MEC-E1005

MODELLING IN APPLIED MECHANICS 2023

Week 20 SINGING BOWL FREQUENCY

Wed 10:15-12:00 Finite element analysis (JF)

QUARTZ SINGING BOWL

ASSIGNMENT

Vibration of a structure can be thought of superposition of modes harmonic variation in time with certain frequencies. In engineering work and design of, the lowest frequencies (or even the lowest one) are important as they determine, e.g., the response for external excitation. Singing bowls are designed to vibrate with a certain frequency under excitation.

In the modelling assignment, the aim is to find the effect of geometrical and material parameters to the lowest frequency of a singing bowl. The starting point is a generic expression predicted by dimension analysis. First, a simplified model is used for qualitative understanding. After that, analysis by FEM is used to study the effects of the geometrical parameters defining the shape of the bowl. The predicted lowest frequency is compared with the experimental one.

IDEALIZATION AND PARAMETERIZATION

The bowl geometry is simplified by considering cylindrical and ellipsoidal parts of constant thickness. Material is linearly elastic and isotropic homogeneous quartz crystal. The effect of the supporting rubber ring is omitted as negligible (no external forces nor displacement constraints).

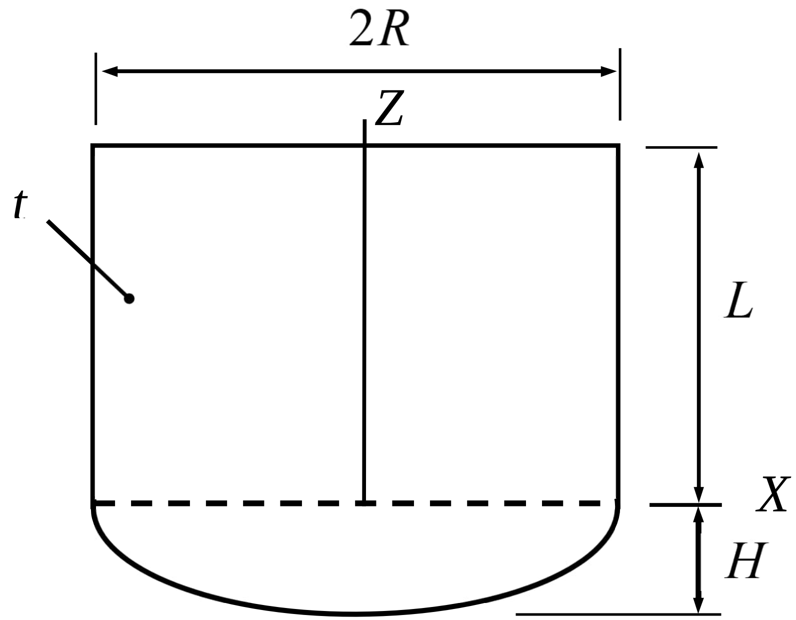


Table 1. Geometrical and material parameters of the structure.

Parameter	symbol	value
Radius	R	0.17 m
Height	L	0.15 m
Rounding	H	0.10 m
Thickness	t	8 mm
Density	ρ	2200 kg/m^3
Young's modulus	E	72 GPa
Poisson's ratio	ν	0.17

DIMENSION ANALYSIS

Assuming that the relevant quantities determining the lowest (non-zero) frequency f are Young's modulus E, Poisson's ratio ν, density ρ, and geometrical parameters t, H, L, R, dimension analysis implies the relationship

$$f\,R\sqrt{\frac{\rho}{E}} = \alpha(\frac{L}{R},\frac{H}{R},\frac{t}{R},\nu), \tag{1}$$

where the expression on the right-hand side is to be found by a more detailed analysis or/and additional assumptions. For example, assuming that the proper mass and stiffness measures inside the square root are the mass per unit area and the bending rigidity

$$\pi_1 = \alpha(\pi_2,\pi_3,\pi_4,\nu) \text{ where } \pi_1 = f\,\frac{R^2}{t}\sqrt{\frac{\rho}{E}},\ \ \pi_2 = \frac{L}{R},\ \ \pi_3 = \frac{H}{R} \text{ and } \pi_4 = \frac{t}{R}. \tag{2}$$

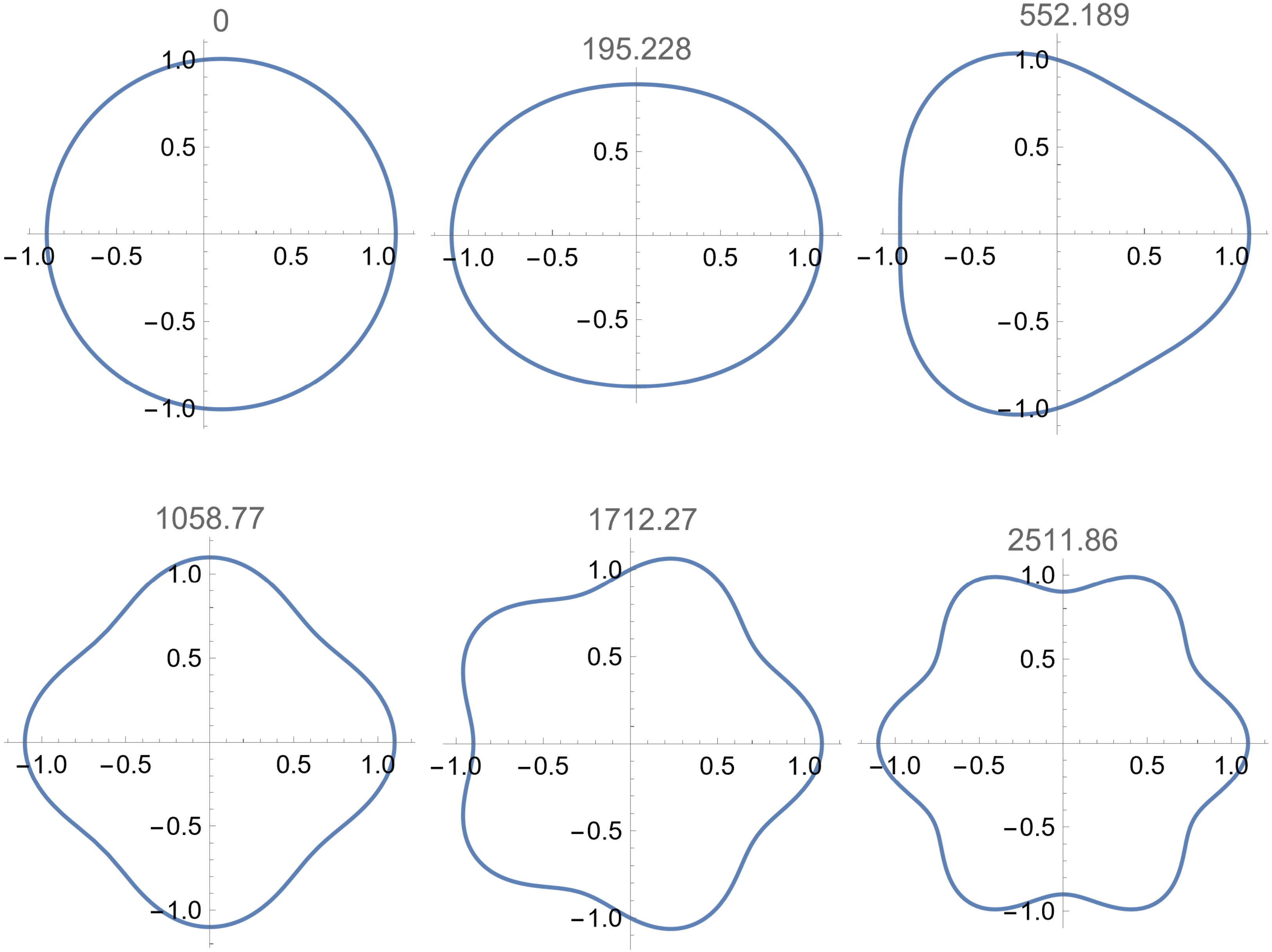
ANALYTICAL METHOD

Analytical solution considers only the cylindrical part without the closed end. The curved beam equations imply the vibration equation

$$\left(\frac{\partial^6 u}{\partial \phi^6} + 2\frac{\partial^4 u}{\partial \phi^4} + \frac{\partial^2 u}{\partial \phi^2}\right) + \frac{\rho A R^4}{EI}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 u}{\partial \phi^2} - u\right) = 0,$$

where $u(\phi,t)$ is the displacement in the angular direction (not transverse but tangential to the mid-surface). Equation assumes that the solution depends only on the angular coordinate ϕ and that the mid-curve is inextensible. The smallest non-zero angular velocity of the free vibrations follows by using the trial solution $u(\phi,t) = \sin i\phi \sin \omega t$

$$\omega_i = \sqrt{\frac{EI}{\rho A R^4}}\, i(i^2 - 1)\sqrt{\frac{1}{i^2+1}} \quad \text{giving} \quad f_{\min} = \frac{1}{2\pi}\sqrt{\frac{EI}{\rho A R^4}}\frac{6}{\sqrt{5}}.$$



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CURVED BEAM EQUATIONS (MEC-E8003)

$$\begin{Bmatrix} \dfrac{dN}{ds} - Q_n\kappa + b_s \\ \dfrac{dQ_n}{ds} + N\kappa - Q_b\tau + b_n \\ \dfrac{dQ_b}{ds} + Q_n\tau + b_b \end{Bmatrix} = 0, \quad \begin{Bmatrix} \dfrac{dT}{ds} - M_n\kappa + c_s \\ \dfrac{dM_n}{ds} + T\kappa - M_b\tau - Q_b + c_n \\ \dfrac{dM_b}{ds} + M_n\tau + Q_n + c_b \end{Bmatrix} = 0,$$

$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = \begin{Bmatrix} EA(\dfrac{du}{ds} - v\kappa) \\ GA(\dfrac{dv}{ds} + u\kappa - w\tau - \psi) \\ GA(\dfrac{dw}{ds} + v\tau + \theta) \end{Bmatrix}, \text{ and } \begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = \begin{Bmatrix} GI_{rr}(\dfrac{d\phi}{ds} - \theta\kappa) \\ EI_{nn}(\dfrac{d\theta}{ds} + \phi\kappa - \psi\tau) \\ EI_{bb}(\dfrac{d\psi}{ds} + \theta\tau) \end{Bmatrix}.$$

In circular geometry, $\tau = 0$ $\kappa = 1/R$, and $s = R\phi$. Assuming that the beam is inextensible in the axial direction, the Bernoulli beam equations written for the present problem simplify to

$$\begin{Bmatrix} \dfrac{\partial N}{\partial s} - \dfrac{1}{R} Q_n - \rho A \dfrac{\partial^2 u}{\partial t^2} \\ \dfrac{\partial Q_n}{\partial s} + \dfrac{1}{R} N - \rho A \dfrac{\partial^2 v}{\partial t^2} \\ \dfrac{\partial M_b}{\partial s} + Q_n \end{Bmatrix} = 0, \quad \begin{Bmatrix} 0 \\ 0 \\ M_b \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial s} - \dfrac{1}{R} v \\ \dfrac{\partial v}{\partial s} + \dfrac{1}{R} u - \psi \\ EI \dfrac{\partial \psi}{\partial s} \end{Bmatrix}.$$

The first set are the "dynamic equilibrium equations" and the second constitutive ones. Dynamic equations use the inertia "force" interpretation for the distributed external force. Eliminating the axial and shear force resultants from the first set (use the second and third equations to get the first one in terms of the bending moment)

$$R\frac{\partial^3 M_b}{\partial s^3}+\frac{1}{R}\frac{\partial M_b}{\partial s}-\rho A\frac{\partial^2 u}{\partial t^2}+R\rho A\frac{\partial}{\partial s}\frac{\partial^2 v}{\partial t^2}=0.$$

Then, use the Bernoulli and inextensibility constraints to write the bending moment constitutive equation in terms of transverse displacement

$$M_b=EI(\frac{\partial^2 v}{\partial s^2}+\frac{1}{R}\frac{\partial u}{\partial s})=EI(R\frac{\partial^3 u}{\partial s^3}+\frac{1}{R}\frac{\partial u}{\partial s}).$$

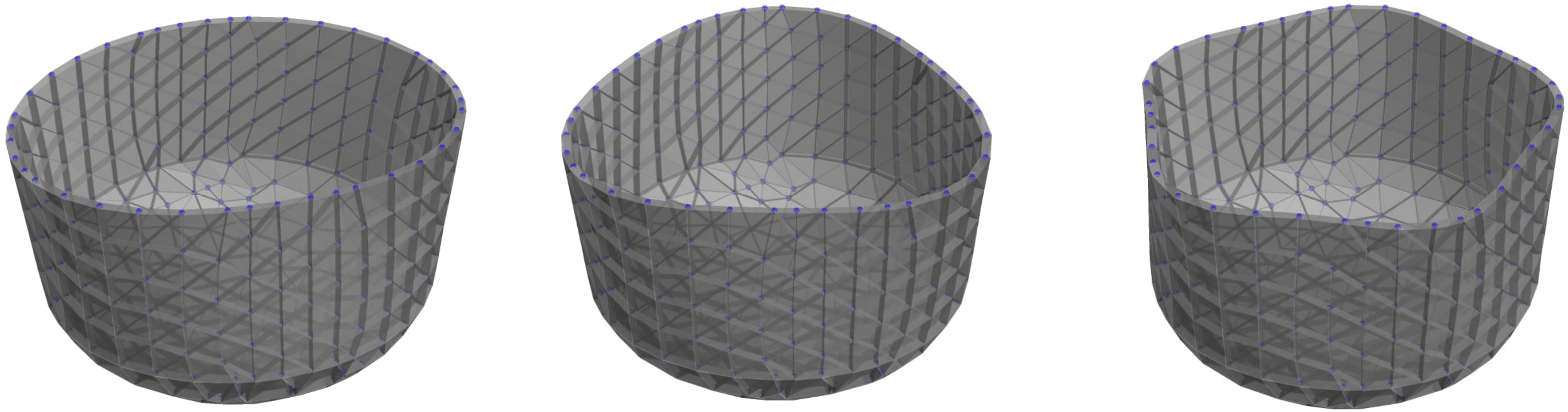
Finally, using the constitutive equation in the “dynamic equilibrium equation” , $s=R\phi$ and inextensibility constraints $v=\partial u/\partial\phi$ gives the vibration equation

$$(\frac{\partial^6 u}{\partial\phi^6}+2\frac{\partial^4 u}{\partial\phi^4}+\frac{\partial^2 u}{\partial\phi^2})+\frac{\rho A R^4}{EI}\frac{\partial^2}{\partial t^2}(\frac{\partial^2 u}{\partial\phi^2}-u)=0.$$

whose solution depends on the dimensionless group $\rho AR^4/EI$ only.

FINITE ELEMENT ANALYSIS

Vibration analysis by the finite element method and solid or plate elements gives the frequency without (too many) simplifying assumptions. However, analysis requires numerical values for all the problem parameters.



As the geometry is too complicated for a (simple enough) analytical solution, recourse to a numerical method seems to be the best choice for finding the effects of the geometrical and material parameters on the lowest frequency. Then one decides the reasonable value ranges and uses FEM to find the value of the right-hand side of equation (2) for each parameter combination and thereby tabulated form of the expression. After that, one may try to compress the data with, e.g., a polynomial approximation. Let us consider the parameter values

$$\frac{L}{R} = \pi_2 \in \tfrac{1}{8}\{1, 2, 4, 6, 8\},$$

$$\frac{H}{R} = \pi_3 \in \tfrac{1}{4}\{1, 2, 4, 6, 8\},$$

$$\frac{t}{R} = \pi_4 \in \frac{1}{100}\{2, 5, 10\}.$$

Table 1. Effects of π_2 (rows) and π_3 (columns) on π_1 when $\pi_4 = 0.02$ and $\nu = 0.17$

	0.125	0.25	0.5	0.75	1.
0.25	0.414649	0.308236	0.202385	0.159268	0.148311
0.5	0.378064	0.269234	0.193258	0.157086	0.151418
1.	0.313051	0.229313	0.173691	0.156169	0.143416
1.5	0.313925	0.206352	0.163639	0.148815	0.140311
2.	0.313471	0.201259	0.15558	0.143836	0.137718

Table 2. Effects of π_2 (rows) and π_3 (columns) on π_1 when $\pi_4 = 0.05$ and $\nu = 0.17$

	0.125	0.25	0.5	0.75	1.
0.25	0.371336	0.273964	0.186617	0.15487	0.14369
0.5	0.332031	0.247969	0.179141	0.152219	0.143352
1.	0.293014	0.212367	0.16547	0.148433	0.13951
1.5	0.298721	0.196079	0.157097	0.14405	0.137512
2.	0.298288	0.193045	0.150848	0.140804	0.135741

Table 3. Effects of π_2 (rows) and π_3 (columns) on π_1 when $\pi_4 = 0.1$ and $\nu = 0.17$

	0.125	0.25	0.5	0.75	1.
0.25	0.322875	0.254326	0.180194	0.152245	0.141411
0.5	0.29277	0.233742	0.172966	0.149656	0.140502
1.	0.272577	0.202263	0.161581	0.145496	0.137692
1.5	0.279105	0.189741	0.154028	0.141885	0.135947
2.	0.2784	0.187647	0.148484	0.139164	0.134467

As an example, the dimensionless groups for Sonic MEINL Energy Crystal Singing Bowl 14" are $\pi_2 = 0.882$, $\pi_3 = 0.588$, and $\pi_4 = 0.047$. Table 2 gives $\alpha = 0.16$ (approximately) and thereby the frequency $f = 253\,\text{Hz}$. A more precise value of $\alpha = 0.162749$ and frequency $f = 257\,\text{Hz}$ can be found by interpolation.

EXPERIMENT

"MEINL Sonic Energy Crystal Singing Bowl 14" is available in Puumiehenkuja 5L (Konemiehentie side of the building). The hall is open for the experiment during the office hours (09:00-16:00) from Thu of week 19 to Thu of week 20.

Record the sound of the bowl and transform the time series to the frequency domain. The lowest frequency of free vibrations is indicated by a peek in the frequency domain representation. You may hit the bowl (gently) to start vibration or try to cause a resonance at the lowest frequency. You may use, for example, Mathematica to record and analyse the sound.

FOURIER TRANSFORM

The Fourier series (various forms exist) can be used to represent a function as the sum of harmonic terms. For example, the sine-transformation pair for a function $a(x)$ $x \in [0, L]$ with vanishing values at the end points is given by

$$\alpha_j = \frac{2}{L}\int_0^L \sin(j\pi\frac{x}{L})a(x)dx \quad j \in \{1,2,\ldots\} \quad \Leftrightarrow \quad a(x) = \sum_{j\in\{1,2,\ldots\}} \alpha_j \sin(j\pi\frac{x}{L}).$$

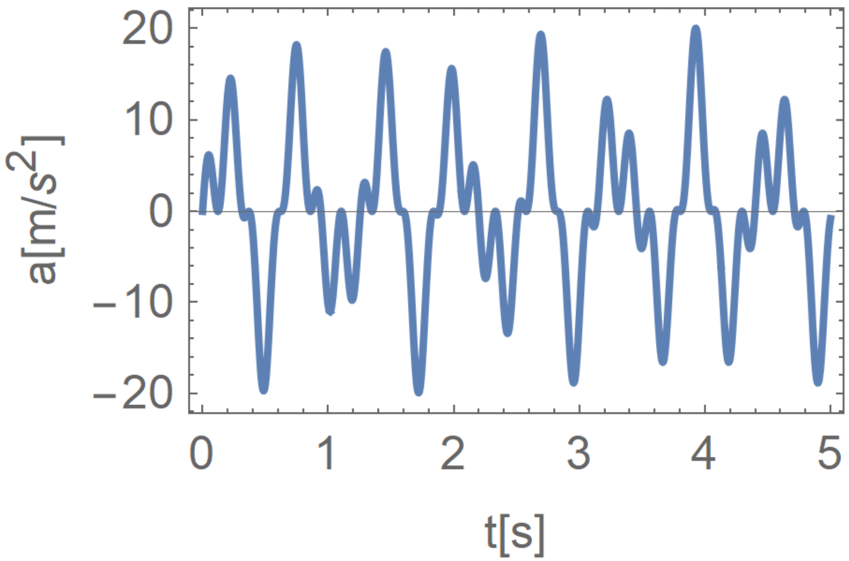
The transformation pair is based on the orthogonality of the modes

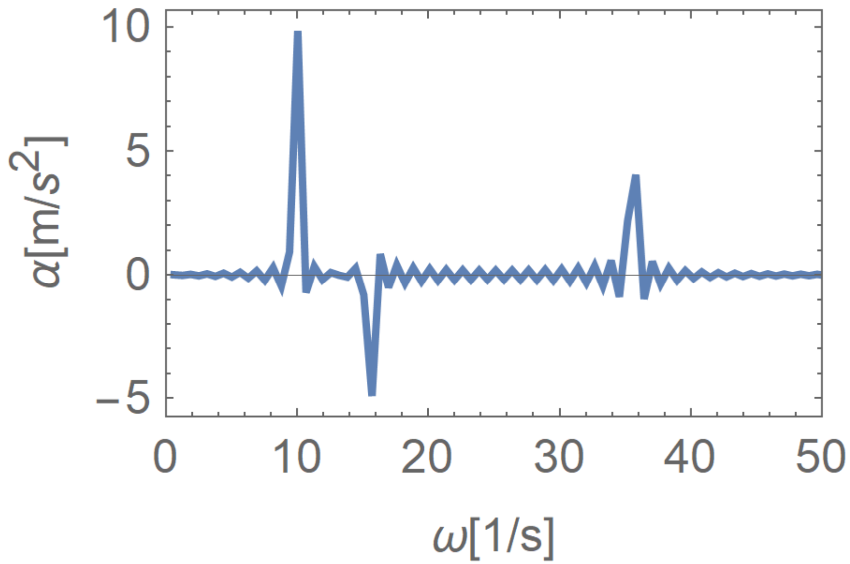
$$\int_0^L \sin(j\pi\frac{x}{L})\sin(l\pi\frac{x}{L})dx = \frac{L}{2}\delta_{jl} \text{ (Kronecker delta).}$$

The transformation (with respect to time) can be used to analyze frequency contents of data, filtering, to find the combination of the terms of the generic series solution for bar and string models satisfying the initial conditions, etc.

FREQUENCY CONTENTS OF DATA

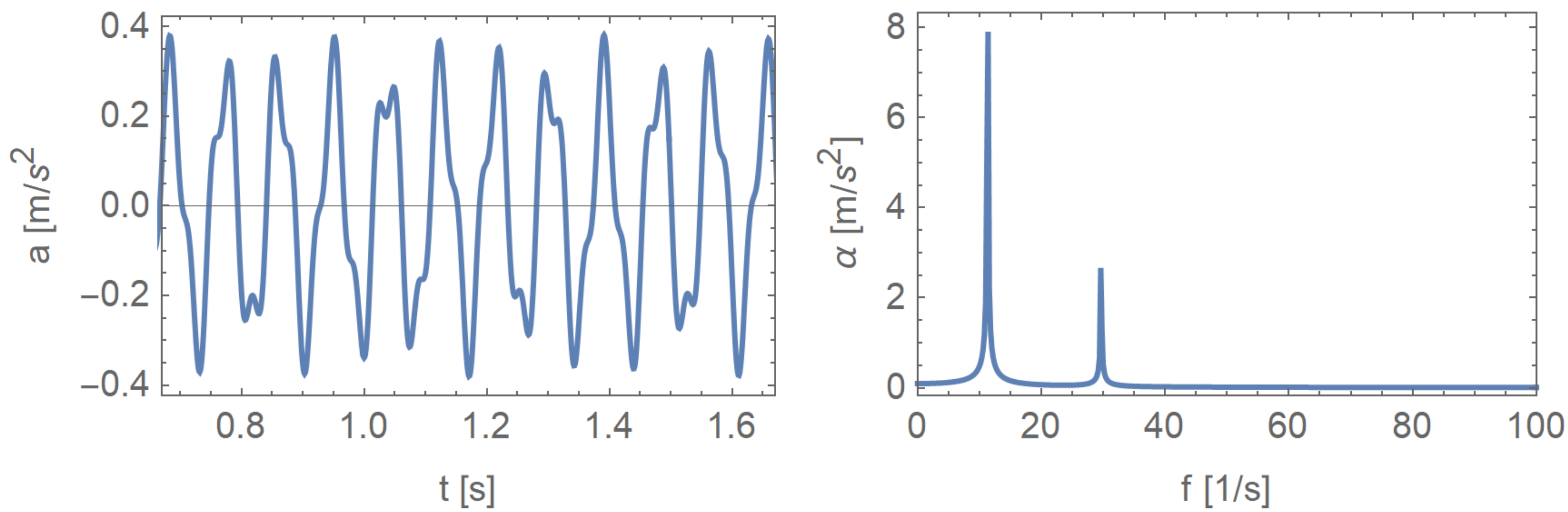
Fourier transform (with respect to time) can be used, e.g., to analyze frequency contents of data, filtering of data etc. As an example, transform (right) of the measured acceleration (left), imply $a(t) = 10\sin(10t) - 5\sin(15.6t) + 5\sin(35.6t)$ (in appropriate units)





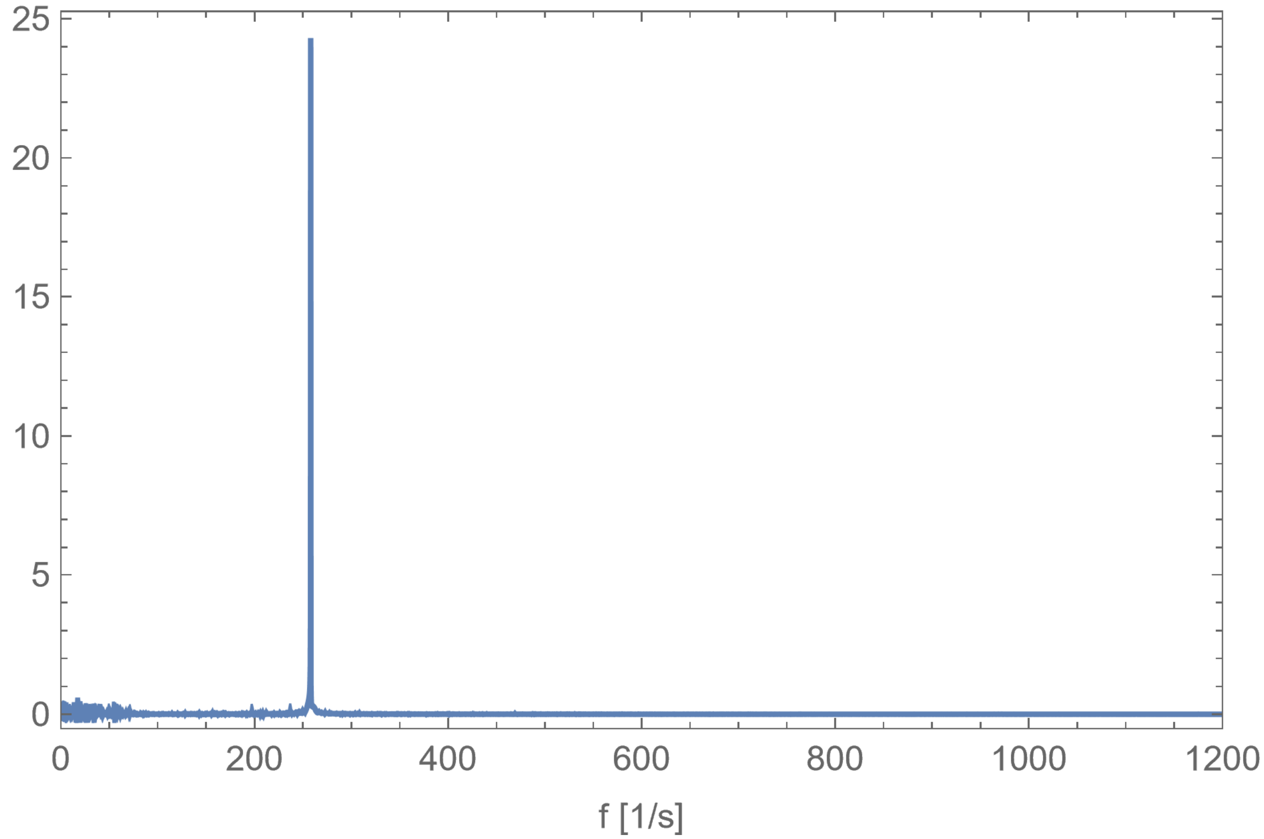
In filtering, one may just omit, e.g., components having frequencies over some value or maybe components of amplitudes of small values depending on the application.

VIBRATION EXPERIMENT



Experimental data may consist of acceleration, pressure etc. time-series. In processing, the data is transformed to frequency domain using the Discrete Fourier Transform (DFT) to find the most significant frequencies.

SOUND ANALYSIS



DESIGN FORMULA

$$f = \frac{t}{R^2}\sqrt{\frac{E}{\rho}}\pi_1(\pi_2, \pi_3, \pi_4, \nu) \text{ and tabulated data for } \pi_1(\pi_2, \pi_3, \pi_4, \nu)$$

Table 4. Comparison of frequencies. The nominal value for the bowl is C4 (261.63 Hz)

f_{sim}[Hz]	f_{exp}[Hz]	f_{des}[Hz]	f_{fem}[Hz]
195	258	257	262