MS-A0111
Differential and integral calculus

Ragnar Freij-Hollanti
Olga Kuznetsova, Aapo Pajala, Aleksi Sundell, Anton Vavilov

October 21, 2019

Teachers

- **Instructor:**
  Ragnar Freij-Hollanti, Y242b, `ragnar.freij@aalto.fi`
- **Teaching assistants**
  H01: Aapo Pajala
  H02: Anton Vavilov
  H03: Olga Kuznetsova, Y249a, `olga.kuznetsova@aalto.fi`
  H04: Aleksi Sundell

Schedule

- **Lectures:**
  Mondays 10-12, Jeti
  and
  Wednesday 10-12, Jeti
- **Exercises:**
  2 times a week, see schedule on course homepage.
  Session 1: Exploratory problems and Additional problems.
  Session 2: Additional problems and Homework problems.

Grading

- Alternative 1 (recommended):
  - **Final exam (60%):** Written exam Wednesday 23.10., 16:30-19:30.
  - **Homework (40%):** Reported under Assignments on mycourses.aalto.fi. Problems presented on mycourses.aalto.fi the previous friday.
- Alternative 2: Re-exam in December or May (100%)

Literature

- **James Stewart**, Calculus: Early transcendentals, 7th edition.
- **David Guichard and friends**, Single variable calculus: Early transcendentals.
  - PDF (Entire book): https://www.whitman.edu/mathematics/calculus/calculus.pdf
  - HTML and Chapter-by chapter: https://www.whitman.edu/mathematics/calculus/
- **Slides** Updated on mycourses.aalto.fi after every lecture.

Course content

- Sequences and series (week 1)
  - Sequences and their limits
  - Series and convergence tests
- Derivatives (week 2-3)
  - Standard functions and continuity
  - Derivatives and how to compute them
  - Extreme values and asymptotes
  - Taylor polynomials and power series
- Integrals (week 4-5)
  - Integrals and the fundamental theorem of calculus
  - Partial fractions and integration by parts
  - Unbounded integrals and applications
- Differential equations (week 5-6)
  - First order differential equations - linear and separable
  - Second order differential equations - homogeneous and inhomogeneous

Number classes

- Natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$
- $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \dots\}$
- Integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- Rationals $\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}\}$
- Real numbers $\mathbb{R}$.
  - Can be thought of as infinite decimal expansions.
  - Constructed (for example) via Cauchy sequences of rationals.
  - Contains all rationals, and (many) other numbers, like $\sqrt{p}$, π, e...
  - In fact, "most" real numbers are not rational.

Supremum axiom

Axiom

Every non-empty set of real numbers that has an upper bound, also has a **least** *upper bound in* $\mathbb{R}$.

- For example, the set $S = \{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, \ldots\}$ has a least upper bound in $\mathbb{R}$, namely π.

Supremum axiom

Axiom

Every non-empty set of real numbers that has an upper bound, also has a **least** *upper bound in* $\mathbb{R}$.

- For example, the set $S = \{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, \dots\}$ has a least upper bound in $\mathbb{R}$, namely π.
- In contrast, S has no least upper bound in $\mathbb{Q}$, because for any rational approximation $\frac{p}{q} < \pi$ of π, there is another rational approximation $\frac{p'}{q'} < \pi$ that is better ("include more decimals").

Supremum axiom

Axiom

Every non-empty set of real numbers that has an upper bound, also has a **least** *upper bound in* $\mathbb{R}$.

- The least upper bound is called *supremum*, and may or may not be contained in the set.
- For example, the sets

$$T = \{x \in \mathbb{R} : x^2 < 2\} \text{ and } T' = \{x \in \mathbb{R} : x^2 \leq 2\}$$

have the same supremum,

$$\sup(T) = \sup(T') = \sqrt{2},$$

but $\sqrt{2} \in T'$, $\sqrt{2} \notin T$.

Supremum axiom

Axiom

Every non-empty set of real numbers that has an upper bound, also has a **least** *upper bound in* $\mathbb{R}$*.*

Axiom

Every non-empty set of real numbers that has a lower bound, also has a **largest** *lower bound in* $\mathbb{R}$*.*

- Proof on the blackboard.
- This is called the *infimum* of the set.
- For example, the set $S = \{4, 3.2, 3.15, 3.142, 3.1416, 3.14160, 3.141593, \dots\}$ has a largest lower bound in $\mathbb{R}$, namely π.

Number sequences

- A number sequence is an infinite sequence of numbers

$$(a_n)_{n\in\mathbb{N}} = (a_n)_1^\infty = (a_1, a_2, a_3, \dots).$$

- A number sequence can also be thought of as a function $f : \mathbb{N} \to \mathbb{R}$, where $a_n = f(n)$.
  - $(1, 2, 3, 4, \dots)$: $a_n = n$.
  - $(1, 2, 4, 8, \dots)$: $a_n = 2^{n-1}$.

Number sequences

- Sometimes a sequence is given *recursively* or *inductively*:
  - Fibonacci sequence $(1, 1, 2, 3, 5, 8, \dots)$ is defined by

$$f_n = f_{n-1} + f_{n-2} \qquad \text{(for } n \geq 3).$$

  - Then we also need *starting values* $f_1 = f_2 = 1$.
  - In fact, f_n can also be written in *closed form* as

$$f_n = \frac{1}{\sqrt{5}}(\phi^n - \frac{(-1)^n}{\phi^n}),$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the *golden ratio*. Proving this is beyond the scope of this course.

Induction proofs

- A proof technique that is very useful for number sequences (but also in many other parts of mathematics)
- **Goal:** Prove a statement $P(n)$ for all natural numbers $n \in \mathbb{N}$.
- **Technique:**
  - First (base case) prove the first case $P(1)$ (or sometimes $P(0)$).
  - Then (induction step) prove that, for an arbitrary $m \in \mathbb{N}$,
    IF $P(m)$ holds, THEN $P(m+1)$ also holds.
  - These two steps together prove that the statement $P(n)$ holds for any $n \in \mathbb{N}$.

$$P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow P(4) \Rightarrow \cdots .$$

Induction proofs

Example

Let a_n be recursively defined by $a_1 = 1$ and $a_{n+1} = 2a_n + 1$. Then $a_n = 2^n - 1$ for all $n \in \mathbb{N}$.

Proof.

- Base case: $a_1 = 1 = 2^1 - 1$, so the statement is true for $n = 1$.
- Induction step: Assume (*induction hypothesis*) that $a_m = 2^m - 1$. Then

$$a_{m+1} \overset{def}{=} 2a_m + 1 \overset{IH}{=} 2 \cdot (2^m - 1) + 1 = 2^{m+1} - 2 + 1 = 2^{m+1} - 1,$$

  so the statement is also true for $n = m + 1$.
- It follows that the statement $a_n = 2^n - 1$ is true for all $n \in \mathbb{N}$.

□

Induction proofs

Example

Recall that the Fibonacci numbers are defined by $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$. For all $n \in \mathbb{N}$ holds $f_n < 2^n$.

Proof.

- Base case: $f_1 = 1 < 2 = 2^1$ and $f_2 = 1 < 4 = 2^2$.
- Induction step: Assume (*induction hypothesis*) that $f_m < 2^m$ and $f_{m-1} < 2^{m-1}$. Then

$$f_{m+1} \overset{def}{=} f_m + f_{m-1} \overset{IH}{<} 2^m + 2^{m-1} < 2 \cdot 2^m = 2^{m+1},$$

  so the statement is also true for $n = m + 1$.
- It follows that the statement $f_n < 2^n$ is true for all $n \in \mathbb{N}$.

□

Properties of sequences

Definition

A sequence $(a_n)_{n\in\mathbb{N}}$ is called

- *bounded from above* if there is $C \in \mathbb{R}$ s.t. $a_n < C$ for all n.
- *weakly increasing* if $a_n \leq a_{n+1}$ for all n.
- *strongly increasing* if $a_n < a_{n+1}$ for all n.

The notions of *bounded from below*, *weakly decreasing* and *strongly decreasing* are defined analogously (by reversing the inequality signs).

Limits

We are interested in what happens to a sequence when n gets large. What is the *limit* of a_n?

Definition

We say that $(a_n)_{n\in\mathbb{N}}$ *converges* to $L \in \mathbb{R}$, and write

$$a_n \xrightarrow[n\to\infty]{} L \text{ or } \lim_{n\to\infty} a_n = L$$

if for every $\epsilon > 0$ there is $N_\epsilon \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon \text{ whenever } n > N_\epsilon.$$

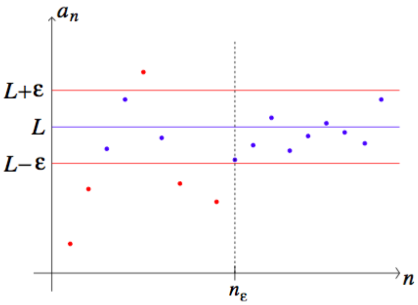
ϵ should be thought of as a very small number, and N_ϵ as a very big integer — the smaller ϵ is, the larger we need to choose N_ϵ.

Limits

Definition

We say that $\lim_{n\to\infty} a_n = L$ if for every $\epsilon > 0$ there is $N_\epsilon \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon \text{ whenever } n > N_\epsilon.$$



Limits

Definition

We say that $\lim_{n\to\infty} a_n = L$ if for every $\epsilon > 0$ there is $N_\epsilon \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon \text{ whenever } n > N_\epsilon.$$

Example

- The sequence $a_n = \frac{1}{n}$ converges to 0.
- Proof: For any $\epsilon > 0$, let $N_\epsilon \geq \frac{1}{\epsilon}$, Then

$$n > N_\epsilon \Longrightarrow |a_n - 0| = a_n = \frac{1}{n} < \frac{1}{N_\epsilon} < \epsilon.$$

Counting with limits

Theorem

Let $a_n \xrightarrow[n\to\infty]{} A$ *and* $b_n \xrightarrow[n\to\infty]{} B$.

Then $(a_n + b_n) \xrightarrow[n\to\infty]{} A + B$.

Proof.

- Fix $\epsilon > 0$.
- Let M_a and M_b be such that $n > M_a \Rightarrow |a_n - A| < \frac{\epsilon}{2}$, and $n > M_b \Rightarrow |b_n - B| < \frac{\epsilon}{2}$.
- Now, if N_ϵ is the largest of M_a and M_b, then

$$n > N_\epsilon \Rightarrow |(a_n + b_n) - (A + B)| \leq |a_n - A| + |b_n - B| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Counting with limits

Theorem

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences with

$$a_n \xrightarrow[n\to\infty]{} A \text{ and } b_n \xrightarrow[n\to\infty]{} B.$$

Then:

- $-a_n \xrightarrow[n\to\infty]{} -A.$
- $(a_n + b_n) \xrightarrow[n\to\infty]{} A + B.$
- $(a_n b_n) \xrightarrow[n\to\infty]{} AB.$
- *If* $B \neq 0$, *then* $\frac{a_n}{b_n} \xrightarrow[n\to\infty]{} \frac{A}{B}$.

We just proved the second part of this theorem. The other three parts are proved similarly.

Counting with limits

Example

$$\lim_{n\to\infty} \frac{n^2 - n}{3n^2 + 1} = \lim_{n\to\infty} \frac{n^2(1-\frac{1}{n})}{n^2(3+\frac{1}{n^2})} = \frac{\lim(1-\frac{1}{n})}{\lim(3+\frac{1}{n^2})} = \frac{1}{3}.$$

Counting with limits

Example

$$\lim_{n\to\infty} \sqrt{n^2+n}-n = \lim_{n\to\infty} \frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n}$$
$$= \lim_{n\to\infty} \frac{(n^2+n)-n^2}{\sqrt{n^2+n}+n}$$
$$= \lim_{n\to\infty} \frac{n}{n(\sqrt{1+\frac{1}{n}}+1)}$$
$$= \frac{1}{\lim(\sqrt{1+\frac{1}{n}}+1)} = \frac{1}{2}.$$

Counting with limits

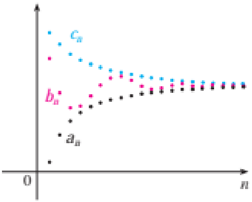
Theorem (‘Squeeze theorem”, or “Lemma of the two policemen”)

Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ be sequences with $a_n \le b_n \le c_n$ for every n and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L.$$

Then

$$\lim_{n\to\infty} b_n = L.$$



Counting with limits

Example

We want to compute

$$\lim_{n\to\infty} \frac{\sin n}{n}.$$

- Note that $\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$.
- But $\frac{-1}{n} \nearrow 0 \swarrow \frac{1}{n}$.
- Thus, by the policemen's lemma, $\frac{\sin n}{n} \to 0$.

Limits

- Not all sequences converge.
- The sequence $(1, 2, 3, 4, \dots)$, given by $a_n = n$, diverges.
- We can also write
$$\lim_{n\to\infty} n = \infty.$$
- We say that
$$\lim_{n\to\infty} a_n = \infty.$$
if for every $M > 0$, there exists $N > 0$ such that $n > N \Rightarrow a_n > M$.
- The sequence $(-1, 1, -1, 1, \dots)$, given by $a_n = (-1)^n$, diverges, and does not tend to infinity.

Limits

Theorem

Let $(a_n)_{n\in\mathbb{N}}$ be a weakly increasing sequence.

- *If a_n is upper bounded, then a_n converges to some $L = \lim a_n \in \mathbb{R}$.*
- *If a_n is not upper bounded, then $a_n \to \infty$.*

- In the first case, the limit is the *least upper bound* of the set $\{a_n : n \in \mathbb{N}\}$. This exists by the supremum axiom.

Limits

Theorem

Let $(a_n)_{n\in\mathbb{N}}$ be a weakly increasing sequence.

- *If a_n is upper bounded, then a_n converges to some $L = \lim a_n \in \mathbb{R}$.*
- *If a_n is not upper bounded, then $a_n \to \infty$.*

- In the first case, the limit is the *least upper bound* of the set $\{a_n : n \in \mathbb{N}\}$. This exists by the supremum axiom.
- For example, $\frac{n-1}{n} \leq \frac{n}{n+1} \leq 1$, so the sequence $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$ is upper bounded and increasing. Thus it has a limit.
- Indeed,

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

Limits

- Even when we know that a sequence converges, it can be difficult to compute its limit.
- Consider the sequence $e_n = (1 + \frac{1}{n})^n$.

$$\begin{aligned} e_1 &= 2 & \\ e_2 &= 9/4 &= 2.25 \\ e_3 &= 64/27 &\approx 2.37 \\ e_4 &= 625/256 &\approx 2.44 \end{aligned}$$

Limits

- Even when we know that a sequence converges, it can be difficult to compute its limit.
- Consider the sequence $e_n = (1 + \frac{1}{n})^n$.

$$\begin{aligned} e_1 &= 2 & \\ e_2 &= 9/4 & = 2.25 \\ e_3 &= 64/27 & \approx 2.37 \\ e_4 &= 625/256 & \approx 2.44 \end{aligned}$$

- One can show that $e_{n-1} < e_n < 3$ for all n.
- So by the theorem about monotone bounded sequences, e_n converges to some number

$$\mathrm{e} = \lim_{n\to\infty} e_n \approx 2.71828.$$

- This is the *natural base* e, which will appear very often in this course.

Speed table

- $$1 \ll \log n \ll n^\alpha \ll e^n \ll n! \ll n^n \text{ for any } \alpha > 0.$$
- By this we mean that the ratios $\frac{1}{\log n}$, $\frac{\log n}{n^\alpha}$, $\frac{n^\alpha}{e^n}$, $\frac{e^n}{n!}$, and $\frac{n!}{n^n}$, all tend to zero.
- Proof: exercise. (or blackboard if time)

Series

- A series is an "infinite sum", like

$$\sum_{i=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots$$

$$\sum_{i=1}^{\infty} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$$\sum_{i=0}^{\infty} \frac{1}{2^i} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

- These have a precise meaning via sequences.

Partial sums

- If $(a_n)_{n\in\mathbb{N}}$ is a number sequence, consider its partial sums

$$s_n = \sum_{i=1}^{n} a_i.$$

- If the sequence $(s_n)_{n\in\mathbb{N}}$ has a limit, then we say that $\sum_{i=1}^{\infty} a_i$ is *convergent*, and write

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} s_n.$$

Example

If $(a_n)_{n\in\mathbb{N}} = (1, 1, 1, \ldots)$, then the sequence of partial sums is $(s_n)_{n\in\mathbb{N}} = (1, 2, 3, \ldots)$. Not convergent.

Partial sums

Example

- If $(a_n)_{n\in\mathbb{N}_0} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$, then $(s_n)_{n\in\mathbb{N}} = (1, \frac{3}{2}, \frac{7}{4}, \dots)$.
- Claim: For $n \in \mathbb{N}_0$ holds

$$s_n = \sum_{i=0}^{n} \frac{1}{2^i} = 2 - \frac{1}{2^n}.$$

- Proof: By induction. (blackboard)

Partial sums

Example

- If $(a_n)_{n\in\mathbb{N}_0} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$, then $(s_n)_{n\in\mathbb{N}} = (1, \frac{3}{2}, \frac{7}{4}, \dots)$.
- Claim: For $n \in \mathbb{N}_0$ holds

$$s_n = \sum_{i=0}^{n} \frac{1}{2^i} = 2 - \frac{1}{2^n}.$$

- Proof: By induction. (blackboard)
- Since $s_n = 2 - \frac{1}{2^n} \to 2$, we get

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} 2 - \frac{1}{2^n} = 2.$$

Arithmetic sums

$$1 \quad + \quad 2 \quad + \quad \cdots \quad + \quad (n-1) \quad + \quad n$$

Arithmetic sums

$$\begin{array}{ccccccccc} 1 & + & 2 & + & \cdots & + & (n-1) & + & n \\ n & + & (n-1) & + & \cdots & + & 2 & + & 1 \\ \hline (n+1) & & (n+1) & & \cdots & & (n+1) & & (n+1) \end{array}$$

- This shows that $2\sum_{i=1}^{n} i = n(n+1)$, so

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

This can also be proven by induction. Exercise.

Arithmetic sums

- An *arithmetic progression* $(a, a+b, a+2b, \ldots, a+nb)$ has $n+1$ terms, first value a, and common difference b.
- Its sum is

$$\sum_{i=0}^{n} a + bi = (n+1)a + b\sum_{i=0}^{n} i$$
$$= (n+1)a + b\frac{n(n+1)}{2}$$
$$= (n+1)\left(a + \frac{nb}{2}\right).$$

- Intuition: The number of terms times the average term.

Geometric sums

- Let r be an arbitrary real number. Then

$$\begin{aligned}&(1 + r + r^2 + \cdots + r^n)(1 - r)\\ =&(1 - r) + (r - r^2) + (r^2 - r^3) + \cdots + (r^n - r^{n+1})\\ =&1 - r^{n+1}.\end{aligned}$$

- Thus, if $1 - r \neq 0$, we have

$$\sum_{i=0}^{n} r^i = 1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

Geometric sums

- A *geometric progression* $(a, ar, ar^2, \dots, ar^n)$ has $n+1$ terms, first value a, and common ratio r.
- For example, it represents the size of a population of fixed growth rate, or the value of a bank account with fixed interest rate, after $0, 1, \dots, n$ years.
- Its sum (if $r \neq 1$) is

$$\sum_{i=0}^{n} ar^i = a \sum_{i=0}^{n} r^i = \frac{a(1-r^{n+1})}{1-r}.$$

Geometric series

Example

$$1+\frac{3}{4}+\frac{9}{16}+\cdots+\frac{3^n}{4^n}=\sum_{i=0}^{n}\frac{3^i}{4^i}=\frac{1-\frac{3}{4}^{n+1}}{1-\frac{3}{4}}\xrightarrow[n\to\infty]{}\frac{1}{1-\frac{3}{4}}=4,$$

so

$$\sum_{i=0}^{\infty}\frac{3^i}{4^i}=4$$

Geometric series

Example

$$1 + \frac{4}{3} + \frac{16}{9} + \cdots + \frac{4^n}{3^n} = \sum_{i=0}^{n} \frac{4^i}{3^i} = \frac{1 - \frac{4}{3}^{n+1}}{1 - \frac{4}{3}} \xrightarrow[n\to\infty]{} \infty,$$

so

$$\sum_{i=0}^{\infty} \frac{4^i}{3^i} = \infty.$$

Geometric series

Theorem

The geometric series

$$a + ar + ar^2 + \cdots = a \sum_{i=0}^{\infty} r^i$$

is

- *Divergent if* $|r| \geq 1$.
- *Convergent, and equal to* $\frac{a}{1-r}$, *if* $-1 < r < 1$

Criteria for convergence

Theorem

- *If* $\sum_{i=0}^{\infty} a_i$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.
- *Conversely, if* $a_n \not\to 0$, *then* $\sum_{i=0}^{\infty} a_i$ *is not convergent.*

- This does not mean that all sequences with $a_n \to 0$ have a convergent sum.

Criteria for convergence

Example

- $\sum \frac{n-1}{n}$ not convergent.
- $\sum \sin n$ not convergent.

Series of powers

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$$
$$\geq \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}}$$
$$= \frac{n}{\sqrt{n}} = \sqrt{n} \xrightarrow[n\to\infty]{} \infty,$$

so the series

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots = \sum_{i=1}^{\infty} \frac{1}{\sqrt{i}} = \infty$$

is *divergent*.

Series of powers

Let $0 < \alpha < 1$. Then

$$\begin{aligned}\sum_{i=1}^{n} \frac{1}{i^\alpha} &= \frac{1}{1^\alpha} + \frac{1}{2^\alpha} + \cdots + \frac{1}{n^\alpha} \\ &\geq \frac{1}{n^\alpha} + \frac{1}{n^\alpha} + \cdots + \frac{1}{n^\alpha} \\ &= \frac{n}{n^\alpha} = n^{1-\alpha} \xrightarrow[n\to\infty]{} \infty,\end{aligned}$$

so the series

$$\frac{1}{1^\alpha} + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \cdots = \sum_{i=1}^{n} \frac{1}{i^\alpha} = \infty$$

is *divergent*.

Comparison criterion

Theorem

Let $\sum_{i=1}^{\infty} b_i$ be a convergent series of positive terms, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that for some $M > 0$, $0 \leq a_k \leq Mb_k$ for every k. Then $\sum_{i=1}^{\infty} a_i$ is convergent, with

$$\sum_{i=1}^{\infty} a_i \leq M \sum_{i=1}^{\infty} b_i.$$

Comparison criterion

Theorem

If $0 \le a_k \le Mb_k$ *for every* k*, then* $\sum_{i=1}^{\infty} a_i \le M \sum_{i=1}^{\infty} b_i$.

Proof.

The inequality

$$s_n = \sum_{i=1}^{n} a_i \le M \sum_{i=1}^{n} b_i \le M \sum_{i=1}^{\infty} b_i$$

holds for every partial sum. So $(s_n)_{n \in \mathbb{N}}$ is an increasing and bounded sequence, so it has a limit $\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} s_n$ that is at most the upper bound $M \sum_{i=1}^{\infty} b_i$. □

Series of powers

$$\begin{aligned}\sum_{i=1}^{n} \frac{1}{i^2} &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \\ &\leq 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \cdots + \frac{1}{n(n-1)} \\ &= 1 + (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{n-1} - \frac{1}{n}) \\ &= 1 + \frac{1}{1} - \frac{1}{n} \xrightarrow[n\to\infty]{} 2.\end{aligned}$$

So

$$\sum_{i=1}^{\infty} \frac{1}{i^2} \leq 2,$$

and in particular $\sum_{i=1}^{\infty} \frac{1}{i^2}$ is convergent.

Series of powers

- $$\sum_{i=1}^{\infty} \frac{1}{i^2} \leq 2,$$
  and in particular $\sum_{i=1}^{\infty} \frac{1}{i^2}$ is convergent.
- In fact,
  $$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6} \approx 1.6449,$$
  but this is MUCH harder to prove.
- Shown by Euler in 1734, after the problem had been asked by Mengoli in 1644.
- Really beautiful geometric explanation: `www.youtube.com/watch?v=d-o3eB9sfls`

Series of powers

Theorem

The sequence

$$\sum_{i=1}^{n} \frac{1}{i^\alpha}$$

is

- *Divergent if* $0 \leq \alpha \leq 1$
- *Convergent if* $1 < \alpha$

- We have already shown the cases $0 \leq \alpha \leq 1$ and $\alpha = 2$.
- The cases $\alpha > 2$ follows from the comparison criterion, as then $\frac{1}{i^\alpha} \leq \frac{1}{i^2}$ for every i.
- The cases $1 < \alpha < 2$ will be treated later in the course.

Limit criterion

Theorem

If $\sum_{n=1}^{\infty} b_n$ *is convergent and*

$$\lim_{n\to\infty} \frac{a_n}{b_n} < \infty,$$

then $\sum_{n=1}^{\infty} a_n$ *is also convergent.*

- Conversely, if $\sum_{n=1}^{\infty} b_n = \infty$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} > 0,$$

  then $\sum_{n=1}^{\infty} a_n$ is also divergent.
- Both these statements are immediate consequences of the comparison criterion.

Limit criterion

Example

- Is $\sum \frac{1}{\sqrt{n^2+2n}}$ convergent or divergent?
- $\sqrt{n^2+2n} = n\sqrt{1+\frac{2}{n}}$.
- $$\frac{1/\sqrt{n^2+2n}}{1/n} = \frac{1}{\sqrt{1+\frac{2}{n}}} \xrightarrow[n\to\infty]{} 1 > 0.$$
- As $\sum \frac{1}{n}$ is divergent, so is $\sum \frac{1}{\sqrt{n^2+2n}}$.

Quotient criterion

- Assume the ratios $\frac{a_{n+1}}{a_n}$ between the terms has a limit

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

- If $\rho > 1$, then the terms do not converge to zero (in fact, they diverge), so the series $\sum a_n = \infty$ is divergent.

Quotient criterion

- Assume the ratios $\frac{a_{n+1}}{a_n}$ between the terms has a limit

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

- If $\rho < 1$, then the inequality $a_n < a_{N_\epsilon}(\rho+\epsilon)^n$ holds for every term a_n where $n > N_\epsilon$, if $0 < \epsilon < 1-\rho$. (Proof: exercise)
- Then

$$\sum_{n=1}^{\infty} a_n < \sum_{i=1}^{N_\epsilon} a_i + a_{N_\epsilon} \sum_{n=N_\epsilon+1}^{\infty} (\rho+\epsilon)^n = \sum_{i=1}^{N_\epsilon} a_i + \frac{a_{N_\epsilon}}{1-\rho-\epsilon} < \infty,$$

so $\sum_{n=1}^{\infty} a_n$ is convergent.

Quotient criterion

- Assume the ratios $\frac{a_{n+1}}{a_n}$ between the terms has a limit

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

- If $\rho = 1$, then the series $\sum a_n$ can either be convergent or divergent.

Example

If $a_n = 1/\sqrt{n}$, then $\frac{a_{n+1}}{a_n} = \frac{\sqrt{n}}{\sqrt{n+1}} \xrightarrow[n\to\infty]{} 1$, and

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

is divergent.

Quotient criterion

- Assume the ratios $\frac{a_{n+1}}{a_n}$ between the terms has a limit

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

- If $\rho = 1$, then the series $\sum a_n$ can either be convergent or divergent.

Example

If $a_n = 1/n^2$, then $\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} \xrightarrow[n\to\infty]{} 1$, and

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent.

Quotient criterion

Theorem

- *If* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, *then* $\sum_{n=1}^{\infty} a_n$ *is convergent.*
- *If* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$, *then* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

Quotient criterion

Example

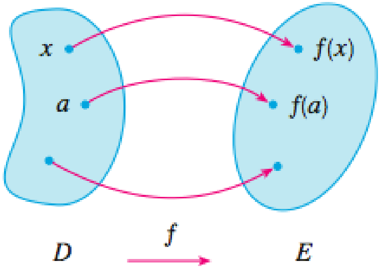
- Is $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ convergent or divergent?
- Let $a_n = \frac{2^n}{n!}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{2}{n+1} \xrightarrow[n\to\infty]{} 0.$$

- By the quotient criterion $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ is convergent.

Functions

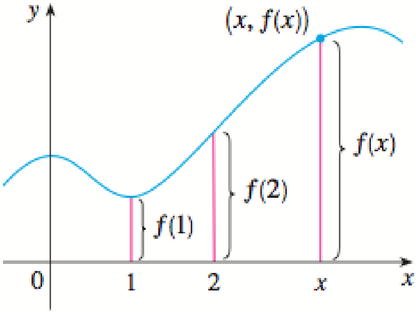
- A function $f : D \to E$ is a rule that assigns, for each element $x \in D$, a unique element $f(x) \in E$.



- D is the *domain* of the function, and E is the *codomain*.

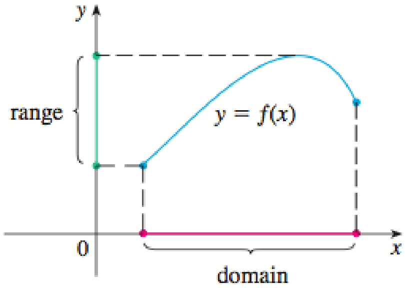
Functions

- A function is often represented by its *graph*, especially when the domain and codomain are both (subsets of) $\mathbb{R}$.



Functions

- The *range* $f(D)$ of the function is the set $\{f(x) : x \in D\}$.



- The range is a subset of E, but not necessarily all of E.

Intervals

If a and b are real numbers, $a \leq b$, then

- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ is a *closed interval*.
- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ is an *open interval*.
- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ and $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$ are *half-open intervals*.

Intervals

We also consider *infinite* and *half-infinite* intervals

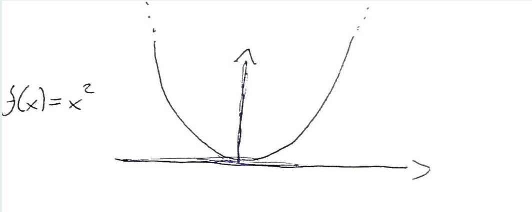
- $[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$.
- $(a, \infty) = \{x \in \mathbb{R} : a < x\}$.
- $(\infty, a] = \{x \in \mathbb{R} : x \leq a\}$.
- $(\infty, a) = \{x \in \mathbb{R} : x < a\}$.
- $(-\infty, \infty) = \mathbb{R}$.

Functions

- The domain and codomain are usually $\mathbb{R}$, or intervals in $\mathbb{R}$.
- If no domain is specified, we assume that the function is defined for every value where its formula "makes sense".

Example

- The function $f(x) = x^2$ has domain $\mathbb{R}$ and range $[0, \infty)$.

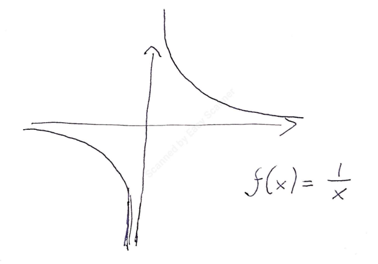


Functions

- The domain and codomain are usually $\mathbb{R}$, or intervals in $\mathbb{R}$.
- If no domain is specified, we assume that the function is defined for every value where its formula "makes sense".

Example

- The function $f(x) = \frac{1}{x}$ has domain and range $\mathbb{R} \setminus \{0\}$.

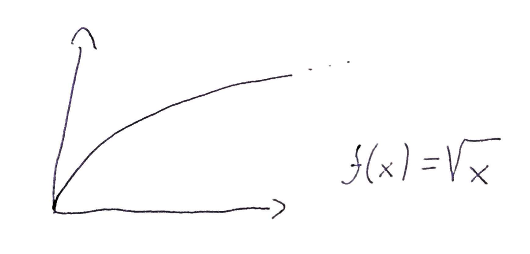


Functions

- The domain and codomain are usually $\mathbb{R}$, or intervals in $\mathbb{R}$.
- If no domain is specified, we assume that the function is defined for every value where its formula "makes sense".

Example

- The function $f(x) = \sqrt{x}$ has domain and range $[0, \infty)$.

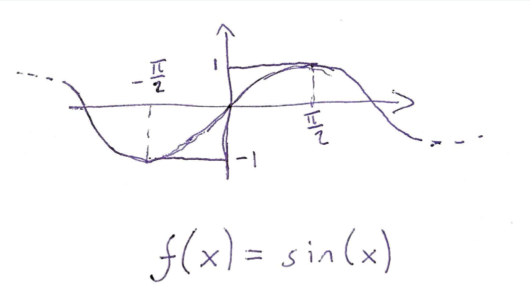


Functions

- The domain and codomain are usually $\mathbb{R}$, or intervals in $\mathbb{R}$.
- If no domain is specified, we assume that the function is defined for every value where its formula "makes sense".

Example

- The function $f(x) = \sin x$ has domain $\mathbb{R}$ and range $[-1, 1]$.

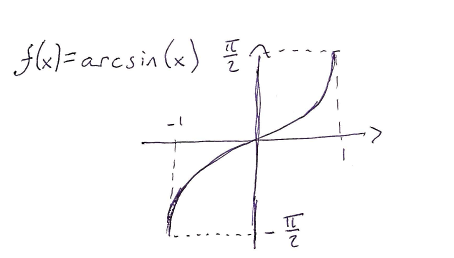


Functions

- The domain and codomain are usually $\mathbb{R}$, or intervals in $\mathbb{R}$.
- If no domain is specified, we assume that the function is defined for every value where its formula "makes sense".

Example

- $f(x) = \arcsin x$ has domain $[-1, 1]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$.



Composition of functions

- Two functions $f : A \to B$ and $g : B \to C$ can be *composed* into a function $g \circ f : A \to C$, $g \circ f(x) = g(f(x))$.
- Sometimes it is easier to analyse one complicated function as a composition of two easier ones.

Example

- The function $h(x) = 2^{x^2+1}$ can be written as $g \circ f$, where $g(y) = 2^y$ and $f(x) = x^2 + 1$.

Composition of functions

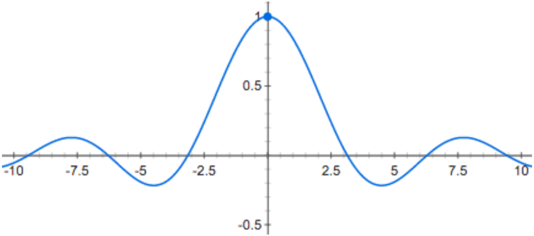
Example

- The function $h(x) = 2^{x^2+1}$ can be written as $g \circ f$, where $g(y) = 2^y$ and $f(x) = x^2 + 1$.
- $$x \overset{f}{\longmapsto} x^2 + 1 \overset{g}{\longmapsto} 2^{x^2+1}.$$
- This is **not** the same as the composition $f \circ g$:

$$x \overset{g}{\longmapsto} 2^x \overset{g}{\longmapsto} (2^x)^2 + 1 = 4^x + 1.$$

Limits of functions

- The function $f(x) = \frac{\sin x}{x}$ is only defined for $x \neq 0$.



- Still, it seems to tend to a limit, marked by the blue dot.

Limits of functions

Definition

We say that a function f *converges* to $L \in \mathbb{R}$ as $x \to a$, and write

$$f(x) \xrightarrow[x \to a]{} L \text{ or } \lim_{x \to a} f(x) = L$$

if for every $\epsilon > 0$ there is $\delta = \delta_\epsilon$ such that

$$|f(x) - L| < \epsilon \text{ whenever } |x - a| < \delta.$$

- Note that this definition does not require that f is defined in a, but only that f is defined in some points *arbitrarily close* to a.

Limits of functions

Definition

$$\lim_{x \to a} f(x) = L$$

means that for every $\epsilon > 0$ there is $\delta = \delta_\epsilon$ such that

$$|x - a| < \delta \implies |f(x) - L| < \epsilon.$$

Example

$$\lim_{x \to 0} \sqrt{x} = 0,$$

because for any $\epsilon > 0$, we can choose $\delta = \epsilon^2$, and get

$$|x - 0| < \delta = \epsilon^2 \implies |\sqrt{x} - 0| < \epsilon.$$

Continuity

Example

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1} x+1 = 2.$$

Limits of functions

Theorem

Let $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$. *Then*

- $\lim_{x\to a} -f(x) = -L$.
- $\lim_{x\to a} f(x) + g(x) = L + M$.
- $\lim_{x\to a} f(x) \cdot g(x) = LM$.
- *If* $M \neq 0$, *then* $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$.

Continuity

- A function is continuous if “you can draw its graph without lifting your pen from the paper”.
- This can be made precise using limits.

Definition

- A function f is *continuous in a* if

$$f(a) = \lim_{x \to a} f(x).$$

- f is *continuous* if it is continuous in all points in its domain.

Continuity

Theorem

- *$f(x) = x^\alpha$ is continuous (except in 0 if $\alpha < 0$.)*
- *$\sin x$ is continuous.*

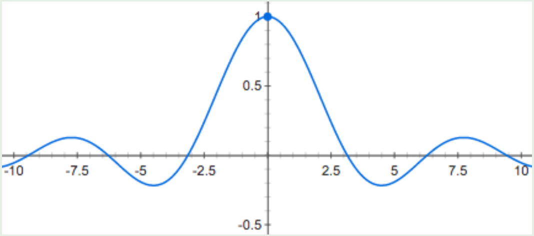
Corollary

- *If f and g are polynomials, then the* rational function *$\frac{f}{g}$ is continuous in the points where $g(x) \neq 0$.*

Continuity

Example

- The function $\frac{\sin x}{x}$ is continuous in $\mathbb{R} \setminus \{0\}$.



- It can be extended to a continuous function on all of $\mathbb{R}$, by letting

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}.$$

Counting with limits

Theorem ('Squeeze theorem", or "Lemma of the two policemen")

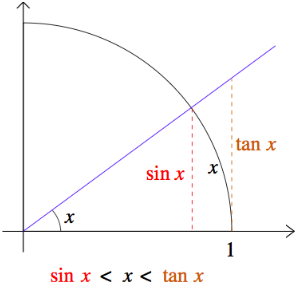
Let f, g and h be functions defined on the same domain D, with $f(x) \leq g(x) \leq h(x)$ for every $x \in D$ and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L.$$

Then

$$\lim_{x \to a} g(x) = L.$$

Standard limits



- Dividing by $\sin x$ we get $1 < \frac{x}{\sin x} < \frac{1}{\cos x} \xrightarrow[x \to 0]{} 1$.
- Squeeze lemma yields $\lim_{x \to 0} \frac{x}{\sin x} = 1$.

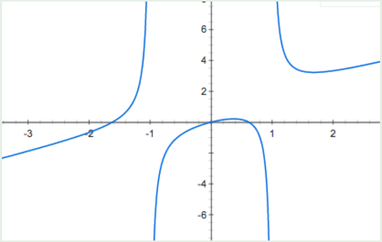
Standard limits

- $\lim_{x\to 0}\frac{\ln(x+1)}{x} = 1$. (later)
- $\lim_{x\to 0} x^x = 1$.
  - Note: $\lim_{x\to 0} x^0 = 1$ but $\lim_{x\to 0} 0^x = 0$.
  - We usually define $0^0 = 1$, but this is only a convention.
- $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \mathrm{e}$. (definition of e).

Continuity

Example

- The function $\frac{x^3+x^2-x}{x^2-1}$ is continuous in $\mathbb{R} \setminus \{-1, 1\}$.



- It can not be extended to a continuous function on all of $\mathbb{R}$, because it has no limit as $x \to -1$ and when $x \to 1$.

One-sided limits

Definition

We say that a function f converges to $L \in \mathbb{R}$ as $x \to a$ *from the right*, and write

$$f(x) \xrightarrow[x \to a^+]{} L \text{ or } \lim_{x \to a^+} f(x) = L$$

if for every $\epsilon > 0$ there is $\delta = \delta_\epsilon$ such that

$$|f(x) - L| < \epsilon \text{ whenever } 0 < x - a < \delta.$$

Convergence from the left, $\lim_{x \to a^-} f(x)$, is defined analogously.

Improper limits

Definition

We say that a function f converges to ∞ as $x \to a$, and write

$$f(x) \xrightarrow[x\to a]{} \infty \text{ or } \lim_{x\to a} f(x) = \infty$$

if for every N there is $\delta = \delta_N$ such that

$$f(x) > N \text{ whenever } |x - a| < \delta.$$

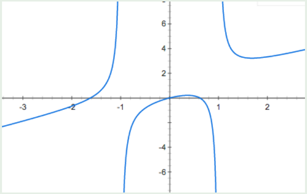
- Convergence to $-\infty$ is defined analogously.
- We can also easily define one-sided improper limits

$$\lim_{x\to a^+} f(x) = \pm\infty \text{ and } \lim_{x\to a^-} f(x) = \pm\infty.$$

Improper limits

Example

- The function $f(x) = \frac{x^3+x^2-x}{x^2-1}$ is continuous in $\mathbb{R} \setminus \{-1, 1\}$.



- Improper limits:
  - $\lim_{x\to -1^-} = \infty$
  - $\lim_{x\to -1^+} = -\infty$
  - $\lim_{x\to 1^-} = -\infty$
  - $\lim_{x\to 1^+} = \infty$

Limits of functions

There are "counting rules" for improper limits as well.

Theorem

Let $\lim_{x\to a} f(x) = L \in \mathbb{R}$ *and* $\lim_{x\to a} g(x) = \infty$. *Then*

- $\lim_{x\to a} -g(x) = -\infty$.
- $\lim_{x\to a} f(x) + g(x) = \infty$.
- *If* $L > 0$, *then* $\lim_{x\to a} f(x) \cdot g(x) = \infty$.
- $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$.

Limits of the form $-\infty + \infty$, $0 \cdot \infty$ and $\frac{\infty}{\infty}$ can not be handled directly with these rules.

Limits of functions

Theorem

Let $\lim_{x\to a} f(x) = b$ *and assume that* g *is continuous in* b. *Then*

$$\lim_{x\to a}(g\circ f)(x) = \lim_{x\to a} g(f(x)) = g(b).$$

Proof.

Blackboard □

- This holds also if $a=\infty$.
- It follows that if $f: A\to B$ and $g: B\to C$ are continuous, then so is $g\circ f: A\to C$.

Limits of functions

Example

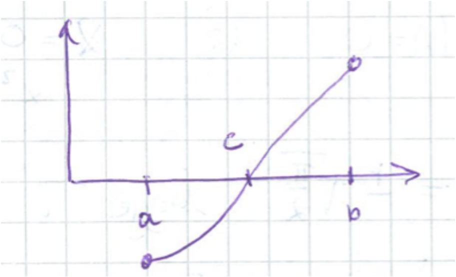
$x \mapsto e^x$ is continuous, so

$$\lim_{x \to 0} \mathrm{e}^{\frac{\sin x}{x}} = \mathrm{e}^{\lim_{x \to 0} \frac{\sin x}{x}} = \mathrm{e}^1 = \mathrm{e}.$$

Intermediate value theorem

Theorem

Assume f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$. Then there is $c \in [a, b]$ with $f(c) = 0$.



Intermediate value theorem

- This yields an *algorithm* to approximate a solution to $f(x) = 0$, with an error of at most ϵ.
- If we know $f(a) < 0 < f(b)$, then there is a solution in $[a, b]$.
- Check the sign of $f(\frac{a+b}{2})$.
- If $f(\frac{a+b}{2}) > 0$, repeat the procedure on $[a, \frac{a+b}{2}]$.
- If $f(\frac{a+b}{2}) < 0$, repeat the procedure on $[\frac{a+b}{2}, b]$.
- Repeat the procedure until the endpoints are less than ϵ apart.

Intermediate value theorem

Example

- Approximate $2^{\frac{1}{3}}$ with an error of at most 0.1.
- Want to find solutions to $f(x) = x^3 - 2 = 0$.
- $f(1) = -1 < 0 < 6 = f(2)$, so $x \in [1, 2]$. Check $f(\frac{3}{2})$.
- $f(\frac{3}{2}) = \frac{27}{8} - 2 = \frac{11}{8} > 0$. Check $f(\frac{1+\frac{3}{2}}{2}) = f(\frac{5}{4})$.
- $f(\frac{5}{4}) = \frac{125}{64} - 2 = \frac{-3}{64} < 0$. Check $f(\frac{\frac{5}{4}+\frac{3}{2}}{2}) = f(\frac{11}{8})$.

Intermediate value theorem

Example

- Approximate $2^{\frac{1}{3}}$ with an error of at most 0.1.
- Want to find solutions to $f(x) = x^3 - 2 = 0$.
- $f(\frac{11}{8}) = \frac{1331}{512} - 2 = \frac{307}{512} > 0$. Check $f(\frac{\frac{5}{4}+\frac{11}{8}}{2}) = f(\frac{21}{16})$.
- $f(\frac{21}{16}) = \frac{9261}{4096} - 2 = \frac{269}{4096} > 0$.
- So $f(\frac{5}{4}) < 0 < f(\frac{21}{16})$.
- $x \in [\frac{5}{4}, \frac{21}{16}]$ and $|\frac{21}{16} - \frac{5}{4}| = \frac{1}{16} < 0.1$.
- This is a rather slow algorithm.

Inverse functions

- $f : A \to B$ is *one-to-one* if for every $y \in B$ there is a **unique** $x \in A$ with $f(x) = y$.
- Strictly increasing functions are one-to-one (if the codomain equals the range.)
- One-to-one functions are also called *bijective* or *invertible*.
- If $f : A \to B$ is one-to-one, then it has an *inverse function* $f^{-1} : B \to A$ such that

$$f(x) = y \iff x = f^{-1}(y).$$

- Warning: Do not mistake the function f^{-1} for the number $f(x)^{-1} = \frac{1}{f(x)}$.

Inverse functions

Example

- $f(x) = x^3$ is invertible $\mathbb{R} \to \mathbb{R}$.
  - $f^{-1}(y) = y^{1/3}$.

Inverse functions

Example

- $f(x) = x^3$ is invertible $\mathbb{R} \to \mathbb{R}$.
    - $f^{-1}(y) = y^{1/3}$.
- $f(x) = x^2$ is not invertible $\mathbb{R} \to [0, \infty)$, as $f(-x) = f(x)$.
- $f(x) = x^2$ is invertible $[0, \infty) \to [0, \infty)$.
    - $f^{-1}(y) = \sqrt{y}$.

Inverse functions

Example

- $\sin x$ is not invertible $\mathbb{R} \to [-1, 1]$, as $\sin(x) = \sin(x + 2\pi)$ and $\sin(x) = \sin(\pi - x)$.
- $\cos x$ is not invertible $\mathbb{R} \to [-1, 1]$, as $\cos(x) = \cos(x + 2\pi)$ and $\cos(x) = \cos(-x)$.
- $\tan x$ is not invertible $\mathbb{R} \to \mathbb{R}$, as $\tan(x) = \tan(x + \pi)$, and since $\tan x$ is not defined when $x \in \{\frac{\pi}{2} + n\pi : n \in \mathbb{Z}\}$.

Inverse functions

Example

- $\sin x$ is invertible $[-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$, with $\sin^{-1}(x) = \arcsin(x)$.
- $\cos x$ is invertible $[0, \pi] \to [-1, 1]$, with $\cos^{-1}(x) = \arccos(x)$.
- $\tan x$ is invertible $(-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, with $\tan^{-1}(x) = \arctan(x)$.

Inverse functions

Example

- The exponential function $x \mapsto \mathrm{e}^x$ is a continuous and increasing function $\mathbb{R} \to (0, \infty)$ defined by

$$\mathrm{e}^x = \begin{cases} (\mathrm{e}^p)^{\frac{1}{q}} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \\ \lim\limits_{\substack{r \to x \\ r \in \mathbb{Q}}} \mathrm{e}^r & \text{otherwise.} \end{cases}$$

- It is one-to-one, with inverse $y \mapsto \ln(y)$.

Inverse functions

Theorem

If A is an interval, $B \subseteq \mathbb{R}$, and $f : A \to B$ is one-to-one and continuous, then the inverse function is also continuous.

Example

The functions $\ln x$, $\arcsin(x)$, $\arccos(x)$ and $\arctan(x)$ are continuous on their respective domains.

Limits of functions

Example

$x \mapsto \ln x$ is continuous, as it is the inverse of $y \mapsto \mathrm{e}^y$. Thus,

$$\lim_{x\to 0} \frac{\ln(x+1)}{x} = \lim_{x\to 0} \ln\left((x+1)^{\frac{1}{x}}\right) = \ln\left(\lim_{x\to 0}(x+1)^{\frac{1}{x}}\right) = \ln(\mathrm{e}) = 1.$$

Derivatives

- How quickly does a function grow?
- This is measured by the slope between two points on the function graph.
$$\frac{\Delta f}{\Delta x} = \frac{f(a+h) - f(x)}{h}.$$
- The *momentary* growth at $a \in \mathbb{R}$ is measured by
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$
- If this limit exists, we say that f is *differentiable* at a.

Derivatives

- In physics, the variable is often denoted by t for time, and the derivative is the speed at which a dependent variable changes.

Example

If $x(t)$ is the location at time t of a particle that moves along a line, let $v(t)$ be its velocity and let $a(t)$ be its acceleration. Then $x'(t) = v(t)$ and $v'(t) = a(t)$.

Derivatives

- Notation: $f'(a) = \frac{df}{dx}|_{x=a} = \frac{d}{dx}f(a) = Df(a) = \dot{f}(a)$.
- The notation $\dot{f}$ is only used in physics, and only when the independent variable is time.

Derivatives

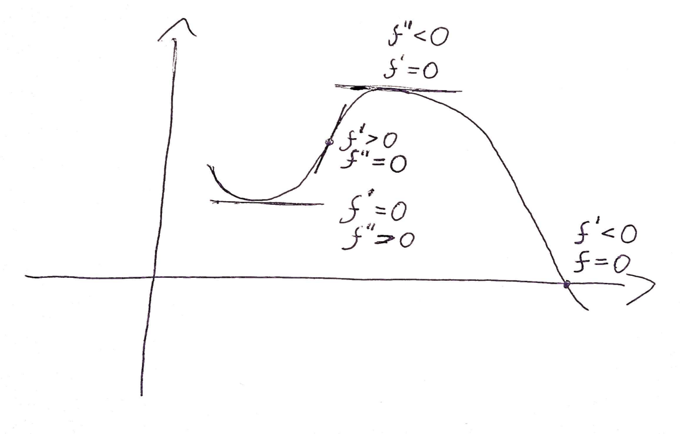
- If $f : A \to \mathbb{R}$ is differentiable on A, then its derivative $Df = f'$ is also a function $A \to \mathbb{R}$.
- If f' also happens to be differentiable on A, then we can study the *second derivative*,

$$f'' = D^2 f = \frac{d^2 f}{dx^2},$$

which measures the rate of change of f'.

Derivatives

Example



Rules of derivation

- Sums:

$$\begin{aligned}(f+g)'(x) &= \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h} \\ &= f'(x)+g'(x).\end{aligned}$$

- Scalar multiplication: If $c \in \mathbb{R}$, then

$$\begin{aligned}(cf)'(x) &= \lim_{h\to 0} \frac{cf(x+h)-cf(x)}{h} \\ &= c \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= cf'(x)\end{aligned}$$

Rules of derivation

Products:

$$
\begin{aligned}
&(fg)'(x) \\
=& \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
=& \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
=& \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x)}{h} \\
=& \lim_{h\to 0} f(x+h)g'(x) + g(x)f'(x) \\
=& f(x)g'(x) + f'(x)g(x).
\end{aligned}
$$

Standard derivatives

- $f(x) = k$ constant:

$$f'(x) = \lim_{h \to 0} \frac{k - k}{h} = 0.$$

- $f(x) = x$:

$$f'(x) = \lim_{h \to 0} \frac{(x + h) - x}{h} = 1.$$

Standard derivatives

- $f(x) = x^p$ monomial:
- Product rule:

$$\begin{aligned}\frac{d}{dx}f(x) &= \frac{d}{dx}(x^{p-1}\cdot x)\\ &= x\frac{d}{dx}(x^{p-1}) + x^{p-1}\frac{d}{dx}x\\ &= x\frac{d}{dx}(x^{p-1}) + x^{p-1}\end{aligned}$$

- By induction (blackboard)

$$\frac{d}{dx}f(x) = px^{p-1}.$$

Rules of derivation

- Fractions:

$$\left(\frac{1}{f}\right)'(x) = \lim_{h\to 0}\frac{\frac{1}{f(x+h)}-\frac{1}{f(x)}}{h}$$
$$= \lim_{h\to 0}\frac{f(x)-f(x+h)}{h}\frac{1}{f(x)f(x+h)}$$
$$= \frac{-f'(x)}{f(x)^2}$$

- It follows that

$$\left(\frac{f}{g}\right)' = \frac{f'}{g} + f\left(\frac{1}{g}\right)' = \frac{f'}{g} - \frac{fg'}{g^2} = \frac{f'g-fg'}{g^2}.$$

Standard derivatives

- $f(x) = x^{-p} = \frac{1}{x^p}$ inverse monomial:
- Fraction rule:

$$\frac{d}{dx}\left(\frac{1}{x^p}\right) = \frac{-\frac{d}{dx}\left(x^p\right)}{(x^p)^2} = \frac{-px^{p-1}}{x^{2p}} = \frac{-p}{x^{p+1}} = -px^{-p-1}.$$

Standard derivatives

$$\frac{d}{dx}\left(\frac{1}{x^p}\right) = -px^{-p-1}.$$

- So the rule $\frac{d}{dx}x^\alpha = \alpha x^{\alpha-1}$ holds also if $\alpha = -p$ is a negative integer!
- Actually, it holds for any real number α.

Rules of derivation

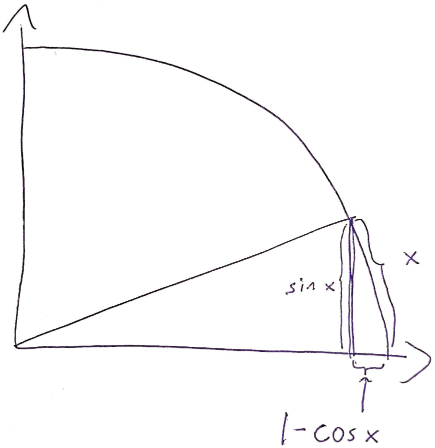
Theorem

Let f and g be differentiable functions, and let c be a constant. Then:

- $(f+g)'(x) = f'(x) + g'(x)$.
- $(cf)'(x) = cf'(x)$.
- $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.
- $\left(\frac{1}{g}\right)'(x) = \frac{-g'(x)}{g(x)^2}$.

Standard derivatives

Recall the standard limits $\frac{\sin x}{x} \xrightarrow[x \to 0]{} 1$ and $\frac{\cos x - 1}{x} \xrightarrow[x \to 0]{} 0$.



Standard derivatives

- $f(x) = \sin x$:
-

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} \\
&= \cos x.
\end{aligned}$$

Standard derivatives

- $f(x) = \cos x$:
-

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\
&= \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h} \\
&= -\sin x.
\end{aligned}$$

Standard derivatives

- $f(x) = \tan x = \frac{\sin x}{\cos x} = \sin x \frac{1}{\cos x}$:
- Product and fraction rule:

$$\begin{aligned} f'(x) &= \sin x \cdot D\left(\frac{1}{\cos x}\right) && + \frac{1}{\cos x} D(\sin x) \\ &= \sin x \frac{-D(\cos x)}{(\cos x)^2} && + \frac{1}{\cos x} D(\sin x) \\ &= \frac{(\sin x)^2}{(\cos x)^2} && + \frac{\cos x}{\cos x} \\ &= \frac{1 - (\cos x)^2}{(\cos x)^2} && + 1 \\ &= \frac{1}{\cos(x)^2}. \end{aligned}$$

Standard derivatives

Theorem

$f(x)$	$f'(x)$
k	0
x^p	px^{p-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\frac{1}{\cos(x)^2}$

Rules of derivation

- Chain rule for compositions:

$$\begin{aligned}(g\circ f)'(x) &= \lim_{h\to 0}\frac{g(f(x+h))-g(f(x))}{h}\\ &= \lim_{h\to 0}\frac{g(f(x+h))-g(f(x))}{f(x+h)-f(x)}\cdot\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{g(f(x+h))-g(f(x))}{f(x+h)-f(x)}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= g'(f(x))f'(x).\end{aligned}$$

- If $f(x+h)-f(x)=0$ for all h close to 0, then we need to modify the proof, but the identity still holds.

Rules of derivation

Example

- Compute the derivative of $f(x) = \sin(x^2)$ in $x = \sqrt{\pi}$.
- Write $f(x) = g \circ h(x)$, where $h(x) = x^2$ and $g(y) = \sin y$.

Rules of derivation

Example

- Compute the derivative of $f(x) = \sin(x^2)$ in $x = \sqrt{\pi}$.
- Write $f(x) = g \circ h(x)$, where $h(x) = x^2$ and $g(y) = \sin y$.
- $h'(x) = 2x$ and $g'(y) = \cos(y)$.
- Thus,
$$f'(x) = g'(h(x)) \cdot h'(x) = \cos(x^2) \cdot 2x.$$
- When $x = \sqrt{\pi}$, we get
$$f'(\sqrt{\pi}) = \cos(\pi) \cdot 2\sqrt{\pi} = -2\sqrt{\pi}.$$

Rules of derivation

Example

- Find an equation for the tangent line to the curve $y = f(x) = \sin(x^2)$ in the point where $x = \sqrt{\pi}$.
- The equation is of the form

$$y = f'(\sqrt{\pi})x + b = -2\sqrt{\pi}x + b$$

  for some b.
- We insert $(x, y) = (\sqrt{\pi}, f(\sqrt{\pi})) = (\sqrt{\pi}, 0)$ and get

$$0 = -2\sqrt{\pi} \cdot \sqrt{\pi} + b = -2\pi + b,$$

  so $b = 2\pi$.

Standard derivatives

- $$f(x) = \mathrm{e}^x = \lim_{y \to 0} (1+y)^{\frac{x}{y}} = \left[t := \frac{y}{x}\right] = \lim_{t \to 0} (1+xt)^{\frac{1}{t}}.$$
- Non-trivial fact:
$$f'(x) = D\left(\lim_{t \to 0} (1+xt)^{\frac{1}{t}}\right) = \lim_{t \to 0} D\left((1+xt)^{\frac{1}{t}}\right).$$
- Chain rule:
$$\begin{aligned} f'(x) &= \lim_{t \to 0} \frac{d}{dx}(1+xt) \cdot \frac{d}{d(1+xt)}(1+xt)^{\frac{1}{t}} \\ &= \lim_{t \to 0} t \cdot \frac{1}{t} \cdot \frac{(1+xt)^{\frac{1}{t}}}{1+xt} \\ &= \lim_{t \to 0} (1+xt)^{\frac{1}{t}} = \mathrm{e}^x \end{aligned}$$

Standard derivatives

Theorem

$f(x)$	$f'(x)$
k	0
x^p	px^{p-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\frac{1}{\cos(x)^2}$
e^x	e^x

Standard derivatives

Example

- $f(x) = a^x = (\mathrm{e}^{\ln a})^x = \mathrm{e}^{\ln a \cdot x}$
- Outer function $y \mapsto \mathrm{e}^y$, inner function $x \mapsto \ln a \cdot x$.
- $f'(x) = \ln a \cdot \mathrm{e}^{\ln a \cdot x} = \ln(a) \cdot a^x$.

Rules of derivation

The chain rule can also be used for compositions of more than two functions.

Example

- Compute the derivative of $\mathrm{e}^{(\tan x)^2}$ with respect to x.
- Write $\mathrm{e}^{\tan x^2} = f \circ g \circ h(x)$, where $h(x) = \tan x$, $g(y) = y^2$, and $f(z) = \mathrm{e}^z$.

Rules of derivation

The chain rule can also be used for compositions of more than two functions.

Example

- Compute the derivative of $\mathrm{e}^{(\tan x)^2}$ with respect to x.
- Write $\mathrm{e}^{\tan x^2} = f \circ g \circ h(x)$, where $h(x) = \tan x$, $g(y) = y^2$, and $f(z) = \mathrm{e}^z$.
- $h'(x) = \frac{1}{\cos(x)^2}$, $g'(y) = 2y$, and $f'(z) = \mathrm{e}^z$.
- $$\begin{aligned}\frac{d}{dx}\mathrm{e}^{(\tan x)^2} &= f'(g(h(x))) \cdot g'(h(x)) \cdot h(x) \\ &= \mathrm{e}^{(\tan x)^2} \cdot 2\tan x \cdot \frac{1}{\cos(x)^2}.\end{aligned}$$

Rules of derivation

A good way to remember (and intuitively understand) the chain rule:

- $$\frac{df(g(x))}{dx} = \frac{df(g(x))}{dg(x)} \cdot \frac{dg(x)}{dx}.$$
- $$\frac{d\mathrm{e}^{(\tan x)^2}}{dx} = \frac{d\mathrm{e}^{(\tan x)^2}}{d(\tan x)^2} \cdot \frac{d(\tan x)^2}{d\tan x} \cdot \frac{d\tan x}{dx}.$$

Derivatives of inverse functions

- Is there a relation between the derivatives of $f : A \to B$ and $f^{-1} : B \to A$?
- Note that the identity function $g(x) = x$ can be written as $g(x) = f(f^{-1}(x))$.
- By the chain rule,

$$1 = g'(x) = f'(f^{-1}(x)) \cdot (f^{-1})'(x).$$

- So $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$.

Rules of derivation

Theorem

Let f and g be differentiable functions, and let c be a constant. Then:

- $(g \circ f)'(x) = g'(f(x))f'(x)$.
- $(f^{-1})'(x) = \frac{1}{f'(y)}$, *where* $f(y) = x$.

Standard derivatives

Example

- $f(x) = \ln x$ is the inverse of $g(y) = \mathrm{e}^y$.
- So
$$f'(x) = \frac{1}{g'(y)} = \frac{1}{e^y},$$
where $g(y) = \mathrm{e}^y = x$.
-
$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

Standard derivatives

Example

- $f(x) = \arcsin x$ is the inverse of $g(y) = \sin y$.
- So
$$f'(x) = \frac{1}{g'(y)} = \frac{1}{\cos y},$$
where $g(y) = \sin y = x$.
- $x^2 + \cos(y)^2 = 1$, so $\cos y = \pm\sqrt{1-x^2}$.
- Since $y = \arcsin x$ is defined to satisfy $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, we have $\cos y > 0$, so $\cos y = \sqrt{1-x^2}$.
- $$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}.$$

Standard derivatives

Example

- $f(x) = \arccos x$ is the inverse of $g(y) = \cos y$.
- So
$$f'(x) = \frac{1}{g'(y)} = \frac{1}{-\sin y},$$
where $g(y) = \cos y = x$.
- $x^2 + \sin(y)^2 = 1$, so $\sin y = \pm\sqrt{1 - x^2}$.
- Since $y = \arccos x$ is defined to satisfy $0 \leq x \leq \pi$, we have $\sin y > 0$, so $\sin y = \sqrt{1 - x^2}$.
- $$\frac{d}{dx} \arccos x = \frac{1}{-\sqrt{1 - x^2}}.$$

Standard derivatives

Example

- $f(x) = \arctan x$ is the inverse of $g(y) = \tan y$.
- So
$$f'(x) = \frac{1}{g'(y)} = \cos(y)^2,$$
where $g(y) = \tan y = x$.
-
$$\cos(y)^2 = \frac{\cos(y)^2}{\sin(y)^2 + \cos(y)^2} = \frac{1}{\tan(y)^2 + 1} = \frac{1}{x^2 + 1}.$$
-
$$\frac{d}{dx} \arctan x = \frac{1}{1 + x^2}.$$

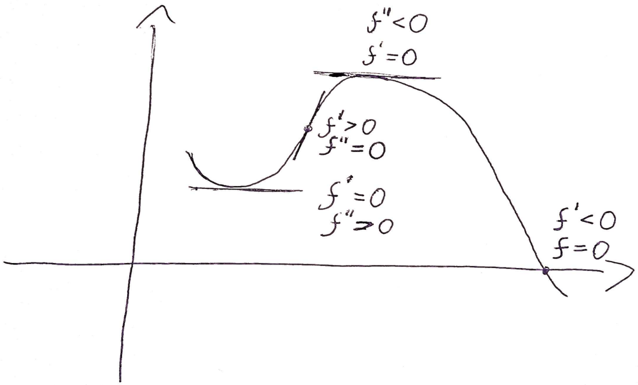
Standard derivatives

Theorem

$f(x)$	$f'(x)$
a^x	$\ln(a) \cdot a^x$
$\ln x$	$\frac{1}{x}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$\frac{1}{-\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$

Extreme values

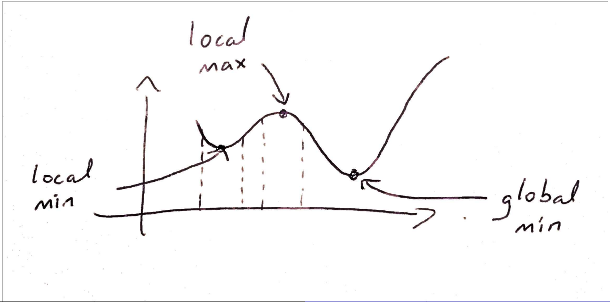
Derivatives can be used to find the extremal values (maximum and minimum) of a function.



Extreme values

Definition

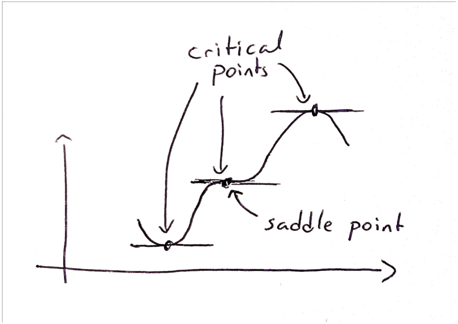
- $a \in A$ is a *local maximum* for the function $f : A \to \mathbb{R}$ if there is an ϵ such that $f(a) \geq f(x)$ for all $x \in (a - \epsilon, a + \epsilon)$.
- $a \in A$ is a *global maximum* for the function $f : A \to \mathbb{R}$ if $f(a) \geq f(x)$ for all $x \in A$.
- Local and global minima are defined analogously.



Critical points

Definition

Assume that $f : A \to \mathbb{R}$ is differentiable in $a \in A$, and that a is not a boundary point of A. Then $a \in A$ is a *critical point* for the function f if $f'(a) = 0$.



Critical points

Theorem

Assume that f is differentiable in a, and that a is not a boundary point of A. If a is a local maximum or a local minimum of f, then $f'(a) = 0$.

- "Proof": If $f'(a) > 0$, then $\frac{f(a+h)-f(a)}{h} > 0$ for all $|h| < \epsilon$ if ϵ is small enough.
- Thus $f(a+h) > f(a)$ for $0 < h < \epsilon$, and $f(a+h) < f(a)$ for $-\epsilon < h < 0$.
- So f can not be neither local min or local max.
- The proof if $f'(a) < 0$ is analogous.

Critical points

Theorem

Assume that $f : A \to \mathbb{R}$ is differentiable in a, and that a is not a boundary point of A. If a is a local maximum or a local minimum of f, then $f'(a) = 0$.

- So to find all the local extreme points of f, we only need to find all critical points, and all points where f is not differentiable.

Optimization

Example

- Find all the local minima and maxima of $f(x) = 3x^4 - 4x^3$.
- Since f is a polynomial, it is differentiable everywhere, so we only need to find points with $f'(x) = 0$.
- $f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1)$.
- The zeroes of f' are the zeroes of its linear factors, so

$$f'(x) = 0 \iff x = 0 \text{ or } x = 1.$$

Optimization

Example (Continued)

- Find all the local minima and maxima of $f(x) = 3x^4 - 4x^3$.
- $$f'(x) = 0 \Longrightarrow x = 0 \text{ or } x = 1.$$
- We study the signs of f' between the critical points:

x		0		1	
$f' = 12x^2(x-1)$	$-$	0	$-$	0	$+$
$f = 3x^4 - 4x^3$	$\searrow$	$\rightarrow$	$\searrow$	$\rightarrow$	$\nearrow$

- So $x = 0$ is a saddle point, $x = 1$ is a local minimum, and there are no local maxima.

Optimization

Example (Continued)

- Find the global minimum and maximum of $f(x) = 3x^4 - 4x^3$, if they exist.
- Since there is no local maximum, there can also be no global maximum.
- Since ("$\infty \cdot \infty$")

$$f = x^3(3x - 4) \xrightarrow[x \to \infty]{} \infty,$$

  f is not even upper bounded.
- The local minimum at 1 is also a global minimum, because f is decreasing to the left of 1, and increasing to the right of 1.

Optimization

Example (Continued)

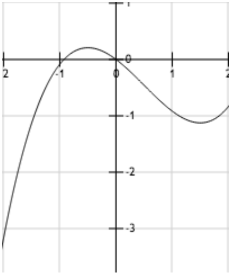
- Find the range of $f(x) = 3x^4 - 4x^3$.
- The smallest value is $f(1) = -1$.
- The function is continuous on $\mathbb{R}$, so its range is an interval.
- The function is not upper bounded, so its range is $[-1, \infty)$.

Optimization

Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous on the closed interval $[a, b]$, then f has a global maximum and a global minimum on $[a, b]$.

- If f is differentiable, then the global extreme points are either obtained in the boundary points a or b, or in the critical points of f.



Optimization

Example

- Find the global minima and maxima of $f(x) = 4x^3 - 6x^2 - 9x$ on the interval $[-2, 2]$.
- First, we find the critical points of f.
- $$f'(x) = 12x^2 - 12x - 9 = 3(4x^2 - 4x - 3) = 3\left((2x-1)^2 - 4\right).$$
- $$f'(x) = 0 \iff 2x - 1 = \pm 2 \iff x = \frac{3}{2} \text{ or } x = -\frac{1}{2}.$$

Optimization

Example (Continued)

- Find the global minima and maxima of $f(x) = 4x^3 - 6x^2 - 9x$ on the interval $[-2, 2]$.
- $$f'(x) = 0 \iff 2x - 1 = \pm 2 \iff x = \frac{3}{2} \text{ or } x = -\frac{1}{2}.$$
- The global extreme points are among
$$\left\{-2, -\frac{1}{2}, \frac{3}{2}, 2\right\}.$$

Optimization

Example (Continued)

- Find the global minimum and maximum of $f(x) = 4x^3 - 6x^2 - 9x$ on the interval $[-2, 2]$.
- The global extreme points are among

$$\left\{-2, -\frac{1}{2}, \frac{3}{2}, 2\right\}.$$

- We compute

$$f(-2) = -38\,, f(-\frac{1}{2}) = \frac{5}{2}\,, f(\frac{3}{2}) = -\frac{27}{2}\,, f(2) = -10.$$

- The minimum value is -38 and the maximum value is $\frac{5}{2}$.

Optimization

Theorem

Assume a is a critical point of f.

- *If $f''(a) < 0$, then a is a local maximum of f.*
- *If $f''(a) > 0$, then a is a local minimum of f.*

- Thus we can often detect whether a critical point is a local maximum or a local minimum, without computing f or f' between the critical points.
- If $f''(a) = 0$, then a can be a local minimum, a local maximum, a saddle point, or neither.

Trigonometric values

Good to remember:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	$-\infty$	0

$$\begin{array}{l|l} \sin -x = -\sin x & \sin(x+\pi) = -\sin x \\ \cos -x = \cos x & \cos(x+\pi) = -\cos x \\ \tan -x = -\tan x & \tan(x+\pi) = \tan x. \end{array}$$

Optimization

Example

- Find all the local minima and maxima of $f(x) = x + 2\sin x$.
- f is differentiable everywhere, so we only need to find points with $f'(x) = 0$.
- $$\begin{aligned} f'(x) &= 1 + 2\cos x = 0 \\ \iff \cos x &= -\frac{1}{2} \\ \iff x &= \frac{2\pi}{3} + 2\pi n \text{ or } x = \frac{-2\pi}{3} + 2\pi n \text{ for } n \in \mathbb{Z}. \end{aligned}$$

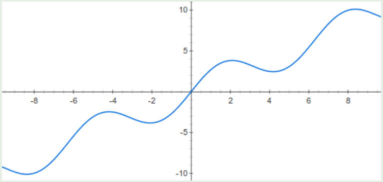
Optimization

Example (Continued)

- $f(x) = x + 2\sin x$, $f'(x) = 1 + 2\cos x$.
- We study the signs of $f''(x) = -2\sin x$ in the critical points:

x	$\frac{2\pi}{3} + 2\pi n$	$\frac{-2\pi}{3} + 2\pi n$
$f''(x)$	$-$	$+$

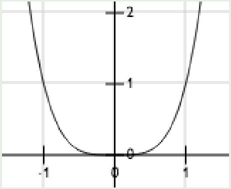
- So $x = \frac{2\pi}{3} + 2\pi n$ is local maximum and $\frac{-2\pi}{3} + 2\pi n$ is a local minimum, for every $n \in \mathbb{Z}$.



Critical points

Example

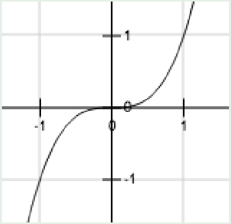
- $f(x) = x^4$
- $f'(0) = f''(0) = 0$
- 0 local minimum, because $f(0) = 0 \leq x^4 = f(x)$ for all x.



Critical points

Example

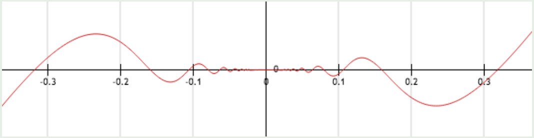
- $f(x) = x^3$
- $f'(0) = f''(0) = 0$
- 0 saddle point, because $f(-x) = -x^3 < 0 \leq x^3 = f(x)$ for all $x > 0$.



Critical points

Example

$$f(x) = \begin{cases} x^4 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



Critical points

Example (Continued)

$$f(x) = \begin{cases} x^4 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

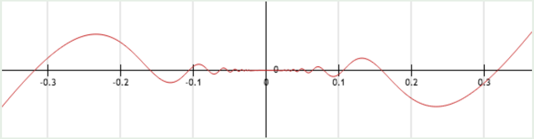
$$f'(x) = \begin{cases} 4x^3 \sin\left(\frac{1}{x}\right) + x^4 \frac{-1}{x^2} \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ \lim_{h\to 0} \frac{h^4 \sin\left(\frac{1}{h}\right) - 0}{h} = 0 & \text{if } x = 0 \end{cases}$$

$$f''(0) = \lim_{h\to 0} \frac{4h^3 \sin\left(\frac{1}{h}\right) - h^2 \cos\left(\frac{1}{h}\right) - 0}{h} = 0$$

Critical points

Example (Continued)

- $$f(x) = \begin{cases} x^4 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
- $f'(0) = f''(0) = 0$, and 0 is neither a local optimum or a saddle point!

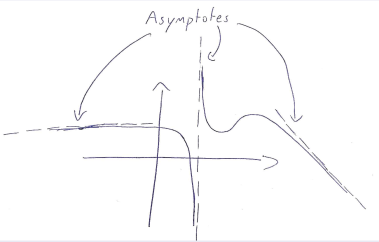


Asymptotes

Definition

An asymptote of the function f is a straight line L such that there is a sequence of points $(x_n, f(x_n))$ on the function graph such that

- Distance between $(x_n, f(x_n))$ and 0 tends to ∞
- Distance between $(x_n, f(x_n))$ and L tends to 0



Asymptotes

- There are two kinds of asymptotes: vertical and slant asymptotes (including horizontal ones).
- Vertical asymptotes $x = c$ exist if

$$f(x) \xrightarrow[x \to c^{\pm}]{} \infty.$$

- Slant asymptotes $y = ax + b$ exist if

$$f(x) - ax \xrightarrow[x \to \pm\infty]{} b.$$

- If $y = ax + b$ is a slant asymptote of f, then $f'(x) \xrightarrow[x \to \pm\infty]{} a$.

Asymptotes

Example

- Plot the function $f(x) = \frac{x^3-1}{x^3-x}$
    - Domain
    - Zeroes
    - Critical points and their types (minima, maxima, saddle...)
    - Asymptotes

Sketching a function graph

Example (Continued)

- Plot the function

$$f(x) = \frac{x^3 - 1}{x^3 - x} = \frac{(x-1)(x^2+x+1)}{(x-1)x(x+1)}.$$

- The domain is the set where the denominator is non-zero, the set $A = \mathbb{R} \setminus \{-1, 0, 1\}$.
- On the domain A we can cancel the factors $x - 1$, so f is equal to

$$\frac{x^2+x+1}{x(x+1)}.$$

Sketching a function graph

Example (Continued)

- On the domain $A = \mathbb{R} \setminus \{-1, 0, 1\}$ f is equal to

$$\frac{x^2 + x + 1}{x^2 + x}.$$

- The numerator is positive for all x, so f is nowhere zero.

Sketching a function graph

Example (Continued)

- On the domain $A = \mathbb{R} \setminus \{-1, 0, 1\}$ f is equal to

$$\frac{x^2 + x + 1}{x^2 + x}.$$

- We get the derivative

$$f'(x) = \frac{(2x+1)(x^2+x) - (x^2+x+1)(2x+1)}{(x^2+x)^2} = \frac{-2x-1}{(x^2+x)^2}.$$

- Since the denominator is strictly positive on A, we have $f'(x) = 0$ if and only if $-2x - 1 = 0$.
- So the only critical point is $x = \frac{-1}{2}$.

Sketching a function graph

Example (Continued)

- $$f'(x) = \frac{(2x+1)(x^2+x)-(x^2+x+1)(2x+1)}{(x^2+x)^2} = \frac{-2x-1}{(x^2+x)^2}.$$

- Sign study:

x		$\frac{-1}{2}$	
f'	$+$	0	$-$
f	$\nearrow$	$\rightarrow$	$\searrow$

- So $x = \frac{-1}{2}$ is a local maximum, with

$$f\left(\frac{-1}{2}\right) = \frac{\frac{-1}{2}^2 + \frac{-1}{2} + 1}{\frac{-1}{2}\left(\frac{-1}{2}+1\right)} = \frac{3/4}{-1/4} = -3$$

Sketching a function graph

Example (Continued)

- On the domain $A = \mathbb{R} \setminus \{-1, 0, 1\}$ f is equal to

$$\frac{x^2 + x + 1}{x(x+1)}.$$

- In the singular points (where f is not defined), we get the limits

$$\lim_{x \to -1^-} f(x) = \lim_{x \to 0^+} f(x) = \infty,$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to 0^-} f(x) = -\infty,$$

and

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 + x + 1}{x(x+1)} = \frac{3}{2}.$$

Sketching a function graph

Example (Continued)

- We have two vertical asymptotes $x = -1$ and $x = 0$, as f has an infinite limit in those points.
- To find slant asymptotes, we must first compute

$$\lim_{x \to \pm\infty} f'(x) = \lim_{x \to \pm\infty} \frac{-2x - 1}{(x^2 + x)^2} = 0.$$

- So any slant polytope has the slope 0, and so it has the form

$$y = b = \lim_{x \to \pm\infty} f(x).$$

Sketching a function graph

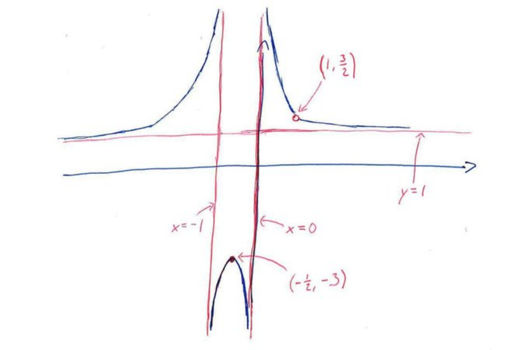
Example (Continued)

- $$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^2+x+1}{x^2+x} = \lim_{x\to\pm\infty} \frac{1+\frac{1}{x}+\frac{1}{x^2}}{1+\frac{1}{x}} = 1.$$
- So we have a unique slant (actually, horizontal) asymptote $y = 1$

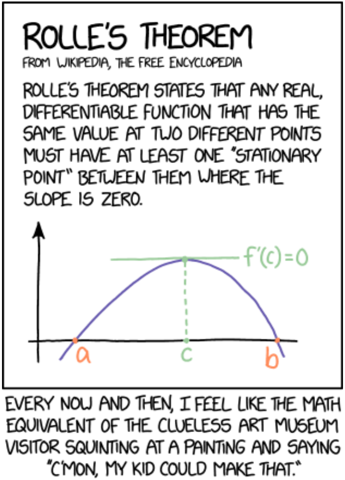
Sketching a function graph

Example (Continued)

$$y = f(x) = \frac{x^3 - 1}{x^3 - x}$$



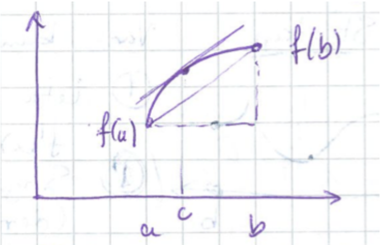
Rolle's Lemma (as told by XKCD)



Mean value theorem

Theorem

If f is differentiable on $[a, b]$, then there is $c \in [a, b]$ with $f'(c)(b - a) = f(b) - f(a)$.



Mean value theorem

Theorem

If f is differentiable on $[a, b]$, then there is $c \in [a, b]$ with $f'(c)(b-a) = f(b) - f(a)$.

- Proven using Rolle's lemma on

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a)$$

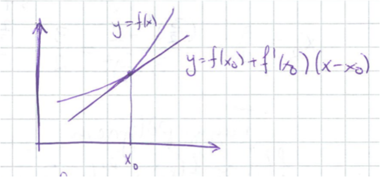
- Used to bound the error term in approximations.

Linear approximation

- Assume we want to compute $f(x)$, knowing $f(x_0)$ for some point x_0
- If x is close to x_0, then

$$f'(x_0) \approx \frac{f(x) - f(x_0)}{x - x_0}$$

- Rewrite $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$.



Newton-Raphson method

- Linear approximation: $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$.
- Again, this suggests an algorithm to approximate $f(x) = 0$.
- Assume $f(x) = 0$, and that $x - |x_0|$ is small. Then $f(x_0) \approx -f'(x_0)(x - x_0)$.
- So $x - x_0 \approx \frac{-f(x_0)}{f'(x_0)}$.
- Set

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

and repeat.

Newton-Raphson method

- Task: Find a such that $f(a) = 0$.
- Algorithm: Start with a point x_0, such that $f'(x)$ has the same sign on all of $[x_0, a]$ (or $[a, x_0]$, if a is to the left of x_0).
- Given x_n, let
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
- Continue for as many steps as you see fit.
- If x_0 is sufficiently close to a, then the the sequence converges to a. Proving that the sequence converges is often rather difficult.

Newton-Raphson method

Example

- Approximate $2^{\frac{1}{3}}$.
- Want to find a such that $f(a) = 0$, where $f(x) = x^3 - 2$.
- $$\frac{f(x)}{f'(x)} = \frac{x^3 - 2}{3x^2} = \frac{x}{3} - \frac{2}{3x^2}$$
- Given x_n in the Newton-Raphson algorithm,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n}{3} + \frac{2}{3x_n^2} = \frac{2}{3}\left(x_n + \frac{1}{x_n^2}\right).$$

Newton-Raphson method

Example

- Approximate $2^{\frac{1}{3}}$.
- Choose $x_0 = 1$.
- $$x_{n+1} = \frac{2}{3}\left(x_n + \frac{1}{x_n^2}\right).$$
- $x_1 = \frac{2}{3}(1+1) = \frac{4}{3}$.
- $x_2 = \frac{2}{3}\left(\frac{4}{3} + \frac{9}{16}\right) = \frac{182}{144} = \frac{91}{72}$.
- We see that
$$\left(\frac{91}{72}\right)^3 \approx 2.019,$$
so already after two iterations we have a good approximation!

Newton-Raphson method

Example

- We prove that $x_3 = \frac{91}{72}$ is a good approximation by the intervediate value theorem:

$$\left(\frac{90}{72}\right)^3 = \frac{5}{4}^3 = \frac{125}{64} < 2 < \left(\frac{91}{72}\right)^3.$$

- $$\frac{90}{72} < 2^{\frac{1}{3}} < \frac{91}{72},$$

so our approximation has an error of at most $1/72 \approx 0.0139$.

Taylor polynomials

- Estimate $f(x)$ close to some point a, where f is easy to evaluate.
  - For now, $a = 0$.
- Easiest possible approximation: $f(x) \approx f(0)$.
- We approximate f by a constant function (degree 0 polynomial)

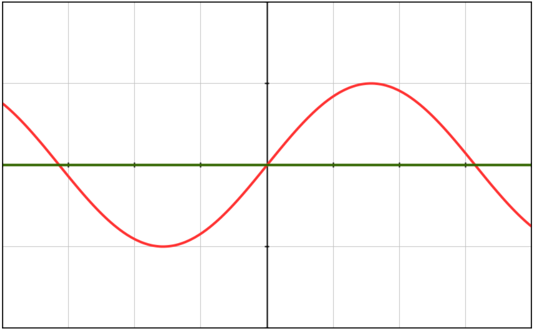
$$f(x) \approx T_0(x) \equiv f(0).$$

- We call this a “degree 0 approximation”.

Taylor polynomials

$$y = T_0(x) = 0$$

$$y = f(x) = \sin x$$



Taylor polynomials

- Next best approximation is linear:

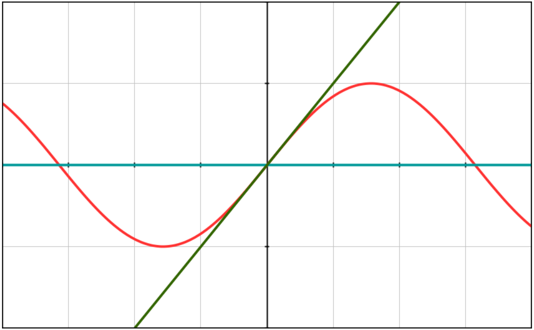
$$f(x) \approx T_1(x) = f(0) + f'(0)x.$$

- T_1 is the *only linear function* such that
  - $T_1(0) = f(0)$
  - $T_1'(0) = f'(0)$

Taylor polynomials

$$y = T_1(x) = x$$

$$y = f(x) = \sin x$$



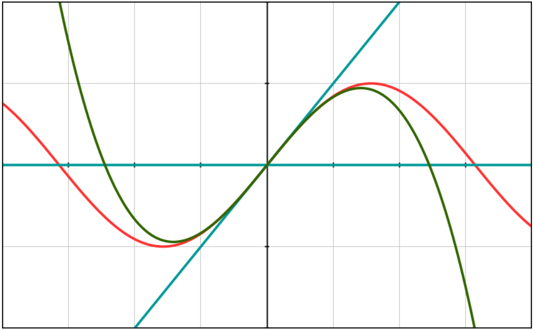
Taylor polynomials

- Goal: find a polynomial $T_n(x)$ of degree n that approximates f as well as possible near 0:
  - $T_n(0) = f(0)$
  - $T_n'(0) = f'(0)$
  - $\cdots$
  - $T_n^{(n)}(0) = f^{(n)}(0)$
- $T_n(x)$ will be called the *degree n Taylor polynomial of f around* 0.

Taylor polynomials

$$y = T_3(x) = x - \frac{x^3}{6}$$

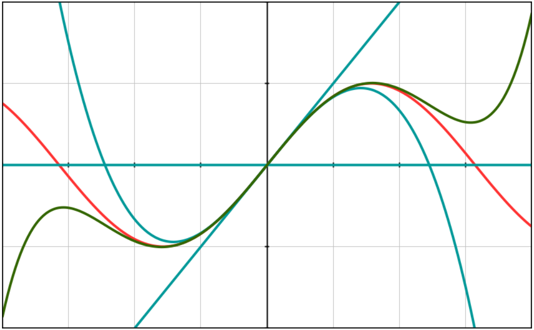
$$y = f(x) = \sin x$$



Taylor polynomials

$$y = T_3(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$$

$$y = f(x) = \sin x$$



Taylor polynomials

- Goal: find a polynomial

$$T_n(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \cdots + c_n x^n$$

of degree n that approximates f as well as possible near 0.

- Derivatives:

$$\begin{aligned}
T_n(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 \cdots + c_n x^n \\
T_n'(x) &= c_1 + 2c_2 x + 3c_3 x^2 \cdots + nc_n x^{n-1} \\
T_n''(x) &= 2c_2 + 6x + \cdots n(n-1)x^{n-2} \\
&\cdots \\
T_n^{(n)}(x) &= n!c_n
\end{aligned}$$

- Derivatives at 0:

$$\begin{aligned}
f(0) &= T_n(0) &= c_0 \\
f'(0) &= T_n'(0) &= c_1 \\
f''(0) &= T_n''(0) &= 2c_2 \\
&\cdots \\
f^{(n)}(0) &= T_n^{(n)}(0) &= n!c_n
\end{aligned}$$

Taylor polynomials

- So the degree n Taylor polynomial of f around 0 is

$$T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

- This is the best possible degree n approximation of f, in that

$$\frac{f(x) - T_n(x)}{x^n} \xrightarrow[x \to 0]{} 0,$$

assuming $f^{(n)}$ is continuous around 0.

Taylor polynomials

- More generally, the degree n Taylor polynomial of f around a is

$$T_n(x;a) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

- This is the best possible degree n approximation of f, in that

$$\frac{f(x) - T_n(x;a)}{(x-a)^n} \xrightarrow[x\to a]{} 0,$$

assuming $f^{(n)}$ is continuous around a.

Error terms in Taylor polynomials

- Better estimate of $f(x) - T_n(x)$:

Theorem (Taylor's Theorem)

Let T_n be the degree n Taylor polynomial of f around 0. Then

$$f(x) - T_n(x) = \frac{f^{(n+1)}(s)}{(n+1)!}x^{n+1}$$

for some s in the interval between a and x.

- Sketch of proof:
  - Consider $F(s) = f(x) - T_n(x; s)$ as a function of $s \in [0, x]$.
  - Use Rolle's Lemma (or the mean value theorem) on $F(s)$.

Error terms in Taylor polynomials

- The same holds for the Taylor polynomial around any point a:

Theorem (Taylor's Theorem)

Let $T_n(\cdot; a)$ be the degree n Taylor polynomial of f around a. Then

$$f(x) - T_n(x; a) = \frac{f^{(n+1)}(s)}{(n+1)!}(x-a)^{n+1}$$

for some s in the interval between a and x.

Error terms in Taylor polynomials

Theorem (Taylor's Theorem)

Let $T_n(\cdot; a)$ be the degree n Taylor polynomial of f around a. Then

$$f(x) - T_n(x; a) = \frac{f^{(n+1)}(s)}{(n+1)!}(x-a)^{n+1}$$

for some s in the interval between a and x.

Corollary

If $|f^{(n+1)}(s)| < M$ for every $s \in [a, x]$, then

$$|f(x) - T_n(x; a)| < \frac{M(x-a)^{n+1}}{(n+1)!}.$$

Taylor series

Corollary

If x is such that

$$\max_{s\in[a,x]} \frac{|f^{(n+1)}(s)|}{(n+1)!}(x-a)^{n+1} \xrightarrow[n\to\infty]{} 0,$$

then

$$|f(x) - T_n(x;a)| \xrightarrow[n\to\infty]{} 0,$$

and so we can write

$$f(x) = \lim_{n\to\infty} T_n(x;a) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Taylor series

- The expression
$$T(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$
is called the *Taylor series* of f at a.
- By convention: $f^{(0)}(x) = f(x)$ and $0! = 1$.
- The Taylor series does not always converge, and even if it does, it might not converge to $f(x)$.

Taylor series

- If the error term

$$\max_{s\in[a,x]} \frac{|f^{(n+1)}(s)|}{(n+1)!}(x-a)^{n+1} \xrightarrow[n\to\infty]{} 0,$$

  then $T(x)$ converges and $f(x) = T(x)$.
- This criterion holds for all polynomials, trigonometric functions, exponential functions, etc...

Taylor series

- Recall the quotient criterion for series convergence: If

$$\frac{f^{(n+1)}(x)}{(n+1)!}(x-a)^{n+1} \Big/ \frac{f^{(n)}(x)}{(n!)}(x-a)^n$$
$$=\frac{f^{(n+1)}(x)}{(n+1)f^{(n)}(x)}(x-a) \xrightarrow[n\to\infty]{} t \in (-1,1),$$

then

$$T(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

converges.

Taylor series

- If
$$\frac{f^{(n+1)}(x)}{(n+1)f^{(n)}(x)}(x-a) \xrightarrow[n\to\infty]{} t \in (-1,1),$$
then $T(x)$ converges.
- In particular, this holds if
$$|x-a| < \lim_{n\to\infty} \frac{(n+1)f^{(n)}(x)}{f^{(n+1)}(x)}$$

Maclaurin series

- The Taylor series of f at $a = 0$, i.e.

$$T(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k,$$

is also known as the *Maclaurin series* of f.

Error terms in Taylor polynomials

Example

- Estimate $\sin 10^\circ$ using the degree 4 Taylor series around 0.
-

	$f^{(i)}(x)$	$f^{(i)}(0)$
f	$\sin x$	0
f'	$\cos x$	1
f''	$-\sin x$	0
f'''	$-\cos x$	-1
f''''	$\sin x$	0

- So

$$T_4(x) = 0 + 1x + \frac{0}{2}x^2 + \frac{-1}{6}x^3 + \frac{0}{24}x^4 = x - \frac{x^3}{6}$$

Error terms in Taylor polynomials

Example

- $T_4(10°) = T_4(\frac{\pi}{18}) = \frac{\pi}{18} - \frac{\pi^3}{6 \cdot 18^3} \approx 0.173646.$
- Error term
$$f(\frac{\pi}{18}) - T_4(\frac{\pi}{18}) = \frac{f^{(5)}(s)}{120} s^5$$
for some $s \in [0, \frac{\pi}{18}]$, $f(x) = \sin x$.
- $f^{(5)}(s) = \cos(s) \leq 1$ and $s^5 \leq \frac{\pi^5}{18^5}$.
- So the error term is at most $\frac{\pi^5}{120 \cdot 18^5} \approx 0.0000013.$

Maclaurin series of e^x

- Let f be a function such that $f'(x) = f(x)$ and $f(0) = 1$.
- Then $f^{(k)}(0) = 1$ for every k, so the Maclaurin series is

$$T(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

- In particular

$$f(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} = \mathrm{e}.$$

Maclaurin series of e^x

- The Taylor series around a is

$$S(a+b) = f(a) + f'(a)b + f''(a)\frac{b^2}{2} + f'''(a)\frac{b^3}{6} + \cdots$$
$$= f(a) + f(a)b + f(a)\frac{b^2}{2} + f(a)\frac{b^3}{6} + \cdots = f(a)T(b).$$

- But f, T, and S represent the same function so $f(a+b) = f(a)f(b)$.
- So f is an exponential function, with $f(1) = e \implies f(x) = e^x$.

Maclaurin series of $\sin x$

	$f^{(i)}(x)$	$f^{(i)}(0)$
f	$\sin x$	0
f'	$\cos x$	1
f''	$-\sin x$	0
f'''	$-\cos x$	-1
	$\dots$	$\dots$
$f^{(2k)}$	$\pm\sin x$	0
$f^{(2k+1)}$	$(-1)^k \cos x$	$(-1)^k$
- So $\sin x$ has the Maclaurin series

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Maclaurin series of $\cos x$

	$f^{(i)}(x)$	$f^{(i)}(0)$
f	$\cos x$	1
f'	$-\sin x$	0
f''	$-\cos x$	-1
f'''	$\sin x$	0
	$\dots$	$\dots$
$f^{(2k)}$	$(-1)^k \cos x$	$(-1)^k$
$f^{(2k+1)}$	$\pm \sin x$	0
- So $\cos x$ has the Maclaurin series

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

Maclaurin series of $\ln(x+1)$

	$f^{(i)}(x)$	$f^{(i)}(0)$
f	$\ln(x+1)$	0
f'	$(x+1)^{-1}$	1
f''	$-(x+1)^{-2}$	-1
f'''	$2(x+1)^{-3}$	2
f''''	$-6(x+1)^{-4}$	-6
	$\dots$	$\dots$
$f^{(k)}$	$(-1)^{k-1}(k-1)!(x+1)^{-k}$	$(-1)^{k-1}(k-1)!$

- So $\ln(x+1)$ has the Maclaurin series

$$\ln(x+1) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k-1)!x^k}{k!} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k}$$

Standard Maclaurin series

Theorem

$f(x)$	*Series*	*Convergence*
Polynomial $P(x)$	$P(x)$	*All* x
e^x	$\sum_{k\geq 0} \frac{x^k}{k!}$	*All* x
$\sin x$	$\sum_{k\geq 0} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$	*All* x
$\cos x$	$\sum_{k\geq 0} \frac{(-1)^k x^{2k}}{(2k)!}$	*All* x
$\ln(x+1)$	$\sum_{k\geq 1} \frac{(-1)^{k+1} x^k}{k}$	$-1 < x \leq 1.$

Taylor series

- Taylor series can be composed and multiplied

Example

$$\sin(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k (x^2)^{2k+1}}{(2k+1)!} = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots .$$

$$x^2 \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^2 \cdot x^{2k+1}}{(2k+1)!} = x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \cdots .$$

Taylor series

- Taylor series can also be differentiated termwise.

Example

$$\frac{d}{dx}\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k \frac{d}{dx} x^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1) x^{2k}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = \cos x.$$

Taylor series

- One can also use Taylor series to compute limits (as $x \to a$).

Example

- Compute
$$\lim_{x\to 0} \frac{x^2\sin(x^2)}{(1-\cos x)^2}$$
- $$x^2\sin(x^2) = x^2\left(x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \cdots\right) = x^4 + o(x^4).$$
- $$(1-\cos x)^2 = \left(-\frac{x^2}{2} + \frac{x^4}{24} - \cdots\right)^2 = \frac{x^4}{4} + o(x^4).$$

- $f(x) = o(x^p)$ means $\lim_{x\to 0} \frac{f(x)}{x^p} = 0$.

Taylor series

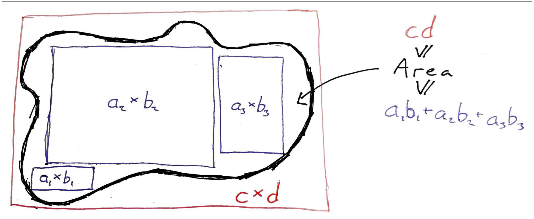
Example (Continued)

- $$x^2\sin(x^2) = x^2\left(x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \cdots\right) = x^4 + o(x^4).$$
- $$(1-\cos x)^2 = \left(-\frac{x^2}{2} + \frac{x^4}{24} - \cdots\right)^2 = \frac{x^4}{4} + o(x^4).$$
- So
$$\lim_{x\to 0}\frac{x^2\sin(x^2)}{(1-\cos x)^2} = \lim_{x\to 0}\frac{x^4 + o(x^4)}{\frac{x^4}{4} + o(x^4)} = \lim_{x\to 0}\frac{1+o(1)}{\frac{1}{4}+o(1)} = 4.$$

Integrals

- What is an area?
- The area of an $a \times b$ rectangle is ab.
- The area of a disjoint union of regions is the sum of their area.
- If a region A fits inside B, then B has larger area than A.
- This is enough to define the area of many regions in the plane.

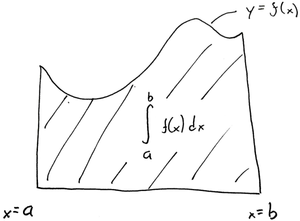
Integrals



- We are interested in the area of the black region R.
- This is smaller than the area of any rectangular region containing R, and larger than the area of any rectangular region contained in R.
- If there is a number A such that we can make both the red and the blue area arbitrarily close to A, then we say that this is the area of the black region.

Integrals

- The integral of a (positive) function f between a and b with respect to x is the area between the x-axis and the function graph.

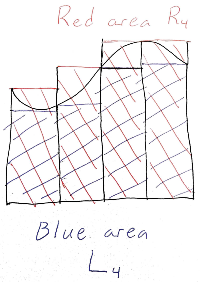


- Notation:

$$\int_a^b f(x)dx.$$

Integrals

- Subdivide the interval from a to b into n parts.
- Look at a function that is $\geq f$ and constant on each interval. Denote the area under it by R_n.
- Look at a function that is $\leq f$ and constant on each interval. Denote the area under it by L_n.



- If

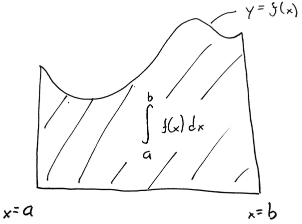
$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n,$$

then f is *integrable*. We call this limit

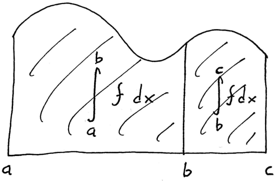
$$\int_a^b f(x)dx.$$

Integrals

- The integral $\int_a^b f(x)dx$ depends on:
  - The function f.
  - The endpoints a and b.
- $$\int_a^b f + g \ dx = \int_a^b f \ dx + \int_a^b g \ dx.$$
- $$\int_a^b cf \ dx = c \int_a^b f \ dx$$
  where $c \in [0, \infty)$ is a constant.

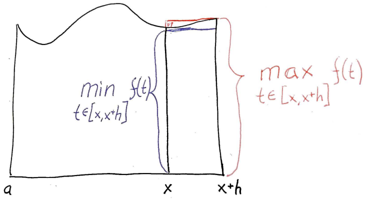


Integrals



- How does $\int_a^b f(x)dx$ depend on the endpoints?
- $$\int_a^b f(t)dt + \int_b^c f(t)dt = \int_a^c f(t)dt.$$

Integrals



$$h \min_{t\in[x,x+h]} f(t) \quad \leq \int_a^{x+h} f(t)dt - \int_a^{x} f(t)dt \leq \quad h \max_{t\in[x,x+h]} f(t)$$

Integrals

$$f(x) \leftarrow \min_{t\in[x,x+h]} f(t) \leq \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} \leq \max_{t\in[x,x+h]} f(t) \rightarrow f(x)$$

So by the policemen's lemma:

$$\frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} \xrightarrow[x\rightarrow 0]{} f(x).$$

Primitive functions

Theorem (Fundamental Theorem of Calculus)

Let f be a continuous function on a neighbourhood of $[a, b]$, and $x \in (a, b)$. Then $\frac{d}{dx} \int_a^x f(t)dt = f(x)$.

Corollary

Let $F : [a, b] \to \mathbb{R}$ be a differentiable function such that $F'(x) = f(x)$ for every $x \in (a, b)$. Then

$$\int_a^b f(x)dx = F(b) - F(a).$$

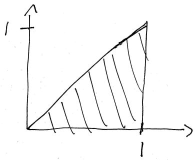
Primitive functions

- So to compute an integral of f, we only need to find a function F whose derivative is f.
- Such a function F is called a *primitive function* of f.
- If F is a primitive function of f, then all primitive functions are given by $F + c$, where c is a constant.

Primitive functions

Example

- Compute $\int_0^1 x\ dx$.
- A primitive function of $f(x) = x$ is $F(x) = \frac{x^2}{2}$.
- So
$$\int_0^1 x\ dx = \left[\frac{x^2}{2}\right]_0^1 \overset{\text{notation}}{=} \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2} - 0 = \frac{1}{2}.$$

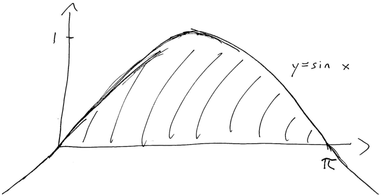


Primitive functions

Example

- Compute $\int_0^\pi \sin x \, dx$.
- A primitive function of $f(x) = \sin x$ is $F(x) = -\cos x$.
- So

$$\int_0^\pi \sin x \, dx = [-\cos x]_0^\pi = -\cos\pi - (-\cos 0) = 1 - (-1) = 2.$$



Integrals of negative functions

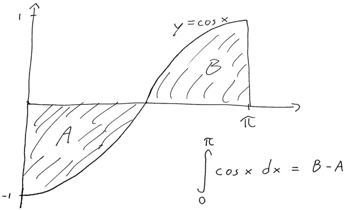
- If $f(x)$ is negative on $[a,b]$ then $\int_a^b$ is the *negative* area between the x-axis and the function graph.
- The standard rules still hold:
  - $\int (f+g)dx = \int fdx + \int gdx$
  - $\int_a^b f \; dx + \int_b^c f \; dx = \int_a^c f \; dx$
  - Fundamental theorem of calculus.

Primitive functions

Example

- Compute $\int_0^\pi \cos x \; dx$.
- A primitive function of $f(x) = \cos x$ is $F(x) = \sin x$.
- So

$$\int_0^\pi \cos x \; dx = [\sin x]_0^\pi = 0 - 0 = 0$$



Primitive functions

- We often denote an arbitrary primitive function of f by $\int f(x)dx$ (without endpoints).

$f(x)$	$\int f(x)dx$	
x^p	$\frac{x^{p+1}}{p+1} + c$	if $p \neq -1$
$\sin x$	$-\cos x + c$	
$\cos x$	$\sin x + c$	
e^x	$e^x + c$	

$f(x)$	$\int f(x)dx$	
$\frac{1}{x}$	$\ln x + c$	if $x > 0$
$\frac{1}{x}$	$\ln(-x) + c$	if $x < 0$
$\frac{1}{1+x^2}$	$\arctan x + c$	
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + c$	

Primitive functions

- WARNING! We can only use primitive functions to compute $\int_a^b f(x)dx$ if f is continuous on all of $[a, b]$.

Example

- $f(x) = \frac{1}{x^2}$ has the primitive function $\frac{-1}{x} + c$.
- $$\left[\frac{-1}{x}\right]_{-1}^{1} = -1 - 1 = -2.$$
- But $f(x)$ is a positive function, so $\int_{-1}^{1} f(x)dx$ can not be negative!

- Next time we will learn to handle integrals of functions with singularities.

Variable substitution

- If F is a primitive function to f, then the chain rule says that

$$\frac{d}{dx}F(g(x)) = f(g(x))g'(x).$$

- Thus,

$$\int f(g(x))g'(x)dx = F(g(x)) + c.$$

- We think of this as substituting x by the variable $g(x)$.
- This is easier to think about formally:

$$\int fdt = \int f\frac{dt}{dx}dx.$$

Variable substitution

Example

- Compute $\int 2x \cos(x^2) dx$.
- We see both the expression x^2 and its derivative $2x$, so we try the substitution $t = x^2$.
-

$$\int 2x \cos(x^2) dx = \begin{bmatrix} t = x^2 \\ dt = 2x \ dx \end{bmatrix} = \int \cos t \ dt$$
$$= \sin t + c = \sin x^2 + c.$$

Variable substitution

Example

- Compute $\int \tan x \ dx$.
- Rewrite $\tan x = \frac{\sin x}{\cos x}$.
- $$\int \tan x dx = \begin{bmatrix} t = \cos x \\ dt = -\sin x \ dx \end{bmatrix} = \int -\frac{1}{t} \ dt$$
  $$= -\ln|t| + c = -\ln|\cos x| + c.$$

Variable substitution

Example

- Compute $\int \frac{dx}{x^2+4}$.
- We know the integral of $\frac{1}{x^2+1}$
- $$\int \frac{dx}{x^2+4} = \int \frac{dx}{4((\frac{x}{2})^2+1)} = \begin{bmatrix} t = \frac{x}{2} \\ dt = \frac{dx}{2} \end{bmatrix}$$
  $$= \int \frac{1}{4}\frac{2dt}{t^2+1} = \frac{1}{2}\arctan t + c = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + c.$$
- More generally,
  $$\int \frac{dx}{x^2+a} = \frac{1}{\sqrt{a}}\arctan\left(\frac{x}{\sqrt{a}}\right) + c$$

Integrating rational functions

- $$\int \frac{dx}{(x+a)} = \ln|x+a| + c$$
- If $n \neq 1$,
$$\int \frac{dx}{(x+a)^n} = -\frac{1}{(n-1)(x+a)^{n-1}} + c$$
- $$\int \frac{dx}{x^2+a} = \frac{1}{\sqrt{a}} \arctan\left(\frac{x}{\sqrt{a}}\right) + c$$
- This allows us to compute the integral of any rational function $\frac{p(x)}{q(x)}$ (where p and q are polynomials).

Integrating rational functions

Example

- Compute $\int \frac{dx}{x^2-1}$.
- Ansatz:

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{a(x-1)+b(x+1)}{(x-1)(x+1)}$$
$$= \frac{a}{x+1} + \frac{b}{x-1}$$

for some a, b.
- $1 = (a+b)x + (-a+b)$, so $a = -b$, $b = \frac{1}{2}$.

Integrating rational functions

Example (Continued)

- Compute $\int \frac{dx}{x^2-1}$.
- $$\begin{aligned}\int \frac{dx}{x^2-1} &= \int \frac{-\frac{1}{2}}{x+1} + \int \frac{\frac{1}{2}}{x-1} \\ &= \frac{1}{2}\left(-\ln|x+1| + \ln|x-1|\right) + c \\ &= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + c\end{aligned}$$

Integrating rational functions

Example

- Compute $\int \frac{3x-4}{x^2-3x+2} dx$.
- $x^2 - 3x + 2 = (x-1)(x-2)$ so we do the Ansatz

$$\frac{3x-4}{x^2-3x+2} = \frac{a}{x-1} + \frac{b}{x-2}.$$

- $3x - 4 = a(x-2) + b(x-1) = (a+b)x + (-2a-b)$.
- $a + b = 3$ and $2a + b = 4$, so $a = 1$ and $b = 2$.

Integrating rational functions

Example (Continues)

- Compute $\int \frac{3x-4}{x^2-3x+2}dx$.
-

$$\int \frac{3x-4}{x^2-3x+2}dx = \int \frac{1}{x-1} + \frac{2}{x-2}dx$$
$$= \ln|x-1| + 2\ln|x-2| + c$$
$$= \ln\left|(x-1)(x-2)^2\right| + c.$$

Integrating rational functions

Example

- Compute $\int \frac{2x^2+x+2}{x^3+x}dx$.
- $x^3+x = x(x^2+1)$.
- Ansatz:
$$\frac{2x^2+x+2}{x^3+x} = \frac{a}{x} + \frac{bx+c}{x^2+1}.$$
Note that we need a linear factor in the numerator over the quadratic denominator x^2+1.
- $2x^2+x+2 = a(x^2+1) + bx^2 + cx = (a+b)x^2 + cx + a$.
- $a+b=2, c=1, a=2$, so $b=0$.

Integrating rational functions

Example (Continued)

- Compute $\int \frac{2x^2+x+2}{x^3+x}dx$.
-

$$\begin{aligned}\int \frac{2x^2+x+2}{x^3+x}dx &= \int \frac{2}{x}+\frac{1}{x^2+1}dx \\ &= 2\ln|x| + \arctan x + c \\ &= \ln(x^2) + \arctan x + c.\end{aligned}$$

Integrating rational functions

Example

- Compute $\int \frac{dx}{x(x-1)^2}$.
- First Ansatz:
$$\frac{1}{x(x-1)^2} = \frac{a}{x} + \frac{bx+c}{(x-1)^2}.$$
- Simplify by breaking out a part of the numerator that is divisible by $x-1$:
$$\frac{1}{x(x-1)^2} = \frac{a}{x} + \frac{b_1}{(x-1)} + \frac{b_2}{(x-1)^2}.$$

Integrating rational functions

Example (Continued)

- $$\frac{1}{x(x-1)^2} = \frac{a}{x} + \frac{b_1}{(x-1)} + \frac{b_2}{(x-1)^2}.$$

- $$\begin{aligned} 1 &= a(x-1)^2 + b_1 x(x-1) + b_2 x \\ &= (a+b_1)x^2 + (-2a - b_1 + b_2)x + a. \end{aligned}$$

- $$\begin{cases} a + b_1 &= 0 \\ 2a + b_1 - b_2 &= 0 \\ a &= 1 \end{cases} \implies \begin{cases} a &= 1 \\ b_1 &= -1 \\ b_2 &= 1 \end{cases}$$

Integrating rational functions

Example (Continued)

- Compute $\int \frac{dx}{x(x-1)^2}$.
-

$$\int \frac{dx}{x(x-1)^2} = \int \frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2} dx$$
$$= \ln|x| - \ln|x-1| - (x-1)^{-1} + c$$
$$= \ln\left|\frac{x}{x-1}\right| - \frac{1}{x-1} + c$$

Integrating rational functions: Ansatzes

- $$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$
- $$\frac{1}{(x-a)(x^2+b)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+b}$$
- $$\frac{1}{(x-a)(x-b)^n} = \frac{A}{x-a} + \frac{B_1}{x-b} + \frac{B_2}{(x-b)^2} + \cdots + \frac{B_n}{(x-b)^n}$$
- $$\frac{1}{(x-a)(x^2+b)^n} = \frac{A}{x-a} + \frac{B_1x+C_1}{x^2+b} + \frac{B_2x+C_2}{(x^2+b)^2} + \cdots + \frac{B_nx+C_n}{(x^2+b)^n}$$

Polynomial division

Theorem

Every rational function $f(x) = \frac{p(x)}{q(x)}$ *can be written*

$$f(x) = a(x) + \frac{r(x)}{q(x)},$$

where a *and* r *are polynomials and* $\deg r < \deg q$*.*

Example

$$\frac{x^3}{x^2-1} = \frac{x(x^2-1)+x}{x^2-1} = x + \frac{x}{x^2-1}.$$

Polynomial division

Theorem

Every rational function $f(x) = \frac{p(x)}{q(x)}$ *can be written*

$$f(x) = a(x) + \frac{r(x)}{q(x)},$$

where a *and* r *are polynomials and* $\deg r < \deg q$*.*

Example

$$\frac{x^4 + 2x}{x^2 - 1} = \frac{x^2(x^2 - 1) + x^2 + 2x}{x^2 - 1} = x^2 + \frac{x^2 + 2x}{x^2 - 1}$$

$$= x^2 + \frac{(x^2 - 1) + 1 + 2x}{x^2 - 1} = x^2 + 1 + \frac{2x + 1}{x^2 - 1}.$$

Integrating rational functions

Example

- Compute $\int \frac{x^3}{x^2-1}dx$.
- Polynomial division and partial fractions:

$$\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1} = x + \frac{x}{(x-1)(x+1)} = x + \frac{a}{x-1} + \frac{b}{x+1}$$

- $x = a(x+1) + b(x-1) = (a+b)x + (a-b)$
- $$\begin{cases} a+b &= 1 \\ a-b &= 0 \end{cases} \implies a = b = \frac{1}{2}.$$

Integrating rational functions

Example

- Compute $\int \frac{x^3}{x^2-1} dx$.
-

$$\begin{aligned}\int \frac{x^3}{x^2-1} dx &= \int x + \frac{1}{2(x-1)} + \frac{1}{2(x+1)} dx \\ &= \frac{x^2}{2} + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + c \\ &= \frac{1}{2}\left(x^2 + \ln\left|x^2-1\right|\right) + c\end{aligned}$$

Integrating rational functions: Recipe

- Do polynomial division and integrate the polynomial term.
- Factorize the denominator
- Assign the correct Ansatz/guess.
- Solve the linear equation to find the coefficients in the numerators.
- Integrate each of the terms.
- The answer should be a sum of powers, logarithms, and arcustangents.

Integration by parts

- The product rule for derivatives says that

$$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$

- Take the primitive function of both sides, and rearrange:

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$

- So instead of integrating $f'g$, we can integrate $g'f$.

Integration by parts

- If F is the primitive function of f:

$$\int f(x)g(x)dx = F(x)g(x) - \int F(x)g'(x)dx$$

- Useful if the integrand is a product of some function f that is easy to integrate, and another function g that is easy to differentiate.
- We often use the notation

$$\int \overbrace{f(x)}^{\uparrow} \underbrace{g(x)}_{\downarrow} dx$$

to indicate the parts when we do integration by parts.

Integration by parts

Example

- Compute $\int x \cos x \; dx$.
- $$\int \underbrace{x}_{\downarrow} \overbrace{\cos x}^{\uparrow} \; dx = x \sin x - \int \sin x \; dx$$
  $$= x \sin x + \cos x + c$$

Integration by parts

Example

- Compute $\int x^2 \ln x \; dx$.
- $$\int \overbrace{x^2}^{\uparrow} \underbrace{\ln x}_{\downarrow} \; dx = \frac{x^3 \ln x}{3} - \int \frac{x^3}{3} \frac{1}{x} \; dx$$
$$= \frac{x^3 \ln x}{3} - \int \frac{x^2}{3} \; dx$$
$$= \frac{x^3 \ln x}{3} - \frac{x^3}{9} + c$$

Integration by parts

Example

- Compute $\int \mathrm{e}^x \cos x \; dx$.
- Both factors are equally easy to both integrate and differentiate, so integration by parts probably does not help. But let's try anyway!

$$\int \overbrace{e^x}^{\uparrow} \underbrace{\cos x}_{\downarrow} \; dx = e^x \cos x - \int -\mathrm{e}^x \sin x \; dx$$

$$= e^x \cos x + \int \mathrm{e}^x \sin x \; dx$$

- That did not help. Let's try again!

Integration by parts

Example

- Compute $\int e^x \cos x \; dx$.
-

$$\int e^x \cos x \; dx = e^x \cos x + \int \overbrace{e^x}^{\uparrow} \underbrace{\sin x}_{\downarrow} \; dx$$

$$= e^x \cos x + e^x \sin x - \int e^x \cos x \; dx$$

- That did not help. Or...wait! We get:

$$2 \int e^x \cos x \; dx = e^x \cos x + e^x \sin x + c.$$

Integration by parts

Example

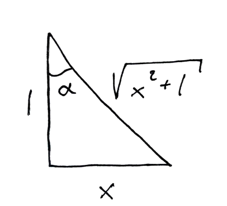
- Compute $\int \ln x \; dx$.
- This does not even look like a product. Still, we can partially integrate!
-

$$\int \ln x \; dx = \int \overbrace{1}^{\uparrow} \underbrace{\ln x}_{\downarrow} \; dx = x \ln x - \int x \frac{1}{x} dx$$
$$= x \ln x - x + c = x(\ln x - 1) + c.$$

Trigonometric substitutions

Example

- Compute $\int \sqrt{1+x^2}\, dx$.
- This integral does not lend itself to any obvious substitution.
- Idea: $\sqrt{1+x^2}$ is the hypothenuse of a right-angled triangle.



$$x = \tan \alpha \qquad \sqrt{1+x^2} = \frac{1}{\cos \alpha} \qquad dx = \frac{d\alpha}{\cos^2 \alpha}.$$

Trigonometric substitutions

Example (continued)

$$x = \tan\alpha \qquad \sqrt{1+x^2} = \frac{1}{\cos\alpha} \qquad dx = \frac{d\alpha}{\cos^2\alpha}.$$

- Now
$$\int \sqrt{1+x^2}\, dx = \int \frac{d\alpha}{\cos^3\alpha}.$$
- An integral that is a rational in terms of trigonometric functions can sometimes be solved by some clever substitution.

Trigonometric substitutions

Example (continued)

- There is one *universal substitution*, that transforms *all* integrals that are rational in terms of trigonometric functions to integrals that are rational in a variable t:
- Recall: $\frac{1}{\cos^2 \beta} = 1 + \tan^2 \beta$.

- $t = \tan \frac{\alpha}{2}$.
- $\sin \alpha = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} = \frac{2t}{1+t^2}$.
- $\cos \alpha = 2 \cos^2 \frac{\alpha}{2} - 1 = \frac{1-t^2}{1+t^2}$.
- $d\alpha = 2 \frac{d \arctan t}{dt} = \frac{2dt}{1+t^2}$.

Trigonometric substitutions

Example (continued)

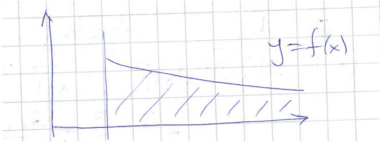
$$x = \tan\alpha \qquad \sqrt{1+x^2} = \frac{1}{\cos\alpha} \qquad dx = \frac{d\alpha}{\cos^2\alpha}.$$

$$t = \tan\frac{\alpha}{2} \qquad \cos\alpha = \frac{1-t^2}{1+t^2} \qquad d\alpha = \frac{2dt}{1+t^2}.$$

- Now
$$\int \sqrt{1+x^2}\, dx = \int \frac{d\alpha}{\cos^3\alpha} = \int \frac{2(1+t^2)^2 dt}{(1-t^2)^3}.$$
- This reduces the integral to a (complicated, but still) rational function, that can be integrated via partial fractions.

Generalized integrals

- $\int_a^\infty f(x)dx$ is a *generalized integral*.



- Problem: there is no "red rectangular area" that contains

$$\{(x,y) \in \mathbb{R}^2 : a \leq x, 0 \leq y \leq f(x)\}.$$

- We define the generalized integral as a limit

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx.$$

Generalized integrals

- The limit
$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$
might not exist, or might be infinite.

Example

- $$\int_0^\infty 1 \; dx = \lim_{b\to\infty} \int_0^b 1 \; dx = \lim_{b\to\infty} b = \infty.$$
- $$\int_0^\infty \cos x \; dx = \lim_{b\to\infty} [\sin x]_0^b = \lim_{b\to\infty} (\sin b - \sin 0)$$
does not converge.

Generalized integrals

Example

- $$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} [\ln x]_1^b = \lim_{b\to\infty} \ln b - 0 = \infty.$$
- If $p \neq 1$, then
$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$
$$= \lim_{b\to\infty} \frac{1}{p-1}\left(1 - b^{1-p}\right) = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p < 1 \end{cases}.$$

Generalized integrals

Theorem

The integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

is $\frac{1}{p-1}$ *if* $p > 1$ *and divergent if* $p \leq 1$.

- For series, we knew that

$$\sum_{n=1}^\infty \frac{1}{n^p}$$

  was convergent if $p > 1$ and divergent if $p \leq 1$.
- For integrals, we get the same result, and in addition we can find exact values if the integral is convergent.

Generalized integrals

- If $f(x) \xrightarrow[x \to a]{} \infty$, then $\int_a^b f(x)dx$ is also a generalized integral.
- We define the generalized integral as a limit

$$\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^b f(x)dx.$$

Generalized integrals

Example

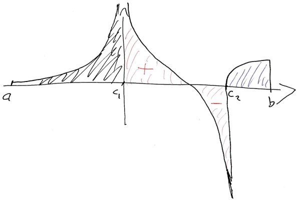
- $$\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0^+} [\ln x]_a^1 = \lim_{a\to 0^+} 0 - \ln a = \infty.$$
- If $p \neq 1$, then

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{a\to 0^+} \left[\frac{x^{1-p}}{1-p}\right]_a^1$$
$$= \lim_{a\to 0^+} \frac{1}{1-p}\left(1 - \frac{1}{a^{p-1}}\right) = \begin{cases} \frac{1}{1-p} & \text{if } p < 1 \\ \infty & \text{if } p > 1 \end{cases}.$$

Generalized integrals

- If f has a singularity in $c \in [a, b]$, we subdivide the interval:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$



- Remember: To use the Fundamental Theorem of Calculus to compute $\int_a^b f(x)dx$, f must be continuous on all of $[a, b]$.

Generalized integrals

- If f has a singularity in $c \in [a, b]$, we subdivide the interval:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Example

- Compute

$$\int_{-\infty}^{\infty} \frac{x}{x^2-1}.$$

Generalized integrals

Example

- Compute
$$\int_{-\infty}^{\infty} \frac{x}{x^2-1}.$$
- Singularities in ± 1.
- $$\int_{-\infty}^{\infty} \frac{x}{x^2-1} = \int_{-\infty}^{-1} \frac{x}{x^2-1} + \int_{-1}^{1} \frac{x}{x^2-1} + \int_{1}^{\infty} \frac{x}{x^2-1}.$$

Generalized integrals

Example (Continued)

- Partial fractions:

$$\frac{x}{x^2-1} = \frac{1}{2(x-1)} + \frac{1}{2(x+1)}.$$

- Primitive function:

$$F(x) = \int \frac{x}{x^2-1} dx = \frac{1}{2} \left(\ln|x-1| + \ln|x+1| \right) + c$$
$$= \frac{1}{2} \ln|x^2-1| + c.$$

Generalized integrals

Example (Continued)

- $$F(x) = \frac{1}{2}\ln|x^2-1| + c$$
- $$\begin{aligned}\int_{-\infty}^{\infty}\frac{x}{x^2-1} &= \int_{-\infty}^{-1}\frac{x}{x^2-1} + \int_{-1}^{1}\frac{x}{x^2-1} + \int_{1}^{\infty}\frac{x}{x^2-1}\\ &= [F(x)]_{-\infty}^{-1^-} + [F(x)]_{-1^-}^{1^-} + [F(x)]_{1^+}^{\infty}\\ &= \lim_{b\to -1^-}F(b) + \lim_{b\to 1^-}F(b) + \lim_{b\to\infty}F(b)\\ &- \lim_{a\to -\infty}F(a) - \lim_{a\to -1^+}F(a) - \lim_{a\to 1^+}F(a)\\ &\text{``}= -\infty - \infty + \infty - \infty + \infty + \infty\text{''}\end{aligned}$$

Generalized integrals

Example (Continued)

- So the integral
$$\int_{-\infty}^{\infty} \frac{x}{x^2-1}$$
is divergent.
- Still, the integrand $f = \frac{x}{x^2-1}$ is odd, meaning that $f(-x) = -f(x)$, so the integral consists of a positive part and a negative part of "equal size".
- But since this "equal size" is infinite, the parts do not cancel.

High school physics

- Free fall (no air resistance)
- $y(t)$ altitude after time t.
- Acceleration

$$a(t) = v'(t) = y''(t) = -g$$

(gravity constant).

High school physics

- $v(t) = \int -g \; dt = -gt + v_0$, $v_0 \in \mathbb{R}$ initial velocity.
- $y(t) = \int v(t) dt = -\frac{gt^2}{2} + v_0 t + y_0$, $y_0 \in \mathbb{R}$ initial altitude.
- This is a *solution* to the *differential equation*

$$y'' = -g.$$

College physics

- With air resistance (no turbulence):
- Now

$$a(t) = v'(t) = y''(t) = -g + ky'$$

(gravity and drag constants, $k < 0$).

- How do you solve a problem like

$$y'' - ky' + g = 0?$$

Ordinary differential equation

- An ordinary differential equation (ODE) is an equation of the form

$$f(t, y, y', y'', \dots) = 0,$$

  where y is a function of *one* independent variable t.
- If only the "variables" $t, y, y', \dots y^{(n)}$ are involved, then the ODE has *order n*.
- A *solution* to an ODE is a formula for all functions y that satisfies the equation.
- We will learn to solve three different kinds of ODEs in this course.

Separable equations

- A first order equation has only t, y, y' involved in the equation.
- First: isolate y' on the left hand side:

$$y' = f(y, t).$$

- If the right hand side has the form

$$f(y,t) = \overbrace{\frac{1}{g(y)}}^{\text{function of } y} \cdot \underbrace{h(t)}_{\text{function of } t},$$

then the equation is *separable*.

Separable equations

- $$y' = \frac{1}{g(y)}h(t) \implies g(y)y'(t) = h(t).$$
- Integrate both sides:
$$\int g(y)dy = \int g(y)y'(t)dt = \int h(t)dt$$
- If G and H are primitive functions of g and h, then
$$G(y) = H(t) + c.$$
- If G is invertible, then $y = G^{-1}(H(t) + c)$.

Separable equations

Example

- Solve $yy' = \mathrm{e}^t$.
- Integrate both sides:

$$\frac{y^2}{2} + a = \int y \ dy = \int yy' \ dt = \int \mathrm{e}^t \ dt = \mathrm{e}^t + b$$

for some constants a and b.

- $$y^2 = 2\mathrm{e}^t + c \implies y = \pm\sqrt{2\mathrm{e}^t + c}.$$

Separable equations

Example

- Solve $y' = 2ty^2$ if $y(0) = 1$.
- Rewrite as $\frac{1}{y^2}\frac{dy}{dt} = 2t$ and integrate both sides:

$$-y^{-1} + a = \int \frac{dy}{y^2} = \int 2t\ dt = t^2 + b$$

for some local constants a and b.

- $$y^{-1} = c - t^2 \implies y = \frac{1}{c - t^2}.$$

Separable equations

Example

- Solve $y' = 2ty^2$ if $y(0) = 1$.
- $$y = \frac{1}{c - t^2}.$$
- $1 = y(0) = \frac{1}{c}$, so $c = 1$ *in the neighbourhood of* 0.
- $$y = \frac{1}{1 - t^2}$$

  when $t \in (-1, 1)$.
- This is called an *initial value problem*.

Linear equations

- The equation
$$y' + f(y)g(t) = h(t)$$
can not be solved in general if $h \not\equiv 0$.
- But we can solve the *linear* equation
$$y' + g(t)y = h(t).$$
- Jungle trick: make the left hand side look like a derivative.

Linear equations

- $$y' + g(t)y = h(t).$$
- $$(\mathrm{e}^{G(t)}y)' = \mathrm{e}^{G(t)}y' + \mathrm{e}^{G(t)}G'(t)y = \mathrm{e}^{G(t)}h(t)$$

  if G is a primitive function of g.
- So
  $$y = \frac{\int \mathrm{e}^{G(t)}h(t)dt}{\mathrm{e}^{G(t)}}.$$

Linear equations

- The equation

$$y' + g(t)y = h(t).$$

  has the solutions

$$y = \frac{\int \mathrm{e}^{G(t)} h(t) dt}{\mathrm{e}^{G(t)}}.$$

- $\mathrm{e}^{G(t)}$ is called the *integrating factor*, where $G(t)$ is *any* primitive function of $g(t)$
- One unknown constant, coming from the primitive function $\int \mathrm{e}^{G(t)} h(t) dt$.

Linear equations

Example

- Solve

$$y' + \frac{y}{t} = t^2.$$

- Integrating factor:

$$\mathrm{e}^{\int \frac{dt}{t}} = \mathrm{e}^{\ln|t|+c} = \mathrm{e}^{c}|t|.$$

- Choose WLOG $c = 0$:

$$(|t|y)' = |t|y' + |t|\frac{y}{t} = |t|t^2.$$

- Multiply by -1 if $t < 0$, we get $(ty)' = t^3$.

Linear equations

Example (Continued)

- Solve
$$y' + \frac{y}{t} = t^2.$$
- $(ty)' = t^3$, so
$$ty = \int t^3 \; dt = \frac{t^4}{4} + c,$$
where c is a local constant.
-
$$y = \begin{cases} \frac{t^3}{4} + \frac{c_1}{t} & \text{if } t > 0 \\ \frac{t^3}{4} + \frac{c_2}{t} & \text{if } t < 0 \end{cases},$$
where c_1 and c_2 are arbitrary constants.

Linear equations

Example

- Solve
$$x^2y' + y = 1,$$
where $y(1) = 0$.
- Isolate y':
$$y' + \frac{1}{x^2}y = \frac{1}{x^2}$$
- Integrating factor:
$$e^{\int \frac{dx}{x^2}} = e^{\frac{-1}{x}+c} \overbrace{=}^{c=0} e^{\frac{-1}{x}}.$$

Linear equations

Example (Continued)

- Solve
$$x^2y' + y = 1,$$
where $y(1) = 0$.
-
$$\left(e^{\frac{-1}{x}}y\right)' = e^{\frac{-1}{x}}y' + \frac{e^{\frac{-1}{x}}}{x^2}y = \frac{e^{\frac{-1}{x}}}{x^2}.$$
-
$$e^{\frac{-1}{x}}y = \int \frac{e^{\frac{-1}{x}}}{x^2}dx = \left[\begin{array}{c} t = \frac{-1}{x} \\ dt = \frac{dx}{x^2} \end{array}\right] = \int e^t dt = e^t + c = e^{\frac{-1}{x}} + c.$$

Linear equations

Example (Continued)

- Solve
$$x^2y' + y = 1,$$
where $y(1) = 0$.
-
$$y = \frac{e^{\frac{-1}{x}} + c}{e^{\frac{-1}{x}}} = 1 - \frac{c}{e^{\frac{-1}{x}}} = 1 - ce^{\frac{1}{x}}.$$
-
$$0 = y(1) = 1 - ce,$$
so $c = e^{-1}$ (when $x > 0$.)

Linear equations

Example (Continued)

- Solve
$$x^2y' + y = 1,$$
where $y(1) = 0$.
- Solution:
$$y = \begin{cases} 1 - e^{\frac{1}{x} - 1} & \text{if } x > 0 \\ 1 - ce^{\frac{1}{x}} & \text{if } x < 0 \end{cases},$$
where c is an arbitrary constant.

Free fall with air resistance

- How do you solve a problem like

$$y'' - ky' + g = 0?$$

  $(k < 0)$
- Rewrite:

$$v' - kv + g = 0,$$

  so we get a first order equation in the unknown $v = y'$.

Free fall with air resistance

- $$v' - kv + g = 0,$$
  where $k < 0$ is the (first order) air resistance coefficient.
- Integrating factor
  $$\mathrm{e}^{\int -k \; dt} = \mathrm{e}^{-kt}$$
  (choose $c = 0$).
- $$\left(\mathrm{e}^{-kt} v\right)' = \mathrm{e}^{-kt} v' - k\mathrm{e}^{-kt} v = -\mathrm{e}^{-kt} g.$$
- $$v = \frac{\int -\mathrm{e}^{-kt} g \; dt}{\mathrm{e}^{-kt}} = \frac{g}{\mathrm{e}^{-kt}} \left(\frac{\mathrm{e}^{-kt}}{k} + c \right) = \frac{g}{k} + c\mathrm{e}^{kt}.$$

Free fall with air resistance

- $$v' - kv + g = 0,$$
  where $k < 0$ is the (first order) air resistance coefficient.

- $$v = \frac{\int -\mathrm{e}^{-kt} g \ dt}{\mathrm{e}^{-kt}} = \frac{g}{\mathrm{e}^{-kt}} \left(\frac{\mathrm{e}^{-kt}}{k} + c \right) = \frac{g}{k} + c\mathrm{e}^{kt}.$$

- $$y = \int v \ dt = \frac{g}{k} t + a\mathrm{e}^{kt} + b$$
  Here, $a = \frac{c}{k}$ is a new unknown constant.

Free fall with air resistance

- $$y = \int v \; dt = \frac{g}{k}t + a\mathrm{e}^{kt} + b$$
- If no velocity when the fall starts, then
$$0 = v_0 = \frac{g}{k} + ak,$$
so $a = -\frac{g}{k^2}$.

Free fall with air resistance

- $$y = \frac{g}{k}t + \frac{g}{k^2}\mathrm{e}^{kt} + b = b + \frac{g}{k}\left(t + \frac{\mathrm{e}^{kt}}{k}\right).$$
- Asymptotically as $t \to \infty$, the altitude is $\approx b + \frac{g}{k}t$. (At least until the poor guy hits the water surface, at which point the resistance coefficient k changes dramatically.)

Particular solutions

- We would like to solve the linear equation

$$y'' + a(t)y' + b(t)y = h(t).$$

- In this course, we will only consider second order equations with constant coefficients

$$y'' + ay' + by = h(t).$$

- We will first find *one particular* solution y_p to the equation, and then show how to get the general solution.

Particular solutions

- Task: find a particular solution to

$$y'' + ay' + by = h(t).$$

- Idea: If h belongs to a nice class of functions that is closed under derivatives, then it makes sense to look for y in the same class.
- Examples:

$$c_p t^p + c_{p-1} t^{p-1} + \cdots + c_0.$$

$$c_0 \cos t + c_1 \sin t.$$

$$\left(c_p t^p + c_{p-1} t^{p-1} + \cdots + c_0\right) \mathrm{e}^x.$$

Particular solutions

Example

- Task: find a particular solution to

$$y'' - y' - 2y = \cos t.$$

- Ansatz:

$$\begin{aligned}
y &= a\cos t + b\sin t\\
y' &= b\cos t - a\sin t\\
y'' &= -a\cos t - b\sin t\\
\hline
y'' - y' - 2y &= (-3a - b)\cos t + (a - 3b)\sin t
\end{aligned}$$

Particular solutions

Example (Continued)

- We assigned $y = a\cos t + b\sin t$.
- Now

$$\cos t = y'' - y' - 2y = (-3a - b)\cos t + (a - 3b)\sin t$$

-

$$\begin{cases} -3a - b &= 1 \\ a - 3b &= 0 \end{cases} \implies \begin{cases} a &= \frac{-3}{10} \\ b &= \frac{-1}{10} \end{cases}$$

- Particular solution:

$$y_p = \frac{-1}{10}\left(3\cos t + \sin t\right).$$

Particular solutions

Example

- Task: find a particular solution to

$$y'' - y' + 2y = t^2 - 1.$$

- Ansatz:

$$\begin{array}{rrrr} y = & at^2 + & bt + & c \\ y' = & & 2at + & b \\ y'' = & & & 2a \\ \hline y'' - y' + 2y = & 2at^2 + & (-2a + 2b)t + & (2a - b + 2c) \end{array}$$

Particular solutions

Example (Continued)

- We assigned $y = at^2 + bt + c$.
- Now

$$t^2 - 1 = y'' - y' + 2y = 2at^2 + (-2a + 2b)t + (2a - b + 2c)$$

- $$\begin{cases} 2a & = 1 \\ -2a + 2b & = 0 \\ 2a - b + 2c & = -1 \end{cases} \implies \begin{cases} a & = \frac{1}{2} \\ b & = \frac{1}{2} \\ c & = -\frac{3}{4} \end{cases}$$

- Particular solution:

$$y_p = \frac{t^2}{2} + \frac{t}{2} - \frac{3}{4}.$$

Particular solutions

Example

- Task: find a particular solution to

$$y'' - y' + 2y = e^{2t}.$$

- Ansatz:

$$\begin{aligned} y &= ae^{2t} \\ y' &= 2ae^{2t} \\ y'' &= 4ae^{2t} \\ \hline y'' - y' + 2y &= 4ae^{2t} \end{aligned}$$

- $4a = 1$, so a particular solution is $y_p = \frac{1}{4}e^{2t}$.

Particular solutions

Example

- Task: find a particular solution to

$$y'' - y' - 2y = e^{2t}.$$

- Ansatz:

$$\begin{aligned} y &= ae^{2t} \\ y' &= 2ae^{2t} \\ y'' &= 4ae^{2t} \\ \hline y'' - y' - 2y &= 0e^{2t} \end{aligned}$$

- So no particular solution of this form.

Particular solutions

Example (Continued)

- Task: find a particular solution to

$$y'' - y' - 2y = e^{2t}.$$

- New ansatz:

$$\begin{aligned} y &= ate^{2t} \\ y' &= (2at + a)e^{2t} \\ y'' &= (4at + 4a)e^{2t} \\ \hline y'' - y' - 2y &= 3ae^{2t} \end{aligned}$$

- $3a = 1$, so a particular solution is $y_p = \frac{t}{3}e^{2t}$.

Particular solutions

- Rule of thumb: Begin with the easiest Ansatz you can think of, and add a factor only if the first one did not work.

Homogeneous solutions

- How do we find *all* solutions to the linear equation

$$y'' + ay' + by = h(t)?$$

- If y_p is one particular solution, and y is another solution, then $y - y_p$ satisfies

$$(y - y_p)'' + a(y - y_p)' + b(y - y_p) = 0$$

Homogeneous solutions

- Let y_p be a particular solution of

$$y'' + ay' + by = h(t).$$

- Let y_h be a general solution of the *homogeneous equation*

$$y'' + ay' + by = 0.$$

- Then a *general* solution of

$$y'' + ay' + by = h(t)$$

is given by $y_p + y_h$.

Homogeneous solutions

Theorem

- *The space of solutions to $y'' + ay' + by = 0$ is two-dimensional.*
- *Thus, if y_1 and y_2 are two different solutions, not constant multiples of each other, then* all *solutions are given by*

$$y = ry_1 + sy_2.$$

- Proven in later courses.
- Intuition: Need to take the primitive function **twice**, so we will get **two** unknown constants.

Homogeneous solutions

- $$y'' + ay' + by = 0$$
- If we find two solutions (not multiples of each other), then we find all.
- Inspired guess:
  $$y = \mathrm{e}^{rx}.$$
- $$y'' + ay' + by = r^2\mathrm{e}^{rx} + ar\mathrm{e}^{rx} + b\mathrm{e}^{rx} = \mathrm{e}^{rx}(r^2 + ar + b).$$

Characteristic equation

- If r is a solution to the equation

$$r^2 + ar + b = 0, \tag{1}$$

then $y = e^{rt}$ is a solution to the differential equation

$$y'' + ay' + by = 0. \tag{2}$$

- We call (1) the *characteristic equation* of (2).

Characteristic equation

- Three cases:
  - The characteristic equation has two distinct real roots: $r^2 - 3r + 2 = (r-2)(r-1)$.
  - The characteristic equation has two distinct complex roots: $r^2 + 1 = (r-i)(r+i)$.
  - The characteristic equation has a double root: $r^2 - 2r + 1 = (r-1)^2$.

Distinct real roots

Theorem

If $r^2 + ar + b = 0$ has two distinct real roots r_1 and r_2, then all solutions to

$$y^2 + ay + b = h(t)$$

are given by

$$A\mathrm{e}^{r_1 t} + B\mathrm{e}^{r_2 t} + y_p,$$

where $A, B \in \mathbb{R}$, and y_p is a particular solution.

Distinct real roots

Example

- Find *all* solutions to the equation

$$y'' - y' - 2y = e^{2t}.$$

- We found (with the ansatz $y = at\mathrm{e}^t$) a particular solution $y_p = \frac{t}{3}\mathrm{e}^{2t}$.
- The characteristic equation

$$0 = r^2 - r - 2 = \left(r - \frac{1}{2}\right)^2 - \frac{1}{4} - 2 = \left(r - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2$$

has the solutions $r_1 = \frac{3}{2} + \frac{1}{2} = 2$ and $r_1 = -\frac{3}{2} + \frac{1}{2} = -1$

Distinct real roots

Example (Continued)

- Find *all* solutions to the equation

$$y'' - y' - 2y = e^{2t}.$$

- The general solution is

$$y_p + \mathrm{e}^{r_1 t} + \mathrm{e}^{r_2 t} = \frac{t}{3}\mathrm{e}^{2t} + A\mathrm{e}^{2t} + B\mathrm{e}^{-t},$$

where A and B are arbitrary real constants.

Distinct complex roots

Theorem

If $r^2 + ar + b = 0$ has two distinct complex roots $\alpha + \beta i$ and $\alpha - \beta i$, then all solutions to

$$y'' + ay' + by = h(t)$$

are given by

$$e^{\alpha t}(A\cos(\beta t) + B\sin(\beta t)) + y_p,$$

where $A, B \in \mathbb{R}$, and y_p is a particular solution.

Distinct complex roots

- "Proof":

$$\mathrm{e}^{(\alpha+\beta i)t} = \mathrm{e}^{\alpha}\mathrm{e}^{\beta i} = \mathrm{e}^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$$

and

$$e^{(\alpha-\beta i)t} = \mathrm{e}^{\alpha}\mathrm{e}^{-\beta i} = \mathrm{e}^{\alpha t}(\cos(-\beta t) + i\sin(-\beta t)) \\ = \mathrm{e}^{\alpha t}(\cos(\beta t) - i\sin(\beta t))$$

are solutions to the equation $y'' + ay' + by = 0$.

- y is a weighted sum of

$$\mathrm{e}^{\alpha t}(\cos(\beta t) + i\sin(\beta t)) \text{ and } \mathrm{e}^{\alpha t}(\cos(\beta t) - i\sin(\beta t))$$

if and only if it is a sum of

$$\mathrm{e}^{\alpha t}\cos(\beta t) \text{ and } \mathrm{e}^{\alpha t}\sin(\beta t))$$

Distinct complex roots

Example

- Find *all* solutions to the equation

$$y'' + 4y = t.$$

- A particular solution is $y = \frac{t}{4}$ (seen by staring, or by assigning an ansatz).
- Characteristic equation:

$$r^2 + 4 = 0 \iff r = \pm 2i$$

Distinct complex roots

Example (Continued)

- Find *all* solutions to the equation

$$y'' + 4y = t.$$

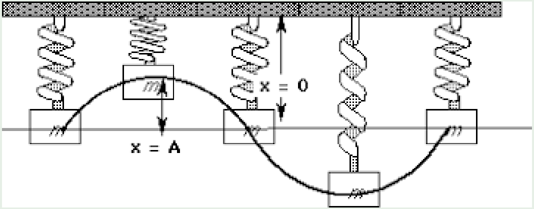
- Characteristic roots $\alpha \pm \beta i$ where $\alpha = 0$, $\beta = 2$.
- The general solution is

$$y = y_p + Ae^{\alpha t}\cos(\beta t) + Be^{\alpha t}\sin(\beta t) = \frac{t}{4} + A\cos(2t) + B\sin(2t),$$

where A and B are arbitrary real constants.

Distinct complex roots

Example



- The (damped) spring equation is $x'' = -kx' - \omega^2 x$, where ω is the spring constant and $k < 2\omega$ is a friction constant.
- By tradition, the location is denoted by $x = x(t)$.

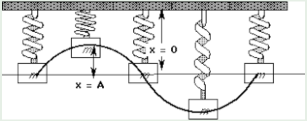
Distinct complex roots

Example (Continued)

- $$x'' + kx' + \omega^2 x = 0$$
- Characteristic equation
$$r^2 + kr + \omega^2 = 0.$$
- Characteristic roots
$$r = \frac{-k \pm \sqrt{k^2 - 4\omega^2}}{2} = \frac{-k}{2} \pm \frac{\sqrt{4\omega^2 - k^2}}{2} i.$$

Distinct complex roots

Example (Continued)



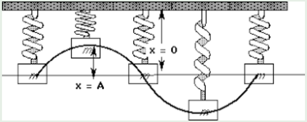
- $$x'' + kx' + \omega^2 x = 0$$
- The general solution is

$$x = \mathrm{e}^{\frac{-kt}{2}} \left(C\cos(t\sqrt{4\omega^2 - k^2}) + D\sin(t\sqrt{4\omega^2 - k^2}) \right),$$

where C and D are arbitrary real constants.

Distinct complex roots

Example (Continued)



- With standard trigonometry we can rewrite

$$x = \mathrm{e}^{\frac{-kt}{2}}\left(C\cos(t\sqrt{4\omega^2-k^2}) + D\sin(t\sqrt{4\omega^2-k^2})\right)$$
$$= \mathrm{e}^{\frac{-kt}{2}} A\sin(t\sqrt{4\omega^2-k^2}+\theta).$$

- $A = \sqrt{C^2+D^2}$ is the initial amplitude, and $\theta = \arcsin(\frac{C}{A})$ is a phase shift.

Double root

- The most difficult case is when the characteristic equation $r^2 + ar + b = (r - \alpha)^2$ has a double root $r_1 = r_2 = \alpha$.
- Then, the set of solutions

$$Ae^{r_1 t} + Be^{r_2 t} = (A + B)e^{\alpha t}$$

is one-dimensional, so we have not yet found all solutions.

Double root

- Assume $r^2 + ar + b = (r-\alpha)^2$. Then $2\alpha + a = 0$.

- Look at $y = te^{\alpha t}$.
-

$$\begin{aligned} y &= t && \cdot e^{\alpha t} \\ y' &= (t\alpha && +1) \cdot e^{\alpha t} \\ y'' &= (t\alpha^2 && +2\alpha) \cdot e^{\alpha t} \end{aligned}$$

$$y'' + ay' + by = \big(t(\alpha^2 + a\alpha + b) + (2\alpha + a)\big) \cdot e^{\alpha t} = 0$$

- So $e^{\alpha t}$ and $te^{\alpha t}$ are two different solutions to $y'' + ay' + by = 0$.

Double root

Theorem

If $r^2 + ar + b = 0$ *has a double root* α, *then all solutions to*

$$y'' + ay' + by = h(t)$$

are given by

$$e^{\alpha t}(A + Bt) + y_p,$$

where $A, B \in \mathbb{R}$, *and* y_p *is a particular solution.*

Initial value problem

Example

- Find a function $y = y(t)$ such that $y'' + 2y' + y = \cos t$ and $y(0) = y'(0) = 0$.

Step 1: Find a particular solution of

$$y'' + 2y' + y = \cos t.$$

(Forget about initial values for now.)

Step 2: Find the general solution of the homogeneous problem

$$y'' + 2y' + y = 0.$$

Step 3: Insert the initial values $y(0) = y'(0) = 0$ to determine the unknown parameters.

Initial value problem

Example (Continued)

Step 1: Find a particular solution of $y'' + 2y' + y = \cos t$.

- Ansatz:

$$\begin{array}{rrr} y = & a\cos t + & b\sin t \\ y' = & b\cos t - & a\sin t \\ y'' = & -a\cos t - & b\sin t \\ \hline y'' + 2y' + y = & 2b\cos t - & 2a\sin t \end{array}$$

- $2b\cos t + -2a\sin t = \cos t$, so $b = \frac{1}{2}$, $a = 0$.
- Particular solution: $y_p = \frac{\sin t}{2}$.

Initial value problem

Example (Continued)

Step 2: Find the general solution of the homogeneous problem $y'' + 2y' + y = 0$.

- Characteristic equation $r^2 + 2r + 1 = (r+1)^2 = 0$.
- Characteristic double root $\alpha = -1$.
- General homogeneous solution: $y_h = \mathrm{e}^{-t}(At + B)$.
- General solution:

$$y = y_p + y_h = \frac{\sin t}{2} + \mathrm{e}^{-t}(At + B)$$

$$y' = \frac{\cos t}{2} + \mathrm{e}^{-t}(-At - B + A).$$

Initial value problem

Example (Continued)

Step 3: Insert the initial values $y(0) = y'(0) = 0$ to determine the unknown parameters.

- $$0 = y(0) = \frac{\sin 0}{2} + \mathrm{e}^{-0}(A \cdot 0 + B) = B,$$
  so $B = 0$.
- $$0 = y'(0) = \frac{\cos 0}{2} + \mathrm{e}^{-0}(-A \cdot 0 - 0 + A) = \frac{1}{2} + A,$$
  so $A = -\frac{1}{2}$.

Initial value problem

Example (Conclusion)

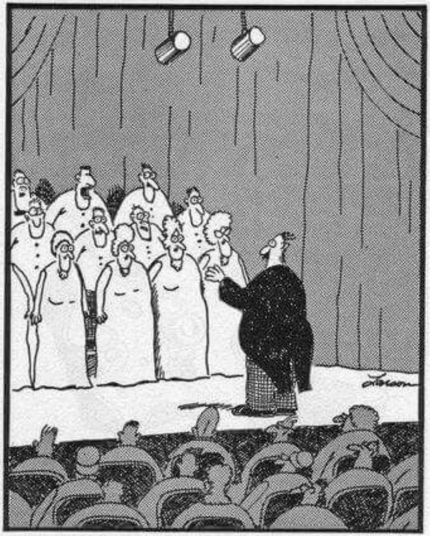
- Task: Find a function $y = y(t)$ such that

$$y'' + 2y' + y = \cos t$$

  and $y(0) = y'(0) = 0$.

- Solution:

$$y = \frac{\sin t}{2} - \frac{1}{2}t\mathrm{e}^{-t} + 0 \cdot \mathrm{e}^{-t} = \frac{\sin t - t\mathrm{e}^{-t}}{2}.$$



In that one split second, when the choir's last note had ended, but before the audience could respond, Vinnie Conswego belches the phrase, "That's all, folks."