MEC-E1005

MODELLING IN APPLIED MECHANICS 2023

Week 17 PERIODIC STRUCTURE DISPLACEMENT

Thu 12:15-13:30 Simplified analysis (K215)

Thu 14:15-15:30 FEM analysis by Mathematica (Y338)

PERIODIC STRUCTURE

ASSIGNMENT

A structure is considered as periodic if one may identify a part which repeats itself in one or several directions. Regularity in that sense can be used to simplify, e.g., displacement analysis with variable number of parts.

The aim is to find the effect of geometrical and material parameters to the displacement-load relationship of a periodic steel structure of the type shown in the figure. The left end is attached to a wall. The right end is loaded by a point force. The starting point is a generic expression predicted by dimension analysis. First, a simplified model is used for a more specific relationship. After that, analysis by FEM is used to study the effects of details. Finally, the displacement-force relationship is compared with an experimental one.

IDEALIZATION AND PARAMETERIZATION

In the idealization, *n* identical parts are connected by welds of triangular shape. The left end is rigidly connected to a wall and load is acting at the right end.

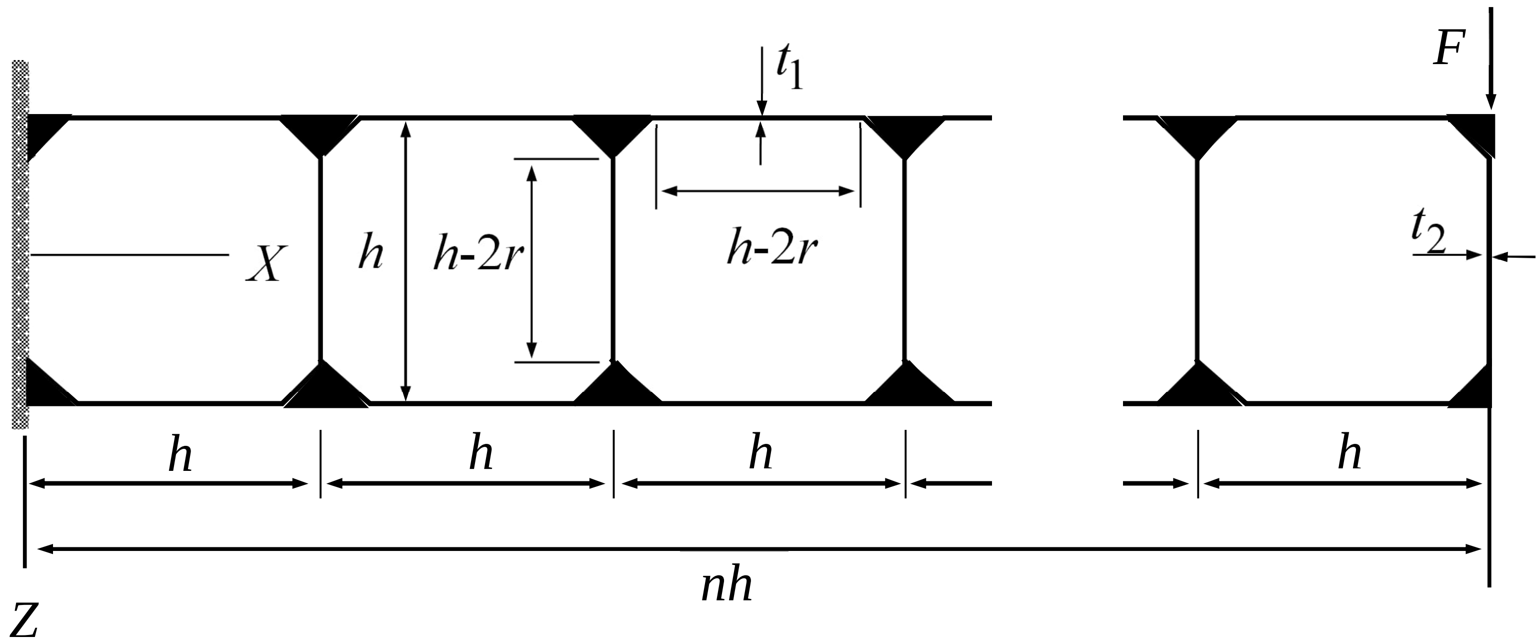


Table 1. Geometrical and material parameters of the structure.

Parameter	symbol	value
Side length	h	
Width	b	
Number of parts	n	
Weld size	r	
Thickness 1	t_1	
Thickness 2	t_2	
Young’s modulus	E	210 GPa

DIMENSION ANALYSIS

Assuming that the effect of the axial deformation of the material is negligible compared to that of bending, dimension analysis, based on quantities E, I_1, I_2, h, r, F, and w, gives the relationship

$$\frac{w}{h} = \frac{Fh^2}{EI_1}\alpha\left(\frac{r}{h}, \frac{I_2}{I_1}, n\right), \tag{1}$$

in which the form of the right-hand side requires a more detailed analysis. The indices in the second moments refer to the two different thicknesses of the material. Equation (1) assumes linearity between displacement and load, vanishing displacement with zero loading, and that the straight parts behave essentially as Bernoulli beams in bending.

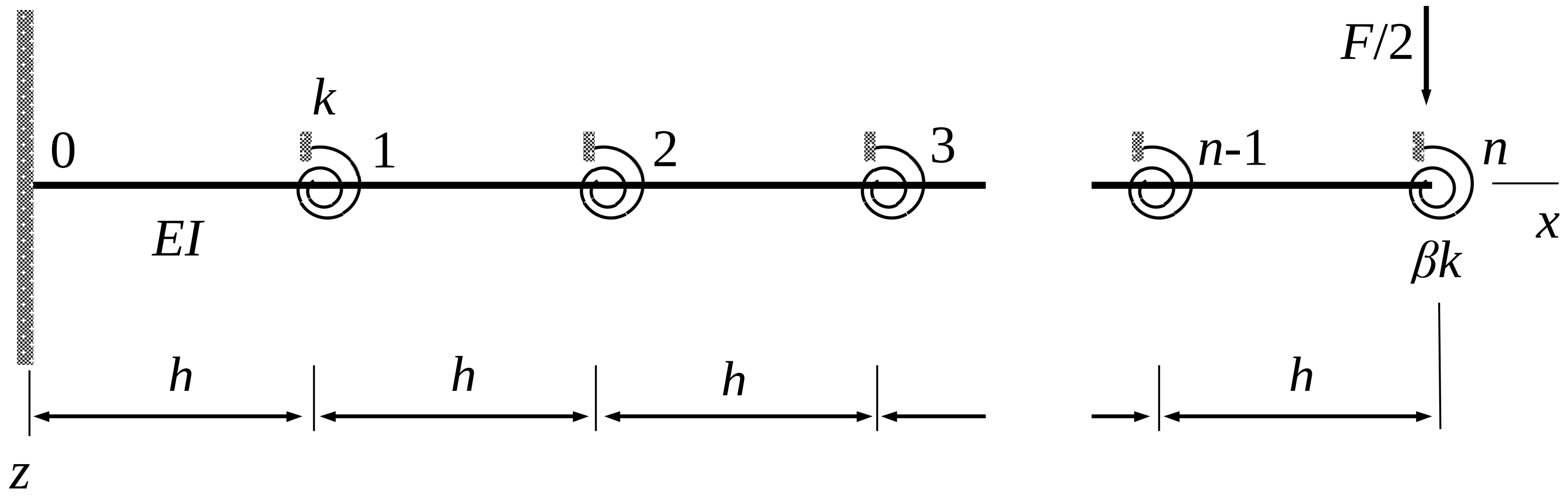
Let us introduce shorthand notations for the dimensionless groups according to

$$\pi_1 = \frac{w}{h}, \quad \pi_2 = \frac{Fh^2}{EI_1}, \quad \pi_3 = \frac{r}{h}, \quad \pi_4 = \frac{I_2}{I_1}$$

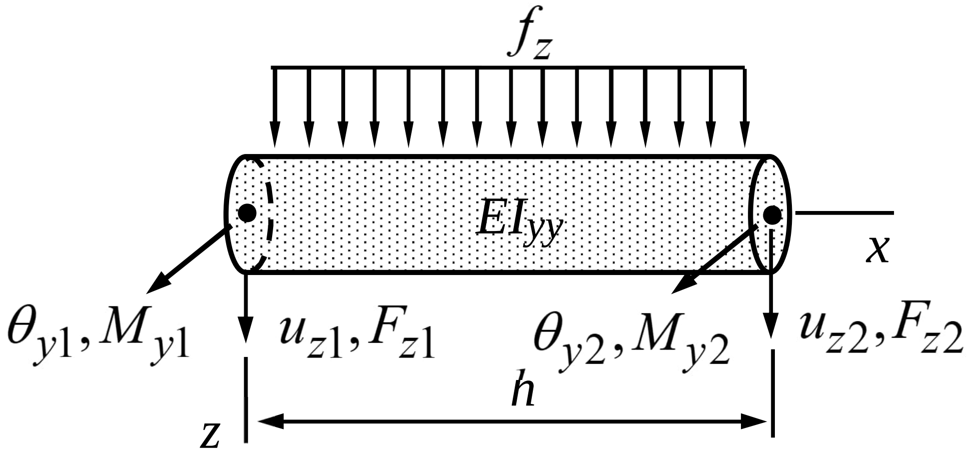
to get $\pi_1 = \pi_2 \alpha(\pi_3, \pi_4, n)$.

SIMPLIFIED ANALYSIS

Simplification assumes that $\pi_3 = r / h = 0$ and $\pi_4 = I_2 / I_1 = 1$. As the end points of the vertical parts translate downwards but not sideways, the vertical parts just increase the bending stiffness of the upper and lower horizontal parts which deform in the same manner. Therefore, one may consider an inextensible Bernoulli beam shown.



BENDING BEAM ELEMENT (MEC-E1050)



$$\begin{Bmatrix} F_{z1} \\ M_{y1} \\ \hdashline F_{z2} \\ M_{y2} \end{Bmatrix} = \frac{EI_{yy}}{h^3} \left[\begin{array}{cc:cc} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ \hdashline -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{array} \right] \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ \hdashline u_{z2} \\ \theta_{y2} \end{Bmatrix} - \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ \hdashline 6 \\ h \end{Bmatrix}$$

the force-displacement relationship of bending beam element!

Notice that the displacements, rotations, forces, and moments are components of vectors of the material coordinate system!

BEAM EEQUATIONS (MEC-E8003)

Timoshenko beam model equilibrium and constitutive equations in component forms

$$\begin{Bmatrix} \dfrac{dN}{dx}+b_x \\ \dfrac{dQ_y}{dx}+b_y \\ \dfrac{dQ_z}{dx}+b_z \end{Bmatrix}=0, \quad \begin{Bmatrix} N \\ Q_y \\ Q_z \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx}-ES_z\dfrac{d\psi}{dx}+ES_y\dfrac{d\theta}{dx} \\ GA(\dfrac{dv}{dx}-\psi)-GS_y\dfrac{d\phi}{dx} \\ GA(\dfrac{dw}{dx}+\theta)+GS_z\dfrac{d\phi}{dx} \end{Bmatrix},$$

$$\begin{Bmatrix} \dfrac{dT}{dx}+c_x \\ \dfrac{dM_y}{dx}-Q_z+c_y \\ \dfrac{dM_z}{dx}+Q_y+c_z \end{Bmatrix}=0, \text{ and } \begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix}=\begin{Bmatrix} -GS_y(\dfrac{dv}{dx}-\psi)+GS_z(\dfrac{dw}{dx}+\theta)+GI_{rr}\dfrac{d\phi}{dx} \\ ES_y\dfrac{du}{dx}-EI_{zy}\dfrac{d\psi}{dx}+EI_{yy}\dfrac{d\theta}{dx} \\ -ES_z\dfrac{du}{dx}+EI_{zz}\dfrac{d\psi}{dx}-EI_{yz}\dfrac{d\theta}{dx} \end{Bmatrix}.$$

BEAM EQUATIONS SIMPLIFIED

As bending moment is not continuous over the torsion springs, solution needs to be found partwise. The equation system for the Bernoulli beam model of the structure (corresponding to $\pi_3 = r/h = 0$ and $\pi_4 = I_2/I_1 = 1$) is given by

$$\frac{dM}{dx} - Q = 0, \quad \frac{dQ}{dx} = 0, \quad M = EI\frac{d\theta}{dx}, \quad \theta = -\frac{dw}{dx} \quad x \in \Omega \setminus I, \tag{2}$$

$$[\![M]\!] = k\theta, \quad [\![Q]\!] = 0, \quad [\![\theta]\!] = 0, \quad [\![w]\!] = 0 \quad x \in I, \tag{3}$$

$$\theta(0) = 0, \quad w(0) = 0, \quad M(L) + k\theta(L) = 0, \quad Q(L) = \frac{1}{2}F, \tag{4}$$

where the bending spring coefficient of the vertical part $k = 6EI/h$ and jump operator $[\![a(x)]\!] = \lim_{\varepsilon \to 0}[a(x+\varepsilon) - a(x-\varepsilon)]$.

Let us start with the equations for a typical part in terms of the quantities at the connection points. The values of the bending moment on the two sides of connection points are denoted by superscripts. Considering part i between points $i-1$ and i

$$M_i^- - M_{i-1}^+ = \frac{1}{2} F h, \;\; M_i^- + M_{i-1}^+ = 2\frac{EI}{h}(\theta_i - \theta_{i-1}),$$

$$w_{i+1} - w_i = -h\theta_i + \frac{h^2}{6}\frac{1}{EI}(2M_i^- + M_{i-1}^+), \;\; M_i^+ - M_i^- = \frac{6EI}{h}\theta_i \quad i \in \{0,1,\ldots,n\},$$

$$w_0 = 0, \quad \theta_0 = 0, \quad M_n^+ = 0.$$

The set of ordinary first order difference equations to the unknown w_i, θ_i, M_i^- and M_i^+ can be solved like a set of ordinary differential equations. Hand calculation is possible, but a bit tedious (one may derive a second order equation for M_i^+ by elimination and solve that first etc.). Use of Mathematica (RSolve) is more straightforward.

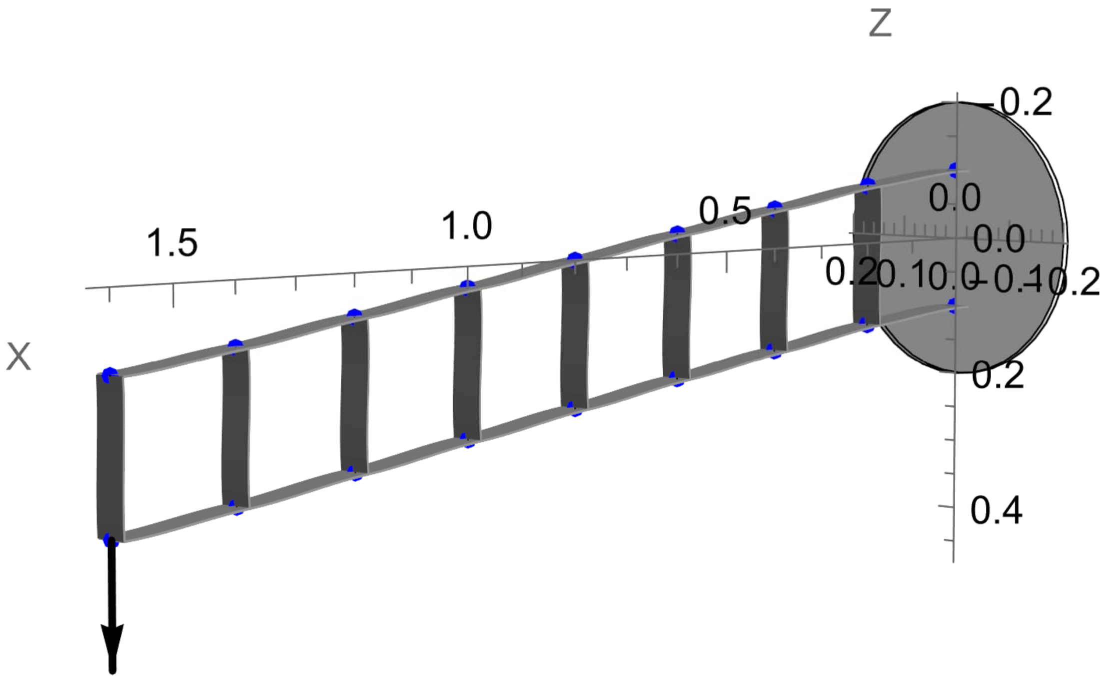
Solving the equations for the end point displacement gives

$$w_n = \frac{Fh^3}{EI}\left[\frac{1}{72}\frac{60-5r_1^n\left(r_1+2\right)-5r_2^n\left(r_2+2\right)}{r_1^n\left(r_1+1\right)+r_2^n\left(r_2+1\right)}+\frac{1}{8}n\right], \quad \text{where} \quad r = 4 \pm \sqrt{15}$$

Expression can be simplified to a form not containing powers of r: s when $n > 2$ (without much error). It may be possible to include the effects of the weld size and stiffer right end spring to the calculations!

FINITE ELEMENT ANALYSIS

Finite element analysis with solid, plate or beam elements give the force-displacement relationship without (too many) simplifying assumptions. However, analysis requires numerical values for all the problem parameters.



Numerical method is just a way to produce displacement-force data for given values of the problem parameters. Studying the effect of *n* means using the method with several selections of *n*. After that, fitting to a polynomial (for example) can be used to get a more compact representation.

Table 2. End point displacement as function of *n*

n	w_n
1	
2	
3	
4	

EXPERIMENT

The set-up is located in Puumiehenkuja 5L (Konemiehentie side of the building). The hall is open for measurements during the office hours (09:00-16:00) from Thu of week 17 to Thu of week 18.

Place a mass on the loading tray and record the displacement shown on the laptop display. Gather enough mass-displacement data to find the displacement-force relationship reliably. For example, you may repeat a measurement with certain loading several times to reduce the effect of random error by averaging etc. Do not overload the structure.