# Lecture 9

Lecturer: G. S. Paraoanu

*Department of Applied Physics, School of Science,*

*Aalto University, P.O. Box 15100, FI-00076 AALTO, Finland*

## I. SUPERCONDUCTING QUANTUM CIRCUITS

General Approach

- Superconducting circuits have quantized energy levels.
- Josephson junctions are non-linear elements which allow us to make the energy spacing non-equidistant.
- We can create a situation where all but 2 energy levels can be ignored creating effectively a quantum two-level system, i.e., a qubit.

## II. SHORT REVIEW: $E_J, E_c, E_L$

$$E_J = \frac{\Phi_0 I_c}{2\pi}(1 - \cos\phi_J)$$
$$E_c = \frac{e^2}{2C}$$
$$E_\ell = \frac{1}{2}(L_g + L_k)I_\ell^2$$

The Charge Qubit

To derive properties of a charge qubit, we follow the "standard" procedure: Start with a Lagrangian and do a Legendre transform.

For a charge qubit there is fixed gate voltage, which we model as an external mode with a well-defined flux $\phi_V = V_g t$, meaning $\dot{\phi}_V = V_g$. Setting $\phi^T = (\phi, \phi_V)$ we write the capacitance matrix

$$C = \begin{bmatrix} C_J + C_g & -C_g \\ -C_g & C_g \end{bmatrix}. \tag{1}$$

From this, we can write the Lagrangian $L = T - V$, i.e.,

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^T C \dot{\phi} + E_J \cos\phi \,, \tag{2}$$

From the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_n} = \frac{\partial \mathcal{L}}{\partial \phi_n} \,. \tag{3}$$

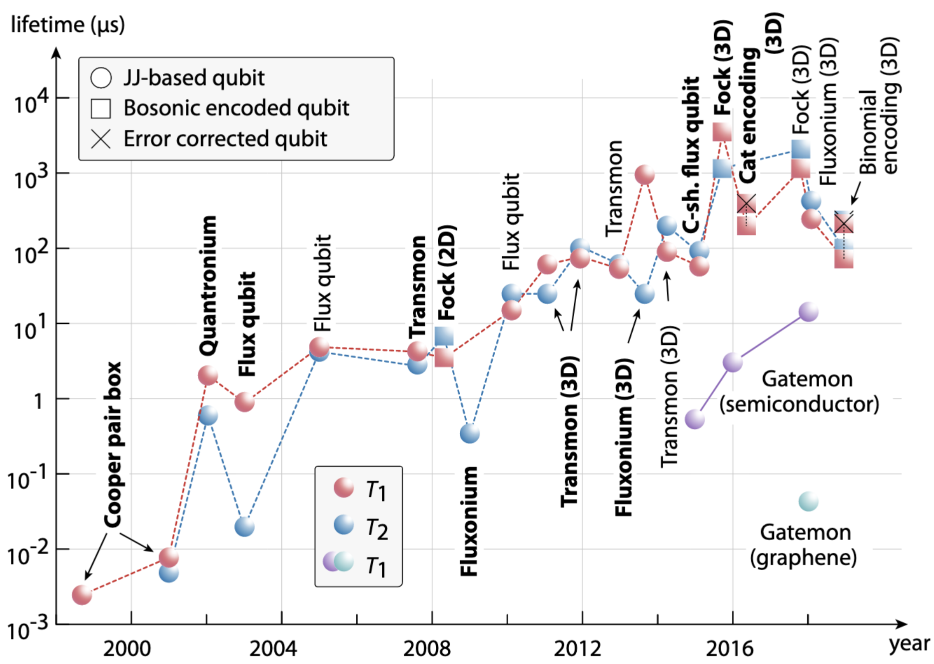
The Hamiltonian of the circuit can be found by a simple Legendre transformation of the Lagrangian. First we define the conjugate momentum to the node flux

$$q_n = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_n} \,, \tag{4}$$

which in vector form becomes

$$q = C\dot{\phi} \,. \tag{5}$$

This requires that the capacitance matrix be invertible.

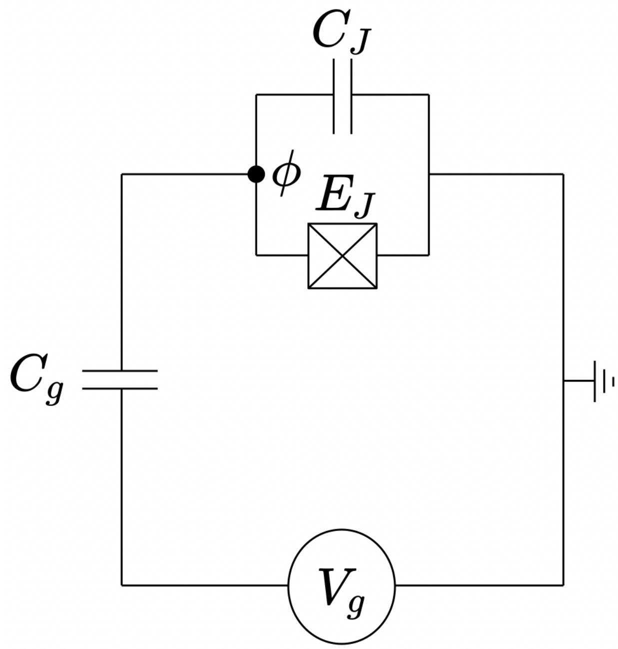



The qubit zoo as shown in arXiv:1905.13641v3.

<u>The charge qubit</u>

The Hamiltonian can now be expressed in terms of the node charges for the kinetic energy and node fluxes for the potential energy, i.e.,

$$\mathcal{H} = \dot{\phi}^T q - \mathcal{L} = \frac{1}{2} q^T C^{-1} q + E_{pot}(\phi) \ . \tag{6}$$




A charge qubit.

Upon solving for $\dot{\phi}$ and applying the Legendre transformation, we find the Hamiltonian

$$\mathcal{H} = \frac{1}{2(C_g + C_J)}(q + C_g V_g)^2 - \frac{C_g V_g^2}{2} - E_J \cos\phi \ . \tag{7}$$

Adopting the conventional notation and defining the effective capacitive energy

$$E_c = \frac{e^2}{2(C_g + C_J)} \ , \tag{8}$$

Further, we quantize the dynamic variables and remove constant terms:

$$\hat{\mathcal{H}} = 4E_c(\hat{n} - n_g)^2 - E_J \cos\hat{\phi} \ , \tag{9}$$

where the offset charge is

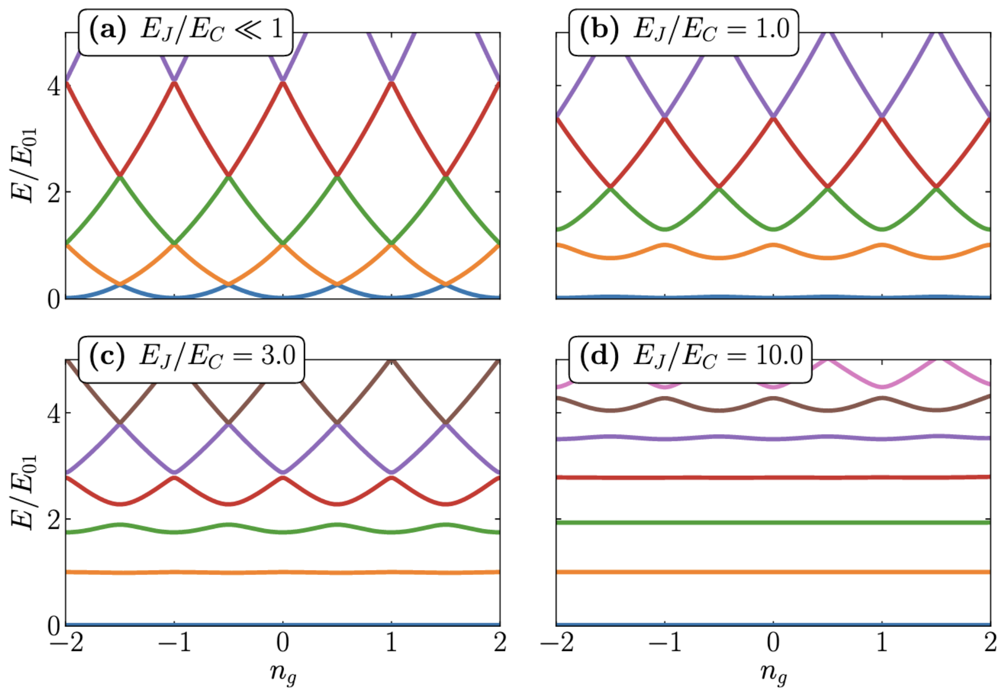
$$n_g = C_g V_g / 2e \ . \tag{10}$$

We now consider certain parameter regimes:

where

| **Energy ratio** | **Effective Hamiltonian** | **Interpretation** |
|---|---|---|
| $E_J/E_c = 0$ | $\hat{\mathcal{H}}_C = 4E_c \sum_{n=-\infty}^{\infty}(n-n_g)^2\|n\rangle\langle n\|$ | Charge states of the capacitor |
| $E_J/E_c = 1.0$ | $\hat{\mathcal{H}} = \hat{\mathcal{H}}_C + \hat{\mathcal{H}}_J$ | Lifted degeneracy, qubit states exist |

$$\hat{\mathcal{H}}_J = -\frac{E_J}{2}\sum_{n=-\infty}^{\infty}\left(|n\rangle\langle n+1| + |n+1\rangle\langle n|\right) . \tag{11}$$




Lowest lying states of a single Cooper pair box as a function of the bias (gate) charge $n_g$.

From arxiv.org/pdf/2103.01225.pdf

We can move to a half-integer voltage offset and ignore all the higher-level states. Then we have a two-level system which is commonly described as

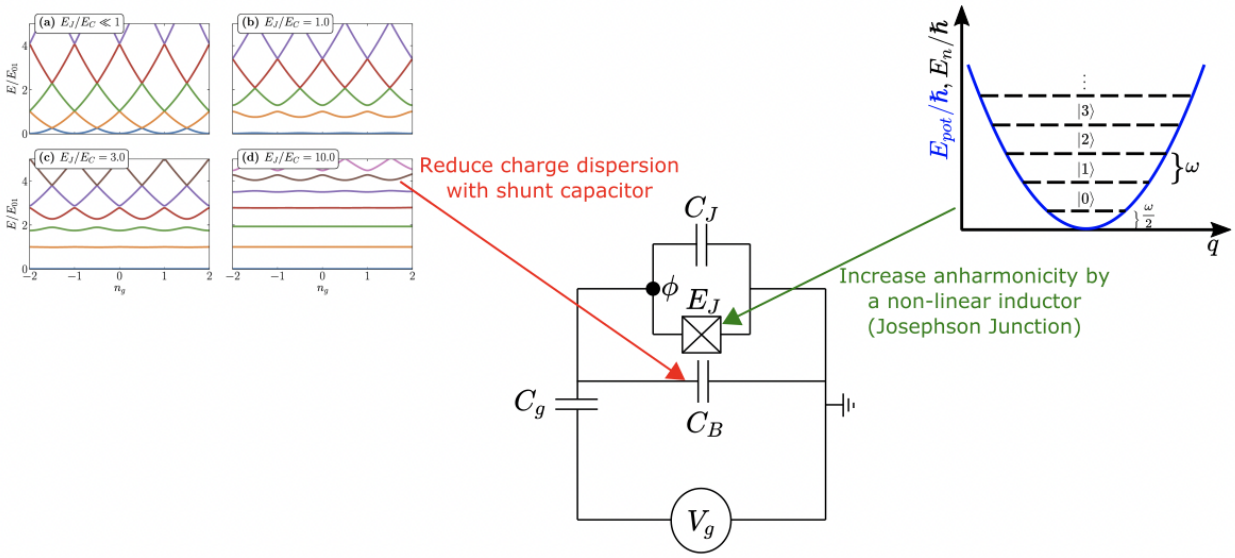
$$\mathcal{H}_q = \frac{\omega_q}{2}\hat{\sigma}_z , \tag{12}$$

where we have used the Pauli z-operator

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{13}$$

for the qubit eigenbasis

$$\psi_{z+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \qquad \psi_{z-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{14}$$




Charge qubit with a large shunt capacitor $C_B$.

As can be seen from the above figure, when we shunt the charge qubit with a large capacitance $C_B$, we are able reduce the charge dispersion. Further, the Josephson junction (non-linear inductor) increases the anharmonicity of the system.

Anharmonic oscillator: The transmon qubit

We have learned that an LC oscillator has equidistant level spacing

$$H = \hbar\omega_r \left( a^\dagger a + \frac{1}{2} \right) . \tag{15}$$

To introduce non-equidistant level spacing (qubits) we use a non-linear inductor (Josesphson junction)

$$L_J = \frac{\Phi_0}{2\pi I_c \cos\phi_J} = L_C \frac{1}{\cos\phi_J} . \tag{16}$$

The Hamiltonian of a capacitively shunted Josephson junction has two components

$$H_{tr} = 4E_c \hat{n}^2 - E_J \cos\hat{\phi} . \tag{17}$$

Here, we have two parameters, i.e., $E_c$ and $E_J$. We have to get rid of one.

To approach quantization, we define the operators

$$\hat{n} = i n_{zpf}(\hat{c} + \hat{c}^\dagger) \ , \tag{18}$$

$$\hat{\phi} = \phi_{zpf}(\hat{c} - \hat{c}^\dagger) \ . \tag{19}$$

Here, $\hat{c}$ is the transmon annihilation operator, so as to distinguish it from the evenly-spaced energy modes of $\hat{a}$

$$\hat{c} = \sum_j \sqrt{j+1} |j\rangle\langle j+1| \ . \tag{20}$$

The pre-factors describe vacuum fluctuations (zero point fluctuations, zpf):

$$n_{zpf} = \left(\frac{E_J}{32E_c}\right)^{1/4} \ , \tag{21}$$

$$\phi_{zpf} = \left(\frac{2E_c}{E_J}\right)^{1/4} \ . \tag{22}$$

To assume $\phi \ll 1$, we choose $C_s \gg C_J$ and Taylor expand the $E_J$ term:

$$E_J \cos(\phi) = \frac{1}{2}E_J\phi^2 - \frac{1}{24}E_J\phi^4 + \mathcal{O}(\phi^6) \ . \tag{23}$$

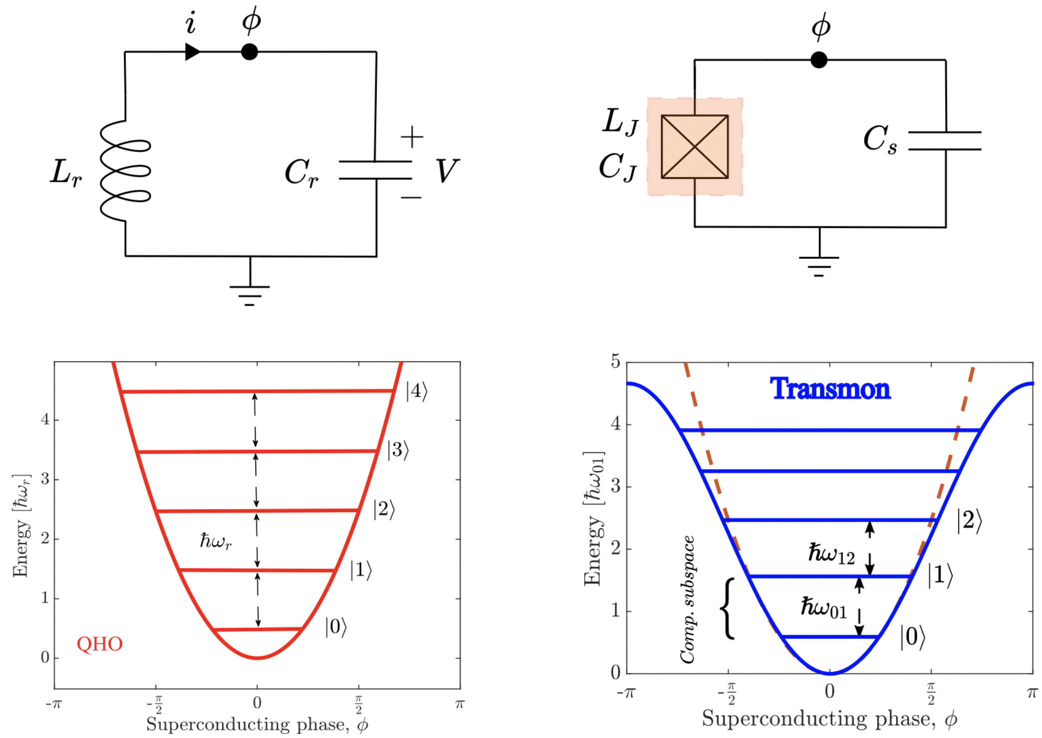
In the $\hat{c}$ basis the complete Hamiltonian reads

$$\begin{aligned} H &= -4E_c n_{zpf}^2(\hat{c} + \hat{c}^\dagger)^2 - E_J\Big(1 - \frac{1}{2}\phi_{zpf}^2(\hat{c} - \hat{c}^\dagger)^2 + \frac{1}{24}\phi_{zpf}^4(\hat{c} - \hat{c}^\dagger)^4 + \cdots\Big) \\ &\approx \sqrt{8E_cE_J}\Big(\hat{c}^\dagger\hat{c} + \frac{1}{2}\Big) - E_J - \frac{E_c}{12}(\hat{c}^\dagger - \hat{c})^4 \ . \end{aligned} \tag{24}$$

Expanding the terms of the transmon operator and dropping the fast-rotating terms (those with an uneven number of $\hat{c}$ and $\hat{c}^\dagger$), and adopting the relations $\hbar\omega_0 = \sqrt{8E_JE_c}$ and $\delta = -E_c$, we find after neglecting constants

$$\hat{H}_{\text{tr}} = \omega_0\hat{c}^\dagger\hat{c} + \frac{\delta}{2}\hat{c}^\dagger\hat{c}^\dagger\hat{c}\hat{c} \ . \tag{25}$$

This is the Hamiltonian of a (non-linear) Duffing Hamiltonian. The energy levels of the system are calculated as




LC oscillator and the transmon qubit along with their corresponding energy levels.

$$\omega_j = \left(\omega - \frac{\delta}{2}\right) j + \frac{\delta}{2} j^2 \ , \tag{26}$$
$$\omega \equiv \omega_0 + \delta \ .$$

Key Takeaway: The transmon qubit is actually a non-linear oscillator and we use the two lowest eigenstates as qubit states. They correspond to having 0 or 1 excitations stored in the system.

Frequency control: The split transmon qubit

Choosing the two lowest eigenstates as qubit states, allows us to treat the transmon as a qubit with Hamiltonian

$$\mathcal{H}_q = \frac{\omega_q}{2} \hat{\sigma}_z \ . \tag{27}$$

Frequency tunability can be created by replacing the single junction with a split junction, i.e., a DC SQUID

$$\omega_q \approx \sqrt{8E_c E_{J0}|\cos(\pi f_{ext})|} \tag{28}$$

Closing the loop: The flux qubit

In contrast to the charge- and transmon qubit, the flux qubit operates in the regime $E_{J0} \gg E_c$ such that magnetic flux is the good quantum variable.

The most common implementation is a closed superconducting loop intersected by three Josephson junctions. One junction has a reduced Josephson energy by a factor $0.5 < \alpha < 1$. This yields the potential

$$U_q = E_{J0}[2 + \alpha - \cos(\phi_1) - \cos(\phi_2) - \alpha\cos(\phi_{ext} + \phi_1 - \phi_2)]\ . \tag{29}$$

Here $\phi_{ext} = 2\pi\Phi_{ext}/\Phi_0$ is the reduced magnetic field threading the qubit loop and $\phi_1$, $\phi_2$ are the phase differences across the two identical junctions. The third phase difference is eliminated due to the boundary condition imposed by flux quantization.

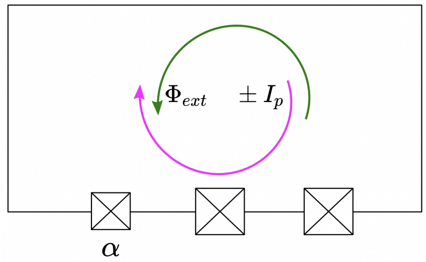
Flux quantization is guaranteed by a persistent circulating current

$$I_p = \pm I_c\sqrt{1 - (2a)^{-2}}\ . \tag{30}$$

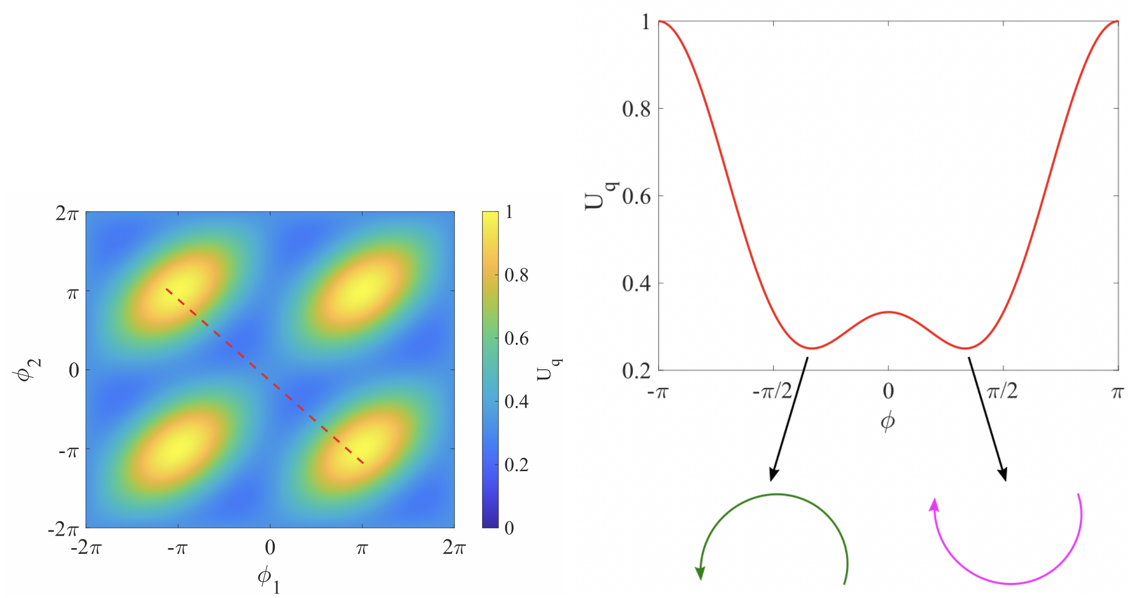
The two degrees of freedom result in a two-dimensional potential. For $\phi_{ext} = \pi$, the potential is symmetric and periodic.

Since the two larger junctions are identical, we are confined to move along the line $\phi_2 = -\phi_1$.

The potential along this line has the form of a double well, where the minima correspond to circulating currents in opposite directions.




A flux qubit with three Josesphon junctions illustrating the superposition of the current states. Two of the junctions have the same Josephson energy $E_{J0}$, whereas, the third junction has an energy lowered by a factor of $\alpha$, with $0.5 < \alpha < 1.0$.



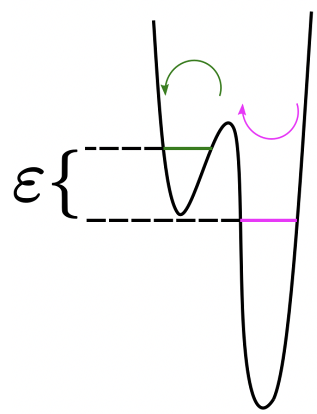
The two-dimensional potential (left panel) of a flux qubit showing the line $\phi_2 = -\phi_1$, and the corresponding double-well potential (right panel) along this line.

Since the potential barrier has a finite height, there is a certain tunneling probability $\Delta$ between the wells.

The tunneling coupling lifts the energy degeneracy between the states, resulting in a level splitting. The resulting energy levels can be used as two qubit states.

A change in magnetic flux bias tilts the potential leading to an additional energy $\varepsilon$.

$$\mathcal{H}_q = \frac{\hbar\Delta}{2}\hat{\sigma}_x + \frac{\hbar\varepsilon}{2}\hat{\sigma}_z = \frac{\hbar}{2}\begin{pmatrix} \varepsilon & \Delta \\ \Delta & -\varepsilon \end{pmatrix} \tag{31}$$




Level splitting due to the tunnel coupling.

Diagonalizing the system Hamiltonian to transform it into the qubit eigenbasis, we have

$$\mathcal{H}_q = \frac{\omega_q}{2}\hat{\sigma}_z \ , \tag{32}$$

where the qubit transition frequency is given as

$$\omega_q = \sqrt{\Delta^2 + \varepsilon^2} \ . \tag{33}$$

Key takeaway: The flux qubit is a closed loop of 3 Josephson junctions where the circulating currents provide energy eigenstates that can be treated as a quantum 2-level system.

Circuit QED

- Qubits can be seen as artificial atoms and resonators as microwave light.
- When we bring them close to each other we create "light-matter" coupling that is treated in the same way as quantum optics.
- Superconducting circuits allow on-chip study of quantum optics in regimes that cannot be reached in nature.

Example: Flux qubit coupled to transmission line resonator.

<u>Generalized light-matter interaction: The quantum Rabi model:</u>

We recall the following physical properties and their quantum mechanical description:

| **Physical System/Parameter** | **Effective Description** |
|---|---|
| Qubit | $\hat{\mathcal{H}}_q = \frac{\omega_q}{2}\hat{\sigma}_z$ |
| Resonator | $H = \hbar\omega_r\left(a^\dagger a + \frac{1}{2}\right)$ |
| Magnetic field | $LI_{vac}(\hat{a} + \hat{a}^\dagger)$ |
| Qubit energy bias | $\varepsilon(\hat{\sigma}_+ + \hat{\sigma}_-)$ |

If we bring the qubit and resonator is close vicinity, we create a mutual inductance leading to a coupling term (interaction Hamiltonian)

$$\hbar g(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^\dagger) \; , \tag{34}$$

where all physical properties are hiding in the coupling constant $g$.

Adding the individual terms for the qubit and resonator results in the system Hamiltonian

$$\mathcal{H}_{\mathrm{QR}} = \frac{\hbar\omega_q}{2}\hat{\sigma}_z + \hbar\omega_r\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \underbrace{\hbar g(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^\dagger)}_{\mathcal{H}_{\mathrm{int}}} \; . \tag{35}$$

The above Hamiltonian is valid in all regimes for $g$.

The counter-rotating and counterintuitive terms $\hat{a}\hat{\sigma}_-$ and $\hat{\sigma}_+\hat{a}^\dagger$ give rise to the so-called Bloch-Siegert shift, which is hard to observe in nature but possible with superconducting systems.

The structure of the quantum Rabi Hamiltonian is such that the physical property <u>parity</u> is conserved. This is a special form of symmetry and one of the most fundamental concepts in physics.

The concept of parity for example gives rise to the selection rules for allowed transitions in atoms. The same selection rules can be observed in superconducting circuits.

Key Takeaway: Superconducting circuits follow the physics of **quantum optics** and can reach **parameter regimes** that are unreachable in nature.

Jaynes-Cummings Model:

- Usually one operates quantum circuits in a practical parameter regime, called strong coupling limit.
- In this limit, the coupling between qubit and electromagnetic field is much stronger as their loss rates but smaller than their eigenfrequencies.
- In this regime, the eigenstates experience a qubit state-dependent energy shift. Detecting this stuff is used for qubit readout.

Rotating wave approximation:

We consider a transmon qubit that is capacitively coupled to a transmission line resonator. We operate in the strong coupling regime where $g \ll \omega_q, \omega_r$.

This allows us to move into the interaction picture (a.k.a. rotating frame) defined as

$$\hat{\mathcal{H}}_{\mathrm{int}}(t) = \frac{\hbar g}{2}\Big(\hat{a}\hat{\sigma}_- e^{-i(\omega_r+\omega_q)t} + \hat{a}^\dagger\hat{\sigma}_+ e^{i(\omega_r+\omega_q)t} + \hat{a}\hat{\sigma}_+ e^{i(-\omega_r+\omega_q)t} + \hat{a}^\dagger\hat{\sigma}_- e^{-i(-\omega_r+\omega_q)t}\Big) \tag{36}$$

This Hamiltonian contains both quickly and slowly oscillating components

$$\omega_r + \omega_q \qquad \omega_r - \omega_q \; . \tag{37}$$

To get a solvable model, the quickly oscillating "counter-rotating" terms are ignored. This is referred to as the rotating wave approximation, and it is valid since the fast oscillating term couples states of comparatively large energy difference.

Transforming back into the Schrödinger picture, the Jaynes-Cummings Hamiltonian is thus

$$\mathcal{H}_{\mathrm{JC}} = \frac{\hbar\omega_q}{2}\hat{\sigma}_z + \hbar\omega_r\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \underbrace{\hbar g(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger)}_{\mathcal{H}_{\mathrm{int}}} \ . \tag{38}$$

With the Jaynes-Cummings Hamiltonian, we can either excite the qubit by absorbing a photon ($\hat{\sigma}_+\hat{a}$) or take one excitation from the qubit and generate a photon ($\hat{\sigma}_-\hat{a}'$).

In the basis of uncoupled excitation number ($n_r$) and qubit eigenstates, the Hamiltonian is transformed to

$$\mathcal{H}_{\mathrm{JC},n} = \frac{\hbar}{2}\begin{pmatrix} 2n_r\omega_r + \omega_q & g\sqrt{n_r+1} \\ g\sqrt{n_r+1} & (n_r+1)\omega_r - \omega_q \end{pmatrix} \ . \tag{39}$$

We diagonalize this Hamiltonian and discuss two parameter regimes: Resonant, i.e. no detuning between qubit and resonator, and off-resonant, i.e. large detuning.

The eigenfrequencies of the Jaynes-Cummings Hamiltonian are given as

$$\omega_{\pm,n} = \left(n_r + \frac{1}{2}\right)\omega_r \pm \frac{1}{2}\sqrt{\delta^2 + 4g^2(n_r+1)} \ . \tag{40}$$

Here, we have defined the detuning $\delta \equiv \omega_q - \omega_r$ and the ground state is $\omega_{-,0} = -\delta/2$.

The new dressed eigenstates of the system are the superposition states

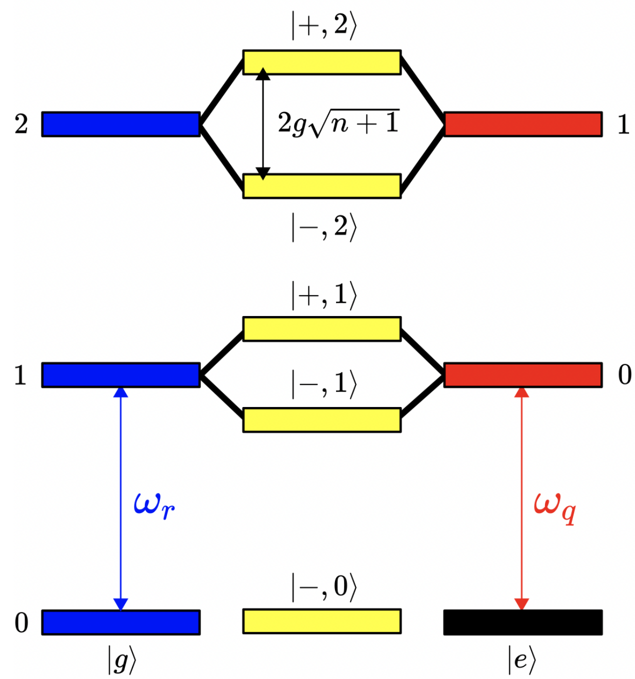
$$|+, n_r\rangle = \cos\Theta_{n_r}|e, n_r\rangle + \sin\Theta_{n_r}|g, n_r\rangle \tag{41}$$

$$|-, n_r\rangle = \cos\Theta_{n_r}|g, n_r+1\rangle - \sin\Theta_{n_r}|e, n_r\rangle \ , \tag{42}$$

where the mixing angle $\Theta_{n_r}$ is a measure of the degree of entanglement between the qubit and resonator states:

$$\Theta_{n_r} = \tan^{-1}(2g\sqrt{n_r+1}/\delta)/2 \ . \tag{43}$$

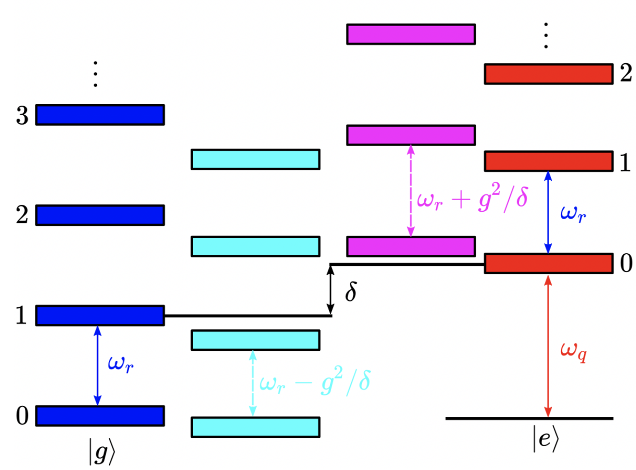
When the qubit and light mode are on resonance, i.e. $\delta \equiv 0$ the mixing angle $\Theta_n = \pi/4$ is maximum and consequently there is strong entanglement.




The resonant regime, i.e. $\omega_q = \omega_r$.

In this regime, a coherent exchange of excitations between qubit and resonator occurs with the vacuum Rabi frequency $2g$. This interaction lifts the degeneracy of the corresponding eigenenergies by $2g\sqrt{n_r+1}$ to new doublet eigenstates.

In the dispersive regime, the detuning between qubit and resonator frequency is much larger than the coupling, i.e., $\delta \gg g$.




The dispersive regime.

In this regime, there is no exchange of excitations anymore but virtual photons mediate a dispersive interaction between qubit and light field. This interaction leads to frequency shifts of the qubit and resonator eigenfrequencies. The dressed states are either more photon-like or more atom-like.

In the atom-like case (close to qubit states), the Hamiltonian can be derived as

$$\mathcal{H}_{\text{disp}} \approx \hbar\omega_r\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)+\frac{\hbar}{2}\Big(\omega_q+\underbrace{2\chi\hat{a}^\dagger\hat{a}}_{\text{AC-Stark shift}}+\underbrace{\chi}_{\text{Lamb shift}}\Big)\hat{\sigma}_z\ . \tag{44}$$

In the photon-like case (close to resonator states), the Hamiltonian can be derived as

$$\mathcal{H}_{\text{disp,r}} \approx \hbar\omega_q\frac{\hat{\sigma}_z}{2}+\hbar(\omega_r+\chi\hat{\sigma}_z)\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)\ , \tag{45}$$

describing the qubit state-dependent resonator frequency, which we use for readout purposes.

Key takeaway: In the dispersive regime of the Jaynes-Cummings model, resonators can be used for qubit readout.

## References


1. https://arxiv.org/pdf/1905.13641.pdf

2. https://arxiv.org/pdf/2103.01225.pdf