MEC-E1005

MODELLING IN APPLIED MECHANICS 2023

Week 18 PERIODIC STRUCTURE DISPLACEMENT

Thu 12:15-14:00 Analytical and numerical methods (JF)

PERIODIC STRUCTURE

ASSIGNMENT

A structure is considered as periodic if one may identify a part which repeats itself in one or several directions. Regularity in that sense can be used to simplify, e.g., displacement analysis with variable number of parts.

The aim is to find the effect of geometrical and material parameters to the displacement-load relationship of a periodic steel structure of the type shown in the figure. The left end is attached to a wall. The right end is loaded by a point force. The starting point is a generic expression predicted by dimension analysis. First, a simplified model is used for a more specific relationship. After that, analysis by FEM is used to study the effects of details. Finally, the displacement-force relationship is compared with an experimental one.

IDEALIZATION AND PARAMETERIZATION

In the idealization, *n* identical parts are connected by welds of triangular shape. The left end is rigidly connected to a wall and load is acting at the right end.

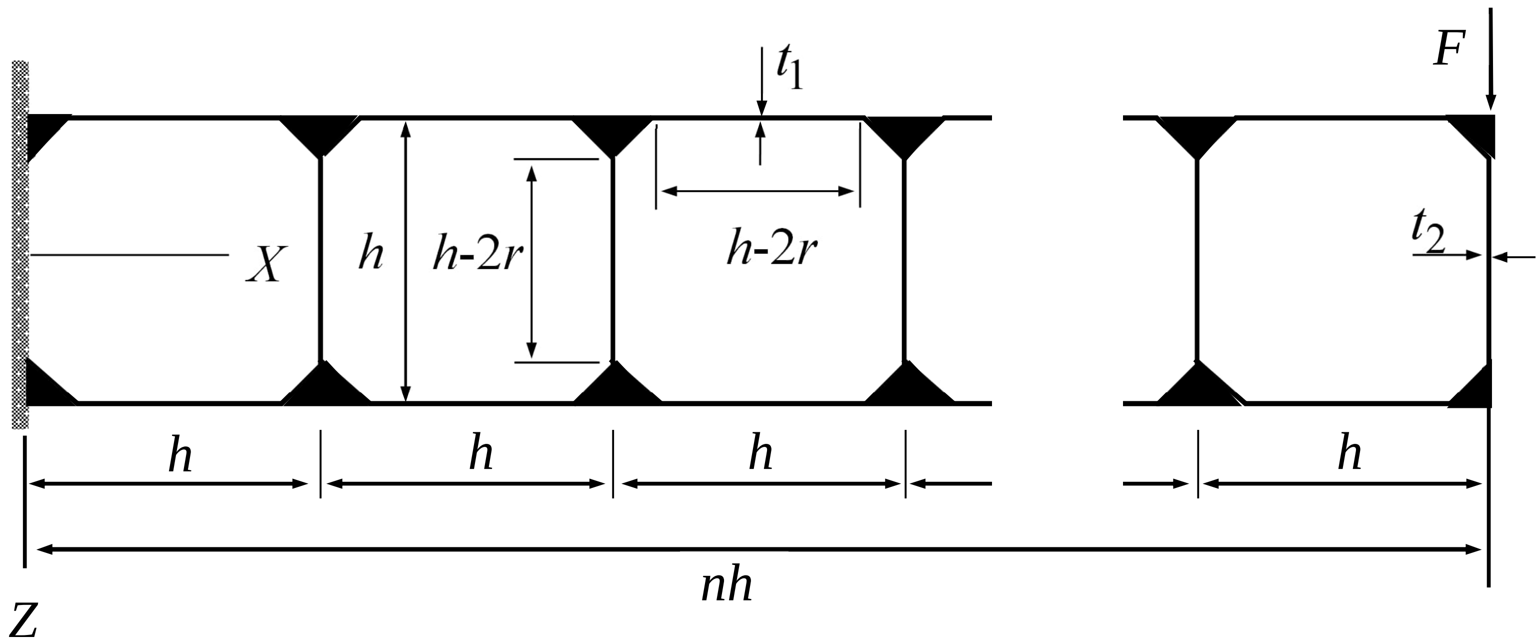


Table 1. Geometrical and material parameters of the structure.

Parameter	symbol	value
Side length	h	0.20 m
Width	b	50 mm
Number of parts	n	5
Weld size	r	4 mm
Thickness 1	t_1	3.1 mm
Thickness 2	t_2	5.2 mm
Young's modulus	E	210 GPa
Acceleration by gravity	g	9.81 m/s^2

DIMENSION ANALYSIS

Assuming (1) linearity between displacement and load, (2) vanishing displacement with zero loading, and (3) that the straight parts behave essentially as Bernoulli beams in bending. dimension analysis on quantities E, I_1, I_2, h, r, F, and w, gives the relationship

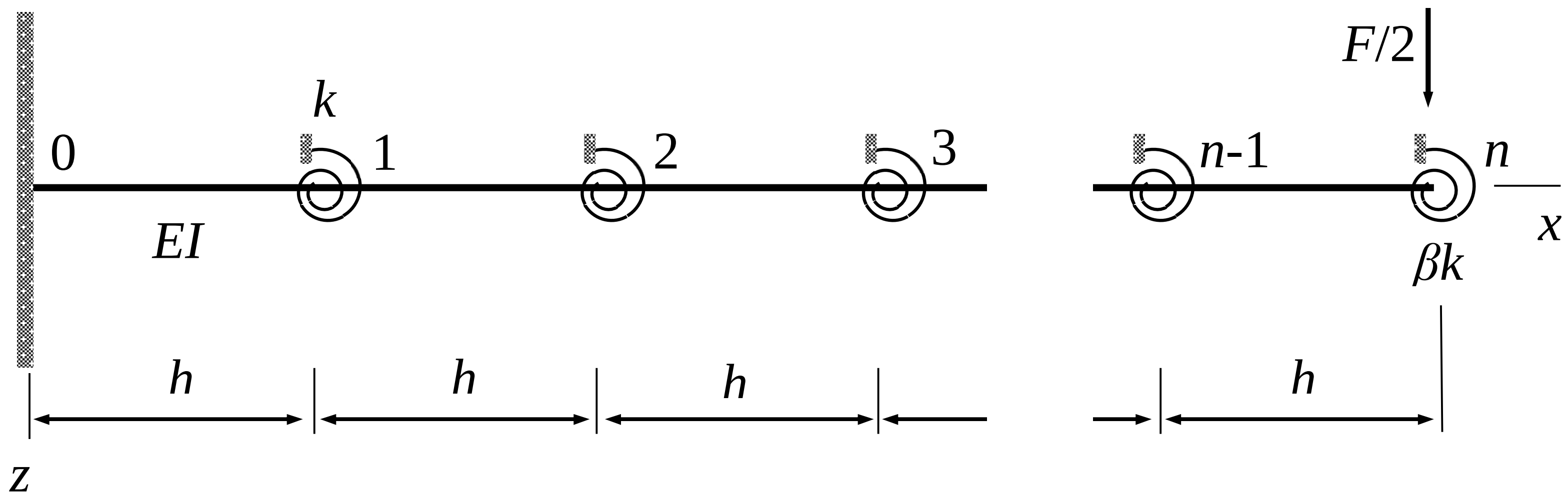
$$\pi_1 = \pi_2 \alpha(\pi_3, \pi_4, n), \text{ where } \pi_1 = \frac{w}{h}, \quad \pi_2 = \frac{Fh^2}{EI_1}, \quad \pi_3 = \frac{r}{h}, \quad \pi_4 = \frac{I_2}{I_1} \tag{1}$$

and $\alpha(\pi_3, \pi_4, n)$ on the right-hand side requires a more detailed analysis. The indices in the second moments refer to the two different thicknesses of the material.

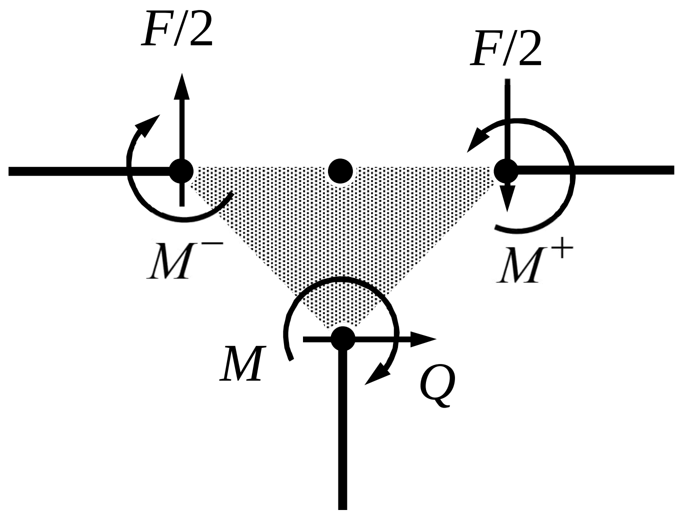
The assumptions and the list of the quantities may need to be reconsidered after experiment and simulation experiments with a model based on less restrictive assumptions.

ANALYTICAL METHOD

Simplification assumes, in addition, that (4) $\pi_4 = I_2 / I_1 = 1$, and (5) welds can be treated as rigid bodies. Then the end points of the vertical parts translate downwards but not sideways, the vertical parts just increase the bending stiffness of the upper and lower horizontal parts which deform in the same manner. Therefore, one may consider a Bernoulli bending beam with torsion springs representing the effect of the vertical parts.



WELD MODEL



The shear forces and bending moments acting on a typical weld of size $r \le h/2$ result into the moment equilibrium

$$M^{+} - M^{-} - Fr + \begin{Bmatrix} r \\ -1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} Q \\ M \end{Bmatrix} = 0.$$

With the assumptions of the simplified analysis, the element contribution of the vertical beam (beam length $d = h - 2r$) gives

$$\begin{Bmatrix} Q \\ M \end{Bmatrix} = \frac{EI}{d^3}\begin{bmatrix} 12 & -6d & -12 & -6d \\ -6d & 4d^2 & 6d & 2d^2 \end{bmatrix}\begin{Bmatrix} -r\theta \\ \theta \\ r\theta \\ \theta \end{Bmatrix} = \frac{EI}{d^3}\begin{Bmatrix} -12d - 24r \\ 6d^2 + 12dr \end{Bmatrix}\theta.$$

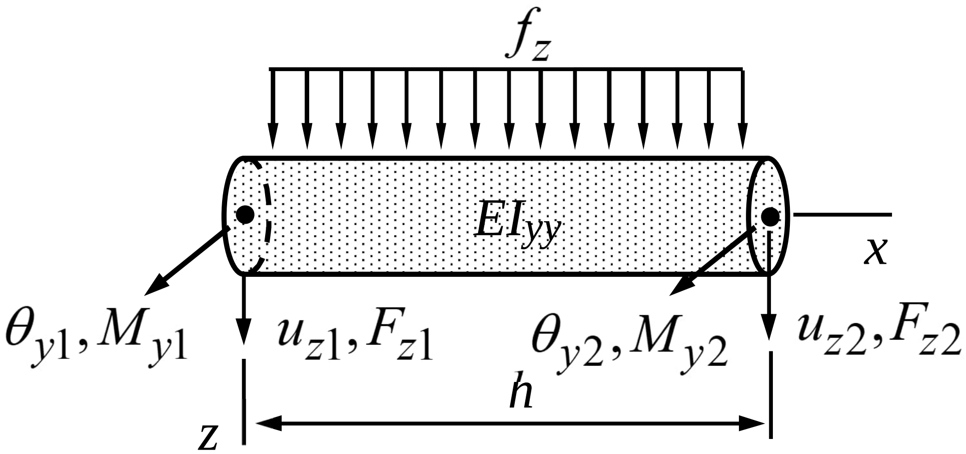
The expression above and the moment equilibrium, gives the jump condition for a non-zero weld size:

$$M^+ - M^- - 6\frac{EI}{d}\varepsilon\theta - Fr = 0 \quad \text{where} \quad \varepsilon = (\frac{h}{d})^2.$$

Jump condition of displacement for a non-zero weld size becomes

$$w^+ - w^- = -2r\theta.$$

BENDING BEAM ELEMENT (MEC-E1050)



$$\begin{Bmatrix} F_{z1} \\ M_{y1} \\ \hdashline F_{z2} \\ M_{y2} \end{Bmatrix} = \frac{EI_{yy}}{h^3} \left[\begin{array}{cc:cc} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ \hdashline -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{array} \right] \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ \hdashline u_{z2} \\ \theta_{y2} \end{Bmatrix} - \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ \hdashline 6 \\ h \end{Bmatrix}$$

the force-displacement relationship of bending beam element!

Notice that the displacements, rotations, forces, and moments are components of vectors of the material coordinate system!

BEAM EQUATIONS (MEC-E8003)

Timoshenko beam model equilibrium and constitutive equations in component forms

$$\begin{Bmatrix} \dfrac{dN}{dx}+b_x \\ \dfrac{dQ_y}{dx}+b_y \\ \dfrac{dQ_z}{dx}+b_z \end{Bmatrix}=0, \quad \begin{Bmatrix} N \\ Q_y \\ Q_z \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx}-ES_z\dfrac{d\psi}{dx}+ES_y\dfrac{d\theta}{dx} \\ GA(\dfrac{dv}{dx}-\psi)-GS_y\dfrac{d\phi}{dx} \\ GA(\dfrac{dw}{dx}+\theta)+GS_z\dfrac{d\phi}{dx} \end{Bmatrix},$$

$$\begin{Bmatrix} \dfrac{dT}{dx}+c_x \\ \dfrac{dM_y}{dx}-Q_z+c_y \\ \dfrac{dM_z}{dx}+Q_y+c_z \end{Bmatrix}=0, \text{ and } \begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix}=\begin{Bmatrix} -GS_y(\dfrac{dv}{dx}-\psi)+GS_z(\dfrac{dw}{dx}+\theta)+GI_{rr}\dfrac{d\phi}{dx} \\ ES_y\dfrac{du}{dx}-EI_{zy}\dfrac{d\psi}{dx}+EI_{yy}\dfrac{d\theta}{dx} \\ -ES_z\dfrac{du}{dx}+EI_{zz}\dfrac{d\psi}{dx}-EI_{yz}\dfrac{d\theta}{dx} \end{Bmatrix}.$$

BEAM MODEL

As bending moment and displacement are not continuous over the welds, solution needs to be found partwise. The equation system for the Bernoulli beam model of the structure (corresponding to $\pi_3 = r / h$ and $\pi_4 = I_2 / I_1 = 1$) is given by

$$\frac{dM}{dx} - Q = 0, \quad \frac{dQ}{dx} = 0, \quad M = EI\frac{d\theta}{dx}, \quad \theta = -\frac{dw}{dx} \tag{2}$$

$$M_i^+ - M_i^- = k\theta_i + Fr, \quad w_i^+ - w_i^- = -2\theta_i r \quad i \in \{1, 2, \ldots, n-1\}, \tag{3}$$

$$\theta_0 = 0, \quad w_0^+ = 0, \quad -M_n^- = k\theta_n + \frac{1}{2}Fr, \quad Q_n = \frac{1}{2}F, \tag{4}$$

where the bending spring coefficient of the vertical part $k = 6\varepsilon EI / d$ and the values of the bending moment and displacement on the two sides of the welds are denoted by superscripts.

The differential equations can be used to find the behavior of all the horizontal parts between the welds. Let us start with the equations for a typical part in terms of the quantities at the welds. Shear force is constant $Q = F/2$.

Considering part i between points $i-1$ and i of length $x_i - x_{i-1} = h - 2r = d$ and placing the origin (temporarily) to point $i-1$

$$\int_0^d \frac{dM}{dx} dx = \int_0^d Qdx \quad \Rightarrow \quad M_i^- - M_{i-1}^+ = \frac{1}{2} Fd,$$

$$\int_0^d Mdx = \int_0^d EI \frac{d\theta}{dx} dx \quad \Rightarrow \quad d \frac{M_i^- + M_{i-1}^+}{2} = EI(\theta_i - \theta_{i-1}),$$

$$-\int_0^d \frac{dw}{dx} dx = \int_0^d \theta dx = (x\theta)_0^d - \int_0^d x \frac{d\theta}{dx} dx = (x\theta)_0^d - \frac{1}{EI} \int_0^d xM dx \quad \Rightarrow$$

$$w_i^- - w_{i-1}^+ = -d\theta_i + \frac{d^2}{6EI}(M_{i-1}^+ + 2M_i^-)$$

The difference equations above hold for all the structural parts so $i \in \{1, \ldots, n\}$.

The set of ordinary first order difference equations to the unknown w_i^-, w_i^+, θ_i, M_i^- and M_i^+ can be solved like a set of ordinary differential equations. Mathematica (RSolve) is handy, but hand calculation is also possible.

HAND CALCULATION (OUTLINE)

One may combine the difference equations to get one second order difference equation for the bending moment

$$M_{i+1}^{+} - 2(1+3\varepsilon)M_i^{+} + M_{i-1}^{+} = \frac{3\varepsilon}{2}Fd \quad i \in \{1, 2\ldots, n-1\}$$

the generic solution to which is given by (generic solution to the homogeneous equation and a particular solution)

$$M_i^{+} = Ar_1^i + Br_2^i - \frac{1}{4}Fd \quad \text{where} \quad r = (1+3\varepsilon) \pm \sqrt{(1+3\varepsilon)^2 - 1}.$$

The values of *A* and *B* can be obtained from boundary conditions, but the generic part is not needed as the particular solution dominates (the solution is linear in *i* except possibly close to the boundaries). Omitting the generic solution, the moment equilibrium equation gives

$$M_i^- = M_{i-1}^+ + \frac{1}{2}Fd = \frac{1}{4}Fd \,.$$

Using the jump conditions for the rotation angle

$$\theta_i = \frac{d}{6\varepsilon EI}(M_i^+ - M_i^- - Fr) = -\frac{Fdh}{12\varepsilon EI}\,.$$

Finally, the difference equation for the transverse displacement and the displacement jump conditions can be combined to get first order difference equation to transverse displacements at the nodes $w_i = (w_i^- + w_i^+)/2$

$$w_i - w_{i-1} = -(r+d)\theta_i - r\theta_{i-1} + \frac{d^2}{6EI}(3M_{i-1}^+ + Fd) = \frac{1}{8}\frac{Fd^3}{EI}$$

Solution to the equation (generic solution to the homogeneous equations and a particular solution)

$$w_i = C + \frac{Fh^3}{EI}\frac{1}{8}(\frac{d}{h})^3 i.$$

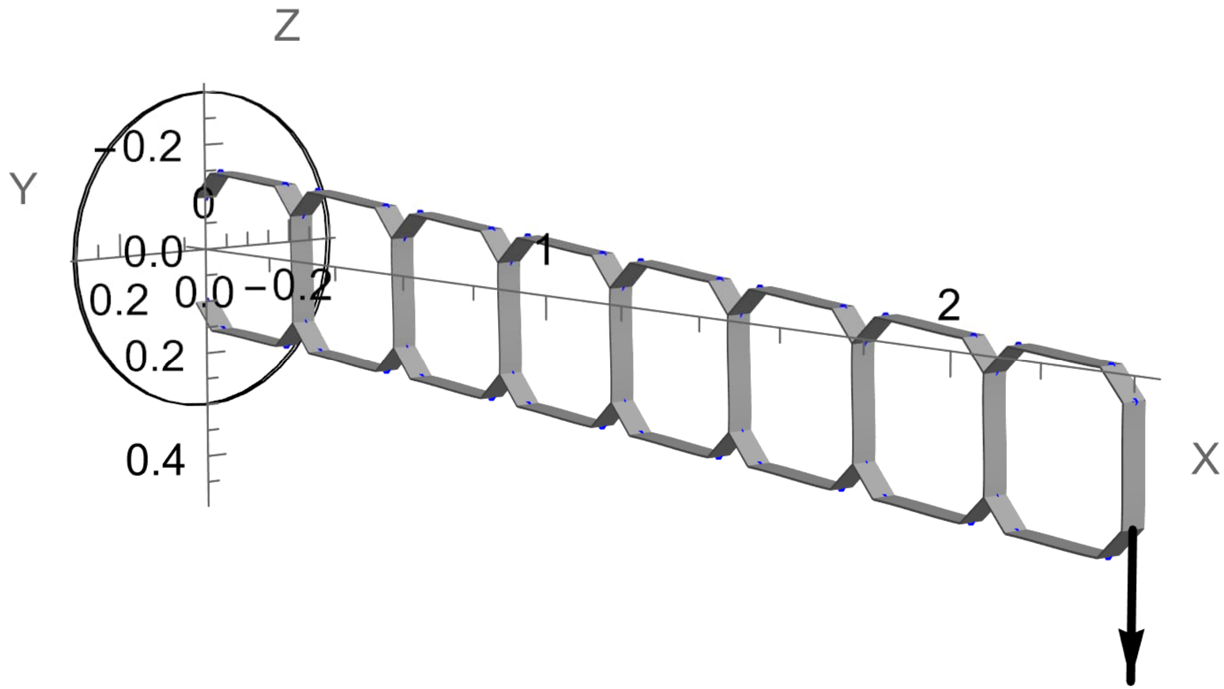
As displacement vanishes when $i = 0$, integration constant $C = 0$. Comparing with the expression by dimension analysis with

$$\alpha(\pi_3, 1, n) = \frac{1}{8}(1 - 2\pi_3)^3 n.$$

As the generic solution to the homogeneous equations to the bending moment is omitted, solutions may not satisfy all the boundary conditions.

FINITE ELEMENT ANALYSIS

Finite element analysis with solid, plate or beam elements give the force-displacement relationship without (too many) simplifying assumptions. However, analysis requires numerical values for all the problem parameters.



Numerical method is just a way to produce displacement-force data for given values of the problem parameters. For example, studying the effect of *n* means using the method with several selections of *n*. Table 2 shows the outcome of a FEM simulation with axially inextensible beams.

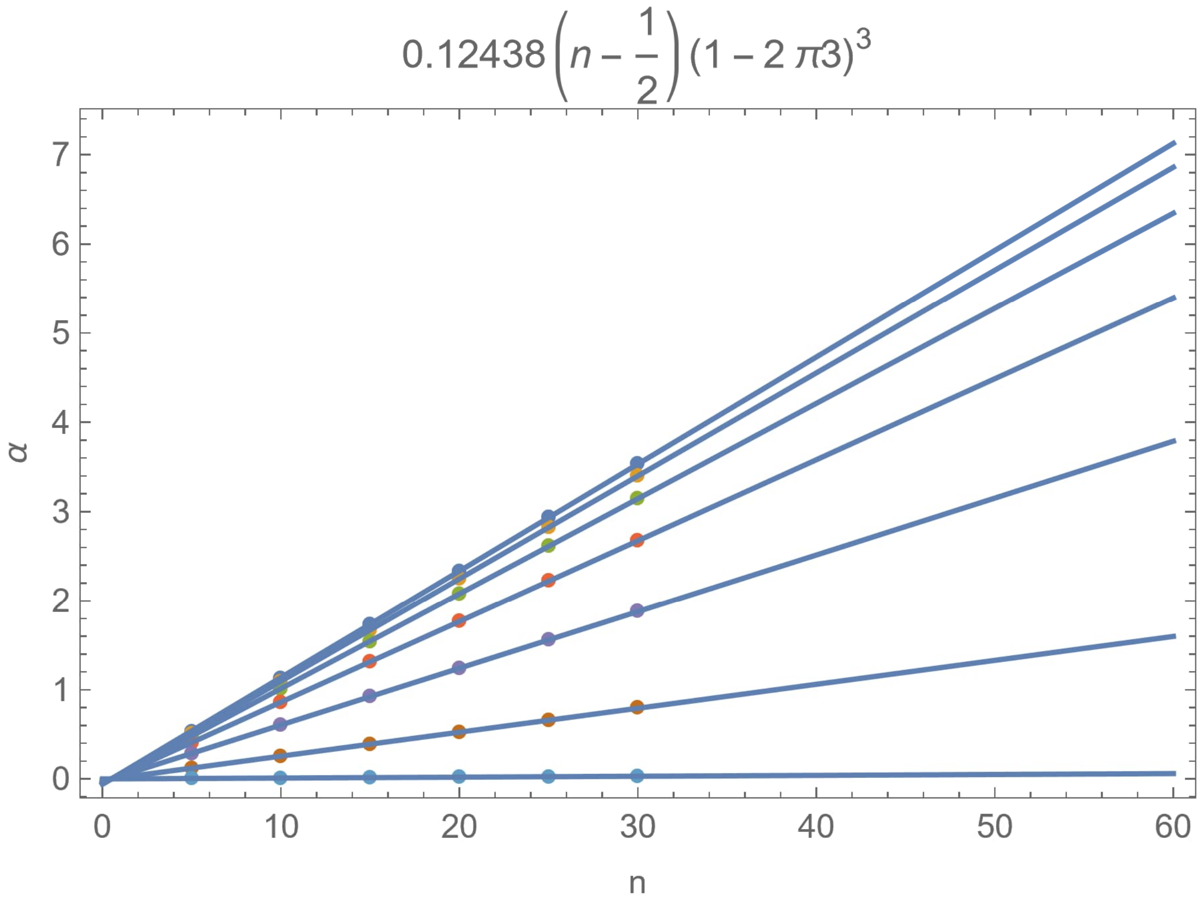
Table 2. Expression $\alpha = \pi_1 / \pi_2$ as function of π_3 (rows) and n (columns).

	5	10	15	20	25	30
0.00625	0.527756	1.12962	1.73149	2.33339	2.93532	3.53729
0.0125	0.508261	1.08755	1.66686	2.2462	2.82556	3.40495
0.025	0.470695	1.00656	1.54244	2.07835	2.61428	3.15024
0.05	0.401112	0.856744	1.31239	1.76806	2.22375	2.67947
0.1	0.282908	0.602915	0.922934	1.24297	1.56303	1.88312
0.2	0.120237	0.255242	0.390256	0.525284	0.660329	0.795396
0.4	0.00449473	0.00949646	0.0145011	0.0195103	0.0245253	0.0295478

The data and $\pi_1 = \pi_2 \alpha(\pi_3, 1, n)$, where $\pi_1 = w / h$, $\pi_2 = Fh^2 / EI$, $\pi_3 = r / h$ give the displacement-force relationship for any selection of the problem parameter (assuming that π_3 and n are in the range of the tabulated data).

DATA COMPRESSION

After that, a polynomial fit (for example) can be used to get a more compact representation which may be used (with caution) also outside the range of the tabulated data:



EXPERIMENT

The set-up is located in Puumiehenkuja 5L (Konemiehentie side of the building). The hall is open for measurements during the office hours (09:00-16:00) from Thu of week 17 to Thu of week 18.

Place a mass on the loading tray and record the displacement shown on the laptop display. Gather enough mass-displacement data to find the displacement-force relationship reliably. For example, you may repeat a measurement with certain loading several times to reduce the effect of random error by averaging etc. Do not overload the structure.

DESIGN FORMULA ($I_2 = I_1 = I$ **)**

$$w = \frac{mgh^3}{EI}\frac{1}{8}(1-2\frac{r}{h})^3(n-\frac{1}{2})$$

Table 3. Comparison of displacements

m [kg]	w_{exp}[mm]	w_{des}[mm]	w_{fem}[mm]
1	1.55	1.65	1.47
2	3.09	3.30	2.93
3	4.67	4.95	4.40
4	6.26	6.60	5.87

DISCUSSION

Effects of π_3 and n without the assumptions that the beams are inecxtensible in the axial direction.