle is kept constant and the pendulum is given an initial position it would swing with constant amplitude. Because of friction there will be damping in the oscillation. The purpose of energy control is to control the pendulum in such a way that the friction is constant.

The potential and kinetic energy of the pendulum is

$$E_p = \frac{1}{2}m_pgL_p(1 - \cos(\alpha)) \tag{4.3}$$

and

$$E_k = \frac{1}{2}J_p\dot{\alpha}^2.$$

The pendulum parameters are described in Section 2 and their values are given in [3]. In the potential energy calculation, we assume the center of mass to be in the center of the link, i.e., $\frac{L_p}{2}$. Adding the kinetic and potential energy together give us the total pendulum energy

$$E = \frac{1}{2}J_p\dot{\alpha}^2 + \frac{1}{2}m_pgL_p(1 - \cos\alpha). \tag{4.4}$$

Taking its time derivative we get

$$\dot{E} = \dot{\alpha}\left(J_p\ddot{\alpha} + \frac{1}{2}m_pgL_p\sin\alpha\right). \tag{4.5}$$

To introduce the pivot acceleration u and eventually, our control variable, solve for $\sin\alpha$ in Equation 4.1 to obtain

$$\sin(\alpha) = \frac{1}{m_pgL_p}(-2J_p\ddot{\alpha} + m_pL_pu\cos(\alpha)).$$

Substitute this into $\dot{E}$, found in Equation 4.5, to get

$$\dot{E} = \frac{1}{2}m_pL_pu\dot{\alpha}\cos\alpha$$

One strategy that will swing the pendulum to a desired reference energy E_r is the proportional control

$$u = (E - E_r)\dot{\alpha}\cos\alpha.$$

By setting the reference energy to the pendulum potential energy, i.e., $E_r = E_p$, the control will swing the link to its upright position. Notice that the control law is nonlinear because the proportional gain depends on the pendulum angle, α, and also notice that the control changes sign when $\dot{\alpha}$ changes sign and when the angle is ± 90 degrees.

For energy to change quickly the magnitude of the control signal must be large. As a result, the following swing-up controller is implemented

$$u = sat_{u_{max}}(\mu(E - E_r)\text{sign}(\dot{\alpha}\cos\alpha)) \tag{4.6}$$

where μ is a tunable control gain and $sat_{u_{max}}$ function saturates the control signal at the maximum acceleration of the pendulum pivot, u_{max}. Taking the sign of $\dot{\alpha}\cos\alpha$ allows for faster switching.

In order to translate the pivot acceleration into servo voltage, first solve for the voltage in Equation 2.4 to get

$$V_m = \frac{\tau R_m}{\eta_g K_g \eta_m k_t} + K_g k_m \dot{\theta}.$$

Then substitute the torque-acceleration relationship given in Equation 4.2 to obtain the following

$$V_m = \frac{R_m m_r L_r u}{\eta_g K_g \eta_m k_t} + K_g k_m \dot{\theta}. \tag{4.7}$$

4.1.3 Self-Erecting Control

The energy swing-up control can be combined with the balancing control in Equation 3.11 to obtain a control law which performs the dual tasks of swinging up the pendulum and balancing it. This can be accomplished by switching between the two control systems.

Basically the same switching used for the balance control in Equation 3.2.6 is used. Only instead of feeding 0 V when the balance control is not enabled, the swing-up control is engaged. The controller therefore becomes

$$u = \begin{cases} K(x_d - x) & |x_2| < \epsilon \\ sat_{u_{max}}(\mu(E - E_r)\text{sign}(\dot{\alpha}\cos\alpha)) & \text{otherwise} \end{cases} \tag{4.8}$$

4.2 Pre-lab Questions

1. Evaluate the potential energy of the pendulum when it is in the downward and upright positions.

2. Compute the maximum acceleration deliverable by the SRV02. Assume the maximum equivalent voltage applied to the DC motor is 5 V such that

$$V_m - K_g k_m \dot{\theta} = 5. \tag{4.9}$$

   The SRV02 motor parameters are given in [2].

3. Find the controller acceleration when the pendulum is initially hanging down and motionless. From a pracitcal viewpoint, what does this imply when the swing-up control is activated?

4. Assume the pendulum is starting to swing from the downward position in the positive direction. Calculate the acceleration the swing-up controller will generate when $\mu = 20$. Does this saturate the controller?

4.3 In-lab Exercises

In this section, you will be implementing the energy-based swing-up controller described in Section 4.1.

1. In the *Rotary Pendulum* LabVIEW project, open the *ROTPEN Swing-Up Control* VI. Make sure the HIL Initialize block in the VI is configured for your data acquisition system, as explained in Section 5.3.

2. **Ensure the modifications you made to the VI in the Controls Laboratory in Section 3.4 have been applied**. Run the VI and verify that the balance control runs fine.

3. The incomplete block diagram is shown in Figure 4.1. Edit the *Formula Node* in the VI to measure the total energy of the pendulum.
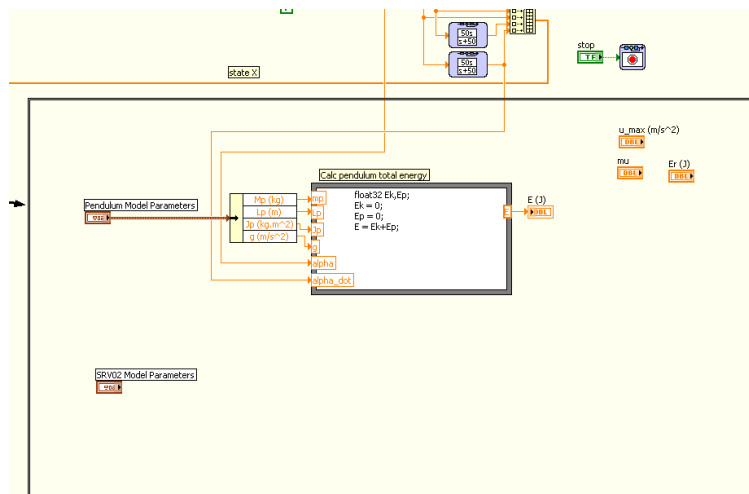

Figure 4.1: Incomplete block diagram in ROTPEN Swing-Up Control VI.

4. Run the VI and rotate the pendulum up to the upright position. While the inverted pendulum is balancing, record the total energy reading displayed in *E (J)* numeric indicator. Is the value as expected?

5. Implement the energy-based swing-up controller. Use the LabVIEW controls *E_r (J)*, *u_max (m/s^2)*, and *mu* that are already included. Make sure you are using the full pendulum angle α, i.e., not the upright based angle used in the feedback for the inverted pendulum balance control.

6. Add the necessary modifications to convert the acceleration generated by the swing-up control to servo voltage. Use the *SRV02 Model Parameters* bundle that is already included in the VI for any of the servo-based attributes you need.

7. Implement the self-erecting control in Equation 4.8, which includes both the swing-up and balance control.

8. Run the VI. Start with the following reference energy, maximum acceleration, and proportional gain parameters:

$$
\begin{aligned}
E_r &= E_p \\
u_{max} &= 5 \text{ m/s}^2 \\
\mu &= 2
\end{aligned}
$$

Make sure the reference energy is set to the pendulum potential energy. The pendulum should be moving back and forth slowly. Gradually increase the u_{max} and/or μ until the pendulum goes up. Do not increase the u_{max} above the maximum acceleration you found for the SRV02 in Section 4.2. When the pendulum swings up to the vertical upright position, the balance controller should engage and balance the link. Show the response of the arm and pendulum angles as well as the control voltage and record the swing-up parameters. Did the swing-up behave with the parameters you expected?

9. Click on the STOP button and close the VI when done. Be careful, as the pendulum will fall down when the VI is stopped.

10. Shut off the power amplifier.

4.4 Results

Fill out Table 4.1 with your answers from your swing-up control lab results.

Description	Symbol	Value	Unit
Pre Lab Questions			
Potential Energy	E_p		J
Maximum Acceleration of SRV02	u_{max}		m/s^2
Implementation			
Balance Control Gain	K		
Reference Energy	E_r		J
Control Maximum Acceleration	u_{max}		m/s^2
Swing-Up Proportional Gain	μ		

Table 4.1: Results

5 SYSTEM REQUIREMENTS

Required Hardware

- Data-aquisition (DAQ) device that is compatible with LabVIEW™ and Quanser Rapid Control Prototyping Toolkit®.

- Quanser SRV02-ET rotary servo. See Reference [2].

- Quanser Rotary Pendulum Module (attached to SRV02). See Reference [3].

- Quanser VoltPAQ power amplifier, or equivalent (e.g. Reference [4] for VoltPAQ User Manual).

Required Software

- NI LabVIEW™

- NI LabVIEW Control Design and Simulation Module

- NI LabVIEW MathScript RT Module or later

- For NI CompactRIO users:

  – NI LabVIEW Real-Time Module
  – NI LabVIEW FPGA Module
  – NI-RIO Drivers

- Quanser Rapid Control Prototyping Toolkit®

5.1 Overview of Files

File Name	Description
Rotary Pendulum Workbook (Student).pdf	This laboratory guide contains pre-lab questions and lab experiments that involve modeling the pendulum, designing and implementing an inverted pendulum balance control, and running an energy-based swing-up control.
Rotary Pendulum (Student).lvproj	LabVIEW project containing the student-based VIs.
ROTPEN Modeling (Student).vi	Creates the inverted pendulum state-space model.
ROTPEN Control Design (Student).vi	Design the state-feedback gain through pole-placement (PP). The model has not been entered (e.g., to be completed by the student).
ROTPEN Control Simulation (Student).vi	Simulates the closed-loop response of the rotary pendulum. The model has not been enterred and the feedback loop is not completed.
ROTPEN Control (Student).vi	Implements the state-feedback balance controller on Quanser Rotary Pendulum (ROTPEN). Control loop not completed.
SRV02 RP Swing Up ((Student).vi	Implements the swing-up control on the Quanser Rotary Pendulum (ROTPEN). Controller not completed.

Table 5.1: Files supplied with the SRV02 Inverted Pendulum laboratory.

5.2 Hardware Setup

Follow these steps to get the system hardware ready for this lab:

1. Make sure the SRV02 is in the *high-gear configuration*.

2. Install the Rotary Inverted Pendulum module on top of the SRV02 gear as shown in [3].

3. Connect the Quanser Inverted Pendulum to the amplifier (e.g. VoltPAQ) and DAQ device as described in [3].

Note: If you are using the NI CompactRIO, then see the SRV02 cRIO User Manual ([5]).

5.3 Software Setup

Follow these steps to get the system ready for this lab:

1. Load the LabVIEW™ software.

2. Open the LabVIEW project called *Rotary Pendulum (Student).lvproj* shown in Figure 5.1.
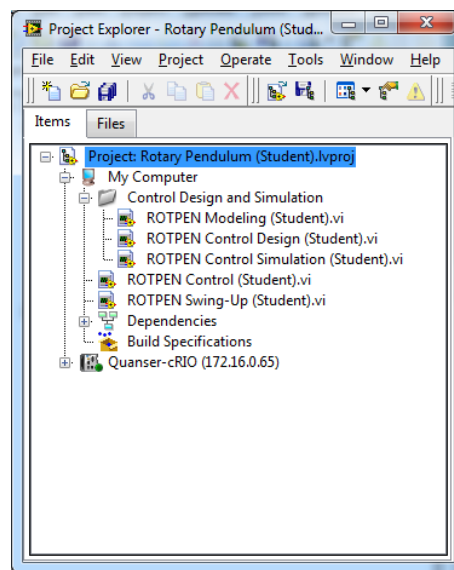

Figure 5.1: Rotary Pendulum LabVIEW Project

3. The *Control Design and Simulation* folder contains the simulation-based VIs that do not require any hardware.

4. To implement the state-feedback balance controller on the Quanser Rotary Pendulum, open ROTPEN Control.vi, shown in Figure 5.2.
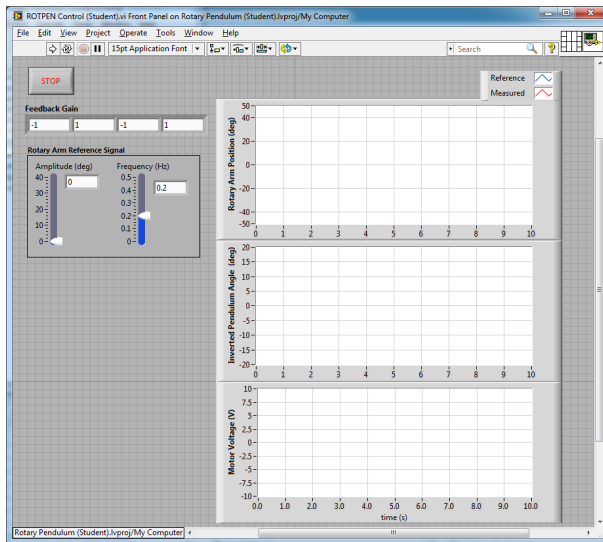
Figure 5.2: Front panel of ROTPEN Control (Student).vi

5. To implement the swing-up control on the Quanser Rotary Pendulum, open one the ROTPEN Swing Up.vi, shown in Figure 5.3.
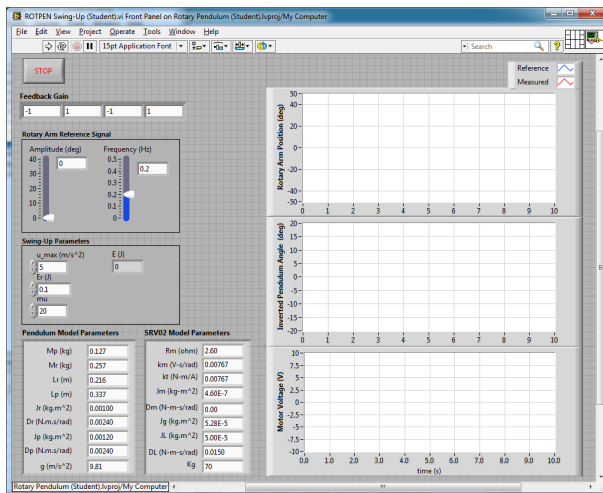

Figure 5.3: Front panel of ROTPEN Swing Up Control (Student).vi

6. **Configure DAQ:** Before running the VI, make sure you set the correct *Board type* (e.g., 'q1_cRIO', 'q2_usb', 'q8_usb', 'qpid', or 'qpid_e') in the HIL Initialize block.

7. **Quanser CompactRIO Users:** Before running the VI, make sure you can connect to your CompactRIO through the Measurement & Automation software. See the SRV02 cRIO User Manual ([5]).

8. **Channel Configuration:** For any of the DAQ-based VIs, the encoder input and ouput channels are set, by default, to match the wiring in the Rotary Pendulum User Manual ([3]). If the wiring is different on your system, make sure the VI uses the correct channels. For instance, if your pendulum encoder is connected to Encoder Input Channel #2 on your DAQ, then set the *HIL Encoder Read* VI to read from channels [0,2] (instead of [0,1]).

6 LAB REPORT

This laboratory contains three experiments, namely,

1. Modeling,

2. Balance Control, and

3. Swing-Up Control.

When you are writing your lab report, follow the outline corresponding to the experiment you conducted to build the *content* of your report. Also, in Section 6.4 you can find some basic tips for the *format* of your report.

6.1 Template for Content (Modeling)

I. PROCEDURE

1. *Model Analysis*

   • Briefly describe the main goal of the simulation.
   • Briefly describe the simulation procedure in steps 2 and 3 in Section 2.3.1.

2. *Calibration*

   • Briefly describe the main goal of the experiment.
   • Briefly describe the experiment procedure (Section 2.3.2)

II. RESULTS
Do not interpret or analyze the data in this section. Just provide the results.

1. State-space representation from step 3 in Section 2.3.1.

2. Provide applicable data collected in this laboratory (from Table 2.1).

III. ANALYSIS
Provide details of your calculations (methods used) for analysis for each of the following:

1. Measured arm response in step 7 in Section 2.3.2.

IV. CONCLUSIONS
Interpret your results to arrive at logical conclusions for the following:

1. Whether the arm and pendulum angles match the model conventions in step 4 of Section 2.3.2, *Sensor calibration*.

2. Whether the control voltage matches the model conventions in step 7 of Section 2.3.2, *Actuator calibration*.

6.2 Template for Content (Balance Control Experiment)

I. PROCEDURE

1. *Control Design*

   - Briefly describe the main goal of the control design.
   - Briefly describe the control design procedure in Step 4 of Section 3.4.1.

2. *Simulation*

   - Briefly describe the main goal of the simulation.
   - Briefly describe the simulation procedure (Section 3.4.2).

3. *Implementation*

   - Briefly describe the main goal of this experiment.
   - Briefly describe the experimental procedure (Section 3.4.3).

II. RESULTS

Do not interpret or analyze the data in this section. Just provide the results.

1. Matrices from step 4 in Section 3.4.1, *Find transformation matrix*.

2. Completed block diagram of *Pole Placement* tab from Step 7 in Section 3.4.1, *Block diagram to generate the control gain*.

3. Response plot from step 4 in Section 3.4.2, *Inverted pendulum balance control simulation*.

4. Response plot from step 5 in Section 3.4.3, *Inverted pendulum control implementation*.

5. Provide applicable data collected in this laboratory (from Table 3.1).

III. ANALYSIS

Provide details of your calculations (methods used) for analysis for each of the following:

1. Step 3 in Section 3.4.1, *Controllability of system*.

2. Step 5 in Section 3.4.2, *Balance control simulation*.

3. Step 6 in Section 3.4.3, *Balance control implementation*.

IV. CONCLUSIONS

Interpret your results to arrive at logical conclusions for each of the following:

1. Step 6 in Section 3.4.1, *Closed-loop system poles*.

2. Step 5 in Section 3.4.2, *Balance control simulation*.

3. Step 6 in Section 3.4.3, *Balance control implementation*.

6.3 Template for Content (Swing-Up Control Experiment)

I. PROCEDURE

1. *Implementation*

    - Briefly describe the main goal of this experiment.
    - Briefly describe the experimental procedure in Step 3 in Section 4.3.
    - Briefly describe the control states and parameters for the swing-up control in Step 5 in Section 4.3.
    - Briefly describe the parameters used to convert torque to voltage in Step 6 in Section 4.3.

II. RESULTS

Do not interpret or analyze the data in this section. Just provide the results.

1. Completed block diagram in 5 in Section 4.3, *Implemented swing-up controller*.

2. Completed block diagram in 7 in Section 4.3, *Implemented self-erecting controller*.

3. Response plot from step 8 in Section 3.4.3, *Self-erecting inverted pendulum control implementation response*.

4. Provide applicable data collected in this laboratory (from Table 4.1).

III. ANALYSIS

Provide details of your calculations (methods used) for analysis for each of the following:

1. Step 4 in Section 4.3, *Potential energy of pendulum*.

IV. CONCLUSIONS

Interpret your results to arrive at logical conclusions for each of the following:

1. Step 8 in Section 4.3, *Swing-up control implementation*.
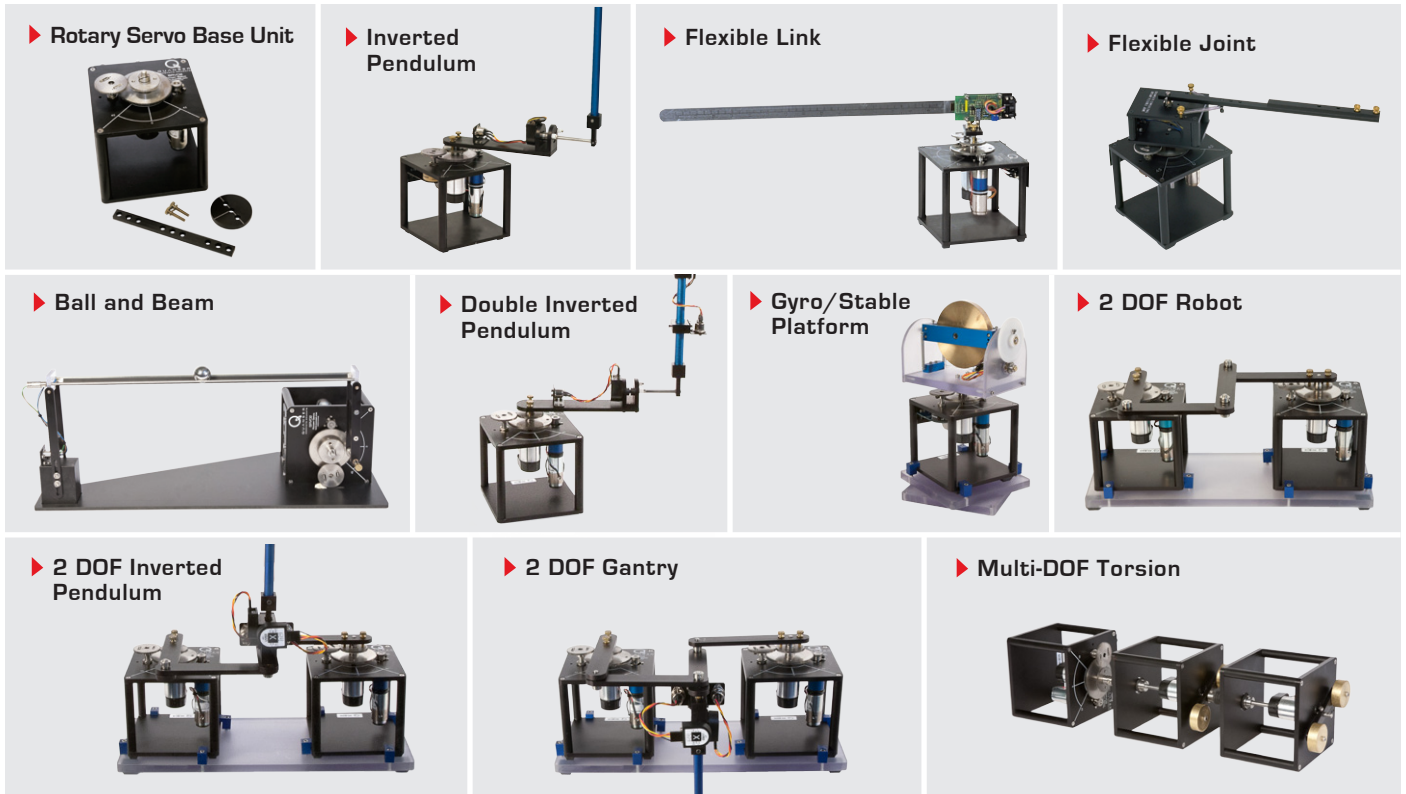
6.4 Tips for Report Format

PROFESSIONAL APPEARANCE

- Has cover page with all necessary details (title, course, student name(s), etc.)

- Each of the required sections is completed (Procedure, Results, Analysis and Conclusions).

- Typed.

- All grammar/spelling correct.

- Report layout is neat.

- Does not exceed specified maximum page limit, if any.

- Pages are numbered.

- Equations are consecutively numbered.

- Figures are numbered, axes have labels, each figure has a descriptive caption.

- Tables are numbered, they include labels, each table has a descriptive caption.

- Data are presented in a useful format (graphs, numerical, table, charts, diagrams).

- No hand drawn sketches/diagrams.

- References are cited using correct format.

REFERENCES

[1] Bruce Francis. Ece1619 linear systems course notes (university of toronto), 2001.

[2] Quanser Inc. *SRV02 User Manual*, 2009.

[3] Quanser Inc. *SRV02 Rotary Pendulum User Manual*, 2010.

[4] Quanser Inc. *VoltPAQ User Guide*, 2010.

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[6] Norman S. Nise. *Control Systems Engineering*. John Wiley & Sons, Inc., 2008.

[7] K. J. Åström and K. Furuta. Swinging up a pendulum by energy control. *13th IFAC World Congress*, 1996.

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