Ex4 solutions 2023

1 Explorative exercises

1.1

There exists m^n functions from $\{1, \ldots, n\}$ to $\{1, \ldots, m\}$ because for each of the n elements we have m choices of where it can be mapped.

1.1.1

For an injective function to exist, we must have that $n \leq m$. When we construct the function, we have m choices for the first element, $m-1$ for the second, and so on until for the final element we have $m-n+1$ choices. Hence the number of injective functions is $\frac{m!}{(m-n)!}$.

1.1.2

For a function to be non-surjective, it has to "miss" some element of $\{1, 2\}$. Since there are only two elements, we have only two such functions, one where everything is mapped to 1, and another where everything is mapped to 2.

1.1.3

The number of functions where the function "misses" $i \in \{1, 2, 3\}$ is 2^n for each i. Summing these up we get $3 \cdot 2^n$. However, this sum double counts all of the functions that map all elements to one element. Hence the final answer is $3 \cdot (2^n - 1)$.

1.2

We can consider the original set to be an increasing ordering and the map (the permutation) as giving the set a new total order by the value they are assigned to.

1.3

1.3.1

The permutation $(\sigma(1), \ldots, \sigma(9) = (2, 7, 5, 6, 9, 3, 8, 4, 1)$ gives the cycle of length 9:

$$1 \to 2 \to 7 \to 8 \to 4 \to 6 \to 3 \to 5 \to 9 \to 1$$

The permutation $(\sigma(1), \ldots, \sigma(9) = (2, 7, 5, 6, 9, 3, 1, 4, 8)$ gives the following cycles

$$1 \to 2 \to 7 \to 1 \text{ and } 3 \to 5 \to 9 \to 8 \to 4 \to 6 \to 3$$

1.3.2

Yes, bijectivity ensures this. Cycles of length one are also cycles.

2 Additional exercises

2.1

There are 26! permutations for the 26 letters. To find the number of permutations which do not contain the words "cats", "snow" or "walk", let us first find the number of permutations which do contain them and then substract that number from 26!.

First, count the number of permutations containing one of the words. We can consider that word to be a solid block and represent it with a new symbol, say π. If the set of letters is Σ, then we now have to count the permutations of $(\Sigma \setminus \{"c","a","t","s"\}) \cup \pi$. That number is 23!.

Let A_1, A_2, A_3 be the sets of permutations containing all permutations which have the word "cats", "snow" and "walk", respectively. Since all words are of length four, we have that $|A_i| = 23!$

For $A_1 \cap A_2$, we only have to consider the block "catsnow". And thus by the previous arguments $|A_1 \cap A_2| = 20!$. The same also holds for $A_2 \cap A_3$, where we consider the block "snowalk", so $|A_2 \cap A_3| = 20!$. The words "cats" and "walk" both contain the letter 'a' in the middle of the word and thus cannot occur at the same time. Hence $|A_1 \cap A_3| = |A_1 \cap A_2 \cap A_3| = \emptyset$. Let Σ^* be the set of all permutations of Σ. By the inclusion-exclusion principle, we get
$|\Sigma^*| - |A_1 \cup A_2 \cup A_3| = |\Sigma^*| - (|A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3|) = 26! - 3 \cdot 23! + 2 \cdot 20!$

2.2

2.2.1

Let's say we first had the pairs $(ab), (cd), (ef)$. Now let's re-assign the pairs, starting with choosing a pair for a. There are 4 choices for a. If a was to be paired with c, then b couldn't be paired with d, since that would leave the pair (ef) untouched. Hence there are 2 choices for b. After this choice there is only one pair left. Thus the answer is $4 \cdot 2 = 8$.

2.2.2

Let us take it as a given that the number of ways to divide a group of $2n$ people into pairs is $f(2n) = \frac{(2n)!}{n!2^n}$. (This can be shown easily).

Now let $A = \{a_1, a_2, \ldots, a_n\}$ be the original set of pairs and let A_i be the set of pairs where the pair a_i is kept the same. It follows that $|A_i| = f(2n-2)$. To get the number of ways to form new pairs without anyone having the same partner, we substract $|A_1 \cup A_2 \cup \cdots \cup A_n|$ from the total number of pairs $f(2n)$.

By the inclusion-exclusion principle

$$|A_1 \cup \cdots \cup A_n| = \sum_{i=0}^{n} |A_i| - \sum_{i \neq j} |A_i \cap A_j| + \cdots - |A_1 \cap A_2 \cap \cdots \cap A_n|$$

The number $|A_i \cap A_j|$ represent the number of pairings where there are the pairs a_i ja a_j. So we find that $|A_i \cap A_j| = f(2n-4)$. The sum over all of these terms is equal to $\binom{n}{2} f(2n-4)$. Same argument can be repeated for k fixed pairs, i.e. $\binom{n}{k} f(2n-2k)$ is the number of pairings where some k pairs are fixed. The final answer becomes

$$f(2n) - |A_1 \cup \cdots \cup A_n| = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(2n-2k)$$

2.3

2.3.1

Product of disjoint cycles:

$$(1362)(2564)(2345) = (13)(5)(264)$$

2.3.2

Two line notation:

$$(1362)(2564)(2345) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 1 & 2 & 5 & 4 \end{pmatrix}$$

2.3.3

Product of transpositions:

$$(1362)(2564)(2345) = (13)(24)(26)$$

2.4

Let $\rho = (123)$ and $\pi = (12)$ in S_3

2.4.1

We see that π^2 is clearly the identity. For ρ^3 we have

$$(123)^3 = (123)(123)^2 = (123)(132) = \iota$$

2.4.2

$S_3 = \{\iota, (12), (13), (23), (123), (132)\}$. We find that

π	(12)
ρ	(123)
ρ^2	(132)
$\pi\rho$	(23)
$\pi\rho^2$	(13)

2.5

2.5.1

$\rho = (abdc)(efhg),\ \ \sigma = (acge)(bdhf),\ \ \tau = (abfe)(cdhg)$

2.5.2

$\rho\sigma = (bce)(dgf),\ \ \sigma\tau = (che)(dfa),\ \ \tau\rho = (fga)(hcb)$

2.5.3

24, see next part.

2.5.4

There are 24 rotational symmetries of a cube, one of which is the identity map. We get 9 other symmetries by the powers of ρ, σ and τ, the symmetric representations of these are already explained in the question. The we have the diagonal axis of symmetry, which keep two the the corners fixed. These are represented with powers of $\rho\sigma$, $\sigma\tau$ and $\tau\rho$ from part b) and the powers of $\tau\sigma$. There is a total of 8 of these symmetries. Finally, we have six symmetries that fix two edges, one such symmetry would be $(ab)(gh)(cf)(de)$, where the fixed edges are (ab) and (gh). The corners of the fixed edges swap and other corners swap diagonally. There are a total of six of these symmetries and they are represented by combinations of powers ρ, τ, and σ mixed together. The example symmetry is given by $\sigma^{-1}\rho^2 = \sigma^3\rho^2 = (ab)(gh)(cf)(de)$.

The final answer is: there are 24 symmetries of a cube (1+9+8+6) and they can all be written as products of the given operations.

2.6

Let $|A \cap B| = |A \cap C| = |B \cap C| = m$. Note that $A \cap B \cap C = \emptyset \ \Rightarrow\ |A \cap B \cap C| = 0$. From the inclusion-exclusion principle it follows that

$$\begin{aligned}
|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
2n &= 3n - 3m \\
n &= 3m
\end{aligned}$$

Thus these condition can only be true when n is divisible by 3.

2.7

2.7.1

See Figure 1 for the set up. Let's start by pairing a. It has three choices. If a is paired with c or e, it leads to only one possible pairing. Since if a chooses c, the only valid choice for b is e. Same holds, if a chooses e. So that's two possible pairings. If a is paired with d, then b has two choices. Thus the number of pairings is 4.

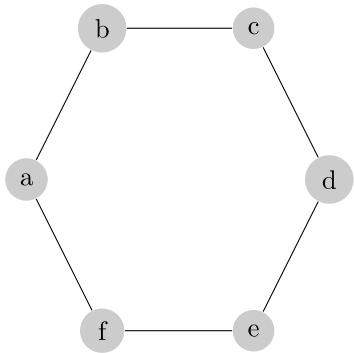

Figure 1: Visualisation of the problem

2.7.2

The answer is

$$\sum_{k=0}^{n}(-1)^k P(n,k)(2n-2k-1)!!,$$

where $P(n,k) = \binom{2n-k}{k} + \binom{2n-k-1}{k-1}$ and $(2n-2k-1)!! = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-2k-1)$. The equation comes from the inclusion-exclusion principle. Here $P(n,k)$ is the number of ways to choose k pairs of adjacent people in the circle. The multiplier $(2n-2k-1)!!$ is the number of ways to pair up the rest of the people.

2.7.3

Overlapping pairs make this problem more difficult and the coefficient more complicated.

2.8

For any element $a \in A$ or any element $b \in B$, let us consider the chain

$$\cdots \to f^{-1}(g^{-1}(a)) \to g^{-1}(a) \to a \to f(a) \to g(f(a)) \to \cdots$$

This chain may terminate to the left if the inverse function does not exist. By injectivity of both f and g, every element has exactly one such chain. Therefore, if an element occurs in the the chain, the chains for these two elements are the same. Thus we have a partition for the set $A \cup B$. Hence it suffices to construct separate bijections for these partitions. If a sequence stops at an element in A, then f is a bijection for it. Similarly, if a sequence stops at an element of B, then g is the bijection. Finally, if the sequence never stops, either one of f or g will work as a bijection.

This is known as the Schröder-Bernstein Theorem.