2

Semialgebraic sets

Algebraic geometry studies geometric objects algebraically. In the language available to us to talk about sets defined algebraically over general fields (using addition, multiplication and equality), we can only express that a polynomial should be zero. This leads to varieties as the basic building blocks of algebro-geometric spaces. But over an ordered field, an additional relation expressing non-negativity is available. Instead of studying polynomial equations, we have access to inequalities, too. Real algebraic geometry is concerned with the geometry of *semialgebraic sets* built up from sets defined by equations and inequalities.

2.1 Basic semialgebraic sets

Definition 2.1. Let $(\mathbb{F}, \leq)$ be an ordered field. A set $K \subseteq \mathbb{F}^n$ defined by $K = \{ x \in \mathbb{F}^n : f_i(x) \bowtie_i 0 \text{ for all } i \}$ with finitely many polynomials $f_i \in \mathbb{F}[x_1, \ldots, x_n]$ is:

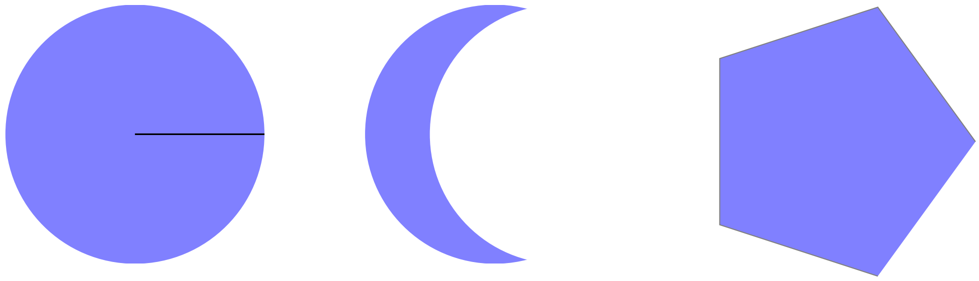
- *basic closed* if all $\bowtie_i$ are $\geq$.
- *basic open* if all $\bowtie_i$ are $>$.
- *primary* if all $\bowtie_i$ are either $>$ or $=$.
- an *algebraic variety* if all $\bowtie_i$ are $=$.

A subset of $\mathbb{F}^n$ is *semialgebraic* if it can be written as a finite boolean combination of basic (open or closed) semialgebraic sets.

By a “boolean combination” above we mean a finite expression involving basic semialgebraic sets as its operands and set union, intersection and complement as its operations. It immediately follows that boolean combinations of semialgebraic sets are semialgebraic.

Algebraic varieties are a special case of basic closed semialgebraic sets, where each polynomial inequality $f_i \geq 0$ which appears in the definition is accompanied by its negative $f_i \leq 0$. If the ambient space is clear from context or irrelevant, we shall abbreviate $\{ x \in \mathbb{F}^n : f_i(x) \bowtie_i 0 \text{ for all } i \}$ to $\{ f_i \bowtie_i 0 \}$.

Remark 2.2. Note that a basic closed set $\{ f_i \geq 0 \} = \bigcap_i \{ f_i \geq 0 \}$ can be written as $\bigcap_i \{ -f_i > 0 \}^{\mathsf{c}}$ which is a boolean combination of basic open sets. In a similar way, basic open sets can be expressed as combinations of basic closed ones, so that in the definition of semialgebraic set one can use basic open and basic closed sets together or restrict use to only one of the two types of basic set.



(a) Slit disc (b) Crescent moon (c) Pentagon

Figure 2.1: Some semialgebraic sets in the plane $\mathbb{R}^2$.

Example 2.3. (1) The open unit ball $B_1^n(0) := \{\, \|x\| < 1 \,\}$ in $\mathbb{R}^n$ is semialgebraic because it can be equivalently defined by the polynomial condition $\|x\|^2 < 1$. This set is basic open. The closed unit ball $\overline{B_1^n(0)} := \{\, \|x\| \le 1 \,\}$ is basic closed and the sphere $S^{n-1} := \{\, \|x\| = 1 \,\}$ is a variety.

(2) The "slit disc" $B_1^2(0,0) \setminus \{\, y = 0,\ 0 \le x < 1 \,\}$ is semialgebraic because it is the difference of two semialgebraic sets. As is the "crescent moon" $B_1^2(0,0) \setminus \overline{B_1^2(1/2, 0)}$. See Figure 2.1.

(3) A *polyhedron* is a subset of $\mathbb{R}^n$ defined by linear equations and inequalities, like the pentagon in Figure 2.1. Polyhedra are a simple kind of semialgebraic set (in particular they are convex) over which linear functions can be efficiently optimized. This is done in an imporant branch of optimization called *linear programming*. △

Example 2.4: Semialgebraic sets in the sciences. (1) **Chemistry:** Consider a chemical reaction $a\,\mathrm{X} + b\,\mathrm{Y} \xrightarrow{\kappa_1} c\,\mathrm{Z} + \mathrm{W}$ in which two substances X and Y in quantities a and b, respectively, yield c units of another substance Z and a unit of some by-product W. The substance Z also decays over time to Y according to $\mathrm{Z} \xrightarrow{\kappa_2} d\,\mathrm{Y}$.

In the *mass action kinetics model*, these reactions happen continuously and proportionally to the concentrations of the chemical species and the *rate constants* κ_1 and κ_2. The evolution in time of the concentrations of the chemical species is controlled by the following system of ordinary differential equations:

$$\dot{x} = -a\kappa_1 x^a y^b, \quad \dot{y} = -b\kappa_1 x^a y^b + d\kappa_2 z,$$
$$\dot{z} = c\kappa_1 x^a y^b - \kappa_2 z, \quad \dot{w} = \kappa_1 x^a y^b.$$

With fixed integer parameters a, b, c, d and rate constants $\kappa_1, \kappa_2 > 0$, the *steady states* of this dynamical system are described by a semialgebraic set:

$$x, y, z, w \ge 0,$$
$$a\kappa_1 x^a y^b = 0, \quad b\kappa_1 x^a y^b = d\kappa_2 z,$$
$$c\kappa_1 x^a y^b = \kappa_2 z, \quad \kappa_1 x^a y^b = 0.$$

(2) **Data science:** A positive definite 3×3 matrix Σ with entries in $\mathbb{R}$ is the covariance matrix of a vector (X, Y, Z) of normally distributed random variables. This matrix encodes the dependence of the random variables algebraically: X and Y are

stochastically independent, $X \perp\!\!\!\perp Y$, if the entry $\Sigma_{X,Y}$ vanishes. More generally, X and Y are *conditionally independent* given Z, written as $X \perp\!\!\!\perp Y \mid Z$, if the subdeterminant $|\Sigma_{XZ,YZ}|$ with rows indexed by X and Z and columns indexed by Y and Z vanishes. Conditional independence means that if the factor Z is controlled, then revealing X gives no information about Y and vice versa.

Consider the statistical model of normally distributed (X, Y, Z) such that both independences are true. This is the semialgebraic set of matrices

$$\begin{pmatrix} p & a & b \\ a & q & c \\ b & c & r \end{pmatrix} \text{ subject to:}$$

$$\left.\begin{aligned} p, q, r &> 0 \\ pq > a^2, pr > b^2, qr &> c^2 \\ pqr + 2abc &> ra^2 + qb^2 + pc^2 \end{aligned}\right\} \text{positive definiteness}$$

$$\left.\begin{aligned} a &= 0 \\ ar &= bc \end{aligned}\right\} \text{independence assumptions}$$

Given this description, it is not hard to see that the polynomial $b \cdot c$ must vanish on this set, which shows the following rule in data science: in a normally distributed random vector, if $X \perp\!\!\!\perp Y$ and $X \perp\!\!\!\perp Y \mid Z$, then Z cannot be dependent on both X and Y.

(3) **Optimization:** Consider the *cardioid curve* in the plane given by the equation $(x^2 + y^2)^2 + 4(x^2 + y^2) - 4y^2$. Given a point $(a, b) \in \mathbb{R}^2$ we are interested in finding the closest point to it on this curve. This leads to an optimization problem

$$\inf \left\{ (x - a)^2 + (y - b)^2 : (x^2 + y^2)^2 + 4(x^2 + y^2) - 4y^2 = 0 \right\}.$$

The Karush–Kuhn–Tucker conditions mandate that the closest point (x_*, y_*) satisfies not only the equation of the cardioid but also

$$\nabla[(x_* - a)^2 + (y_* - b)^2] = \lambda \nabla[(x_*^2 + y_*^2)^2 + 4(x_*^2 + y_*^2) - 4y_*^2]$$

for a parameter $\lambda \in \mathbb{R}$. This can be expressed as the vanishing of the determinant

$$\begin{aligned} &\left| \begin{pmatrix} \nabla[(x_* - a)^2 + (y_* - b)^2] \\ \nabla[(x_*^2 + y_*^2)^2 + 4(x_*^2 + y_*^2) - 4y_*^2] \end{pmatrix} \right| \\ &= 24bx_*^2 + 8bx_*^3 + 16ay_* - 16x_*y_* - 16ax_*y_* - 8x_*^2y_* - \\ &\quad 8ax_*^2y_* + 8by_*^2 + 8bx_*y_*^2 - 8y_*^3 - 8ay_*^3. \end{aligned}$$

The resulting polynomial system with two equations in two variables has finitely many solutions for the point $(a, b) = (-5, 2)$. Numerical algebraic geometry software like `HomotopyContinuation.jl` [BT18] quickly finds approximations to these solutions.

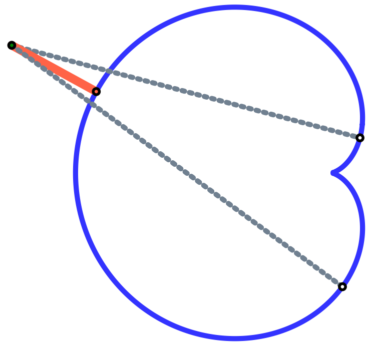


Figure 2.2: The cardioid, a data point and the critical points of the euclidean distance minimization problem. The closest point is marked in red.

```
using HomotopyContinuation
@var x y
a, b = (-5,2)
F = System([(x^2 + y^2)^2 + 4*x*(x^2 + y^2) - 4*y^2,
    24*b*x^2 + 8*b*x^3 + 16*a*y - 16*x*y - 16*a*x*y - 8*x^2*y -
      8*a*x^2*y + 8*b*y^2 + 8*b*x*y^2 - 8*y^3 - 8*a*y^3])
res = solve(F) # (0.4598, 0.5496),   (−3.6747, 1.2746),   (0.1849, −1.7842)
```

In this case there are three non-singular solutions all of which are real. These are only the critical points of the optimization problem, which include the local minima and maxima. However, it is now easy to find the global optimum by computing the distance of (a, b) to each of these finitely many points. The closest one is the second point displayed above with distance 1.51. This situation is pictured in Figure 2.2.

This type of problem is well-studied. It is known that the number of complex solutions to this problem is an invariant of the curve called its *ED degree* and it is attained for generic data (a, b); cf. [DHO+16]. $\triangle$

Lemma 2.5. Every semialgebraic set can be written as a finite union of primary semialgebraic sets.

Proof. We induct on the length of a boolean formula describing the semialgebraic set. Every basic open set is primary. An arbitrary basic closed set is the intersection of sets of the form $\{\, f \geq 0 \,\}$ for a single polynomial f. This set equals $\{\, f > 0 \,\} \cup \{\, f = 0 \,\}$ which is a union of primary sets. The intersection of a union of primary sets can be rewritten using the distributive law into a union of intersections of primary sets. Since primary sets are closed under intersection, basic closed sets satisfy the statement of the lemma.

Now suppose that K and L are finite unions of primary sets, $K = \bigcup_p K_p$ and $L = \bigcup_q L_q$. Their union $K \cup L$ is obviously also of this form. Their intersection equals $K \cap L = \bigcup_{p,q}(K_p \cap L_q)$ where $K_p \cap L_q$ is primary because the primary sets are closed under intersection. Since $\left(\bigcup_p K_p\right)^{\mathsf{c}} = \bigcap_p K_p^{\mathsf{c}}$ and by the previous part of the proof, it remains to show that the complement of a primary set can be expressed as a finite union of primary sets. Let $K = \{ f_i = 0,\ g_j > 0 \}$ be primary. Then

$$\begin{aligned} K^{\mathsf{c}} &= \left(\bigcap_i \{ f_i = 0 \} \cap \bigcap_j \{ g_j > 0 \} \right)^{\mathsf{c}} \\ &= \bigcup_i \{ f_i \neq 0 \} \cup \bigcup_j \{ g_j \leq 0 \} \\ &= \bigcup_i (\{ f_i > 0 \} \cup \{ -f_i > 0 \}) \cup \bigcup_j (\{ -g_j > 0 \} \cup \{ g_j = 0 \}) \end{aligned}$$

is a union of primary sets. □

Remark 2.6. Analogously to the conjunctive normal form of a boolean formula in propositional logic, every semialgebraic set can be written as an intersection of unions of basic closed (or basic open) sets and their complements.

The order topology on an ordered field is familiar from Exercise 1.6. Its definition is canonically extended to $\mathbb{F}^n$: it is the smallest topology whose open sets include the open regions $\{ x : a < x \}$ for $a \in \mathbb{F}^n$ and coordinate-wise comparison. For $\mathbb{F} = \mathbb{R}$ this leads to the usual euclidean topology on $\mathbb{R}^n$. Clearly, polynomials are continuous in the order topology and primary semialgebraic sets are locally closed.

2.2 Semialgebraic functions

Definition 2.7. Let $A \subseteq \mathbb{F}^n$ and $B \subseteq \mathbb{F}^m$ be semialgebraic. A map $f : A \to B$ is *semialgebraic* if its graph $\Gamma(f) := \{ (x, y) \in A \times B : f(x) = y \} \subseteq \mathbb{F}^{n+m}$ is a semialgebraic set. If f is also bijective then it is a *semialgebraic isomorphism*.

Note that if $f : A \to B$ is a semialgebraic isomorphism, then indeed f^{-1} must be a semialgebraic isomorphism as well. Hence, we get a well-defined notion of equivalence for semialgebraic sets.

Example 2.8. (1) Polynomial maps between semialgebraic sets are semialgebraic. If $y = f(x) = {}^{p(x)}/_{q(x)}$ is a rational map, then it is also semialgebraic since its graph is described by $\{ yq(x) = p(x),\ q(x) \neq 0 \}$.

(2) The square root map $\mathbb{R}_+ \to \mathbb{R}_+$ given by $x \mapsto \sqrt{x}$ is semialgebraic because its graph inside $\mathbb{R}_+ \times \mathbb{R}_+$ is $\{ y^2 = x \}$. The negative square root map $\mathbb{R}_+ \to \mathbb{R}_-$, $x \mapsto -\sqrt{x}$ is semialgebraic for the same reason.

(3) The set $\{ 0 < x < {}^1/_2,\ y = 0 \} \cup \{ {}^1/_2 \leq x < 1,\ y = 1 \}$ describes a semialgebraic function between $(0, 1)$ and $\{ 0, 1 \}$ which is not continuous. △

Much more can be said about the geometry of semialgebraic sets and many more functions appearing in the sciences, mathematics and data analysis are semialgebraic. However, to recognize this we need more powerful techniques from real algebra, in particular the Tarski–Seidenberg theorem which states that projections of semialgebraic sets remain semialgebraic. Using this result we see, for example, that the distance

$$\operatorname{dist}(a, B) := \inf \{ \|a - b\| : b \in B \}$$

from a point $a \in \mathbb{R}^n$ to a semialgebraic set $B \subseteq \mathbb{R}^n$ is a semialgebraic function of a. Indeed, $\operatorname{dist}(a, B)$ is the unique real number d such that

$$\big(\forall b \in B : d^2 \leq \|a - b\|^2\big) \ \wedge \ \big(\forall \varepsilon > 0\, \exists b \in B : d^2 \geq \|a - b\|^2 - \varepsilon\big).$$

The existence of such a *quantified boolean formula* is enough to conclude semialgebraicity of the function. The key to these results lies in model theory, a branch of mathematical logic. In the next chapter we prove that the arithmetic theory of a real-closed fields admits quantifier elimination. This technical result will allow us to prove an array of important results and also open the doors to computational real algebraic geometry.

2.3 Exercises

Choose exercises to solve from the list below. Complete the reading part of Section 2.3 before the lecture on May 8. The target value this week is 15 points. By solving more exercises, you can get up to 20 points. Solutions must be submitted on MyCourses by **Thursday, May 11, 12:00**.

2.1 (1) Read Appendix B in preparation for next week's lectures. (2) Fix a polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$. Construct a formula φ in the first-order language of ordered rings which describes the smallest root of f, i.e., φ should have one free variable a such that over any ordered field $\mathbb{F}$ and element $a \in \mathbb{F}$ we have $\mathbb{F} \models \varphi[\mathsf{a}/a]$ if and only if a is the smallest roof of f in $\mathbb{F}$. $\boxed{\text{6 points}}$

2.2 Show that every semialgebraic in $\mathbb{R}$ is a finite union of points and open intervals and conclude that the following sets are not semialgebraic over $\mathbb{R}$: (1) $\mathbb{Z} \subseteq \mathbb{R}$, (2) $\bigcup_{n\in\mathbb{N}} \{ y = nx \} \subseteq \mathbb{R}^2$. Furthermore, show that $\Gamma(\exp) \subseteq \mathbb{R}^2$ is not primary. *(Hint: No polynomial equations hold on $\Gamma(\exp)$ because x and $\exp x$ are algebraically independent functions over $\mathbb{R}$. So, why is $\Gamma(\exp)$ not basic open?)* $\boxed{\text{6 points}}$

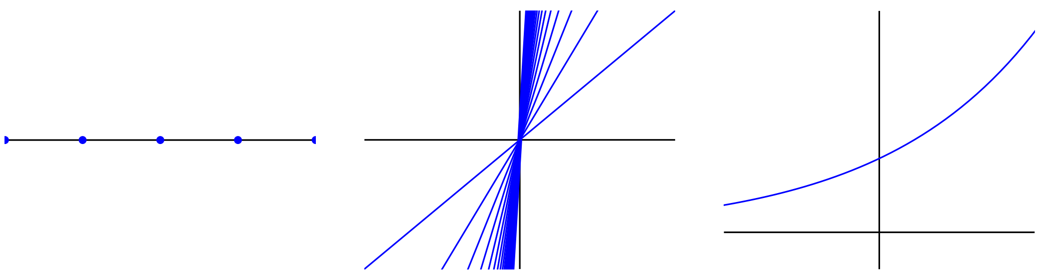


Figure 2.3: The three sets from Exercise 2.2.

2.3 Prove Remark 2.6: every semialgebraic set can be written in the form $\bigcap_i \bigcup_j K_{ij}$ where K_{ij} is basic open or the complement of a basic open set. $\boxed{\text{3 points}}$

2.4 A polynomial $f \in \mathbb{R}[x, y]$ defines a curve $C = \{ f = 0 \}$. A *bottleneck* on C consists of two points $p, q \in C$ such that $p - q$ is orthogonal to the tangent at p and at q. Derive the four equations in four parameters p_x, p_y, q_x, q_y which define the bottlenecks of C. Compute the bottlenecks of the curve $f = (x^3 + xy - 4)^2(x^2 + y^2 - 1) + y^2 - 10$ and find the narrowest one. $\boxed{\text{6 points}}$

2.5 Prove that the set of all triples $(a, b, c) \in \mathbb{R}^3$ for which there exists a solution $x \in \mathbb{R}$ to $ax^2 + bx + c = 0$ is semialgebraic in $\mathbb{R}$. What if $\mathbb{R}$ is replaced everywhere with $\mathbb{Q}$? $\boxed{\text{3 points}}$

2.6 Let $K \subseteq \mathbb{R}^n$ be semialgebraic. Its *vanishing ideal* $\mathcal{I}(K)$ consists, as usual, of all polynomials $f \in \mathbb{R}[x_1, \ldots, x_n]$ such that $f(x) = 0$ for all $x \in K$. Give an example of a real variety for which Hilbert's Nullstellensatz (Theorem A.5) fails. $\boxed{\text{2 points}}$

2.7 Give an example of two semialgebraic sets which are semialgebraically isomorphic but not homeomorphic. $\boxed{\text{3 points}}$

2.8 A polynomial $f \in \mathbb{F}[x, y]$ over an ordered field partitions the plane $\mathbb{F}^2$ into three primary sets: $\{ f > 0 \}$, $\{ f < 0 \}$ and the variety $\{ f = 0 \}$. Plot the following configuration of a conic and a cubic in $\mathbb{R}^2$:

$$f = -4x + 2x^2 - (-3 + y)y,$$
$$g = -3x^3 + x^2(3 + 6y) + x(6 + 25y - 13y^2) - 4y(12 - 7y + y^2)$$

How many real intersection points do these curves have? How many connected components does the open semialgebraic set $\mathbb{R}^2 \setminus (\{ f = 0 \} \cup \{ g = 0 \})$ have? $\boxed{\text{8 points}}$

2.9 Let $(\mathbb{F}, \leq)$ be a euclidean ordered field and consider a general boolean combination of polynomial equations and inequalities defining a semialgebraic set K. Show that there is a variety V over $\mathbb{F}$ such that K is non-empty if and only if V is:

(1) Start by transforming the constraints defining K into a union of primary constraints K_i as in Lemma 2.5 and solve the exercise for the primary semialgebraic sets K_i yielding varieties $V_i \subseteq \mathbb{F}^{n_i}$.

(2) Show that every variety V_i is semialgebraically isomorphic to a variety V_i' defined by a single equation.

(3) Embed all V_i' as varieties V_i'' in a common space $\mathbb{F}^N$ and derive the equations describing $\bigcup_i V_i''$ from the single-equation definition of V_i''.

(4) *Bonus:* Suppose the complexity of a system of polynomial constraints is its length in symbols in the language of rings. What is the running time of each step in your transformation? Can you do it in polynomial time?

(Hint: Over every euclidean field we have that $f(x) > 0$ if and only if there exists some y such that $y^2 \cdot f(x) = 1$. This exercise consists of finding such algebraic reformulations for various set-theoretic operations on semialgebraic sets.)
$\boxed{\text{8 points}}$