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MS-EV0011: Codes over nonstandard alphabets

Problem Set III

Problem 1: Let R be a finite ring, and let $R^\times x$ denote the set of all generators of the ideal $Rx \leq R$, and let μ be the Moebius function on the poset of all principal left ideals of R.

(a) Show that for all $x \in R$ there holds:

$$\dot{\bigcup_{Ry \leq Rx}} R^\times y \;=\; Rx.$$

(b) Use Moebius inversion in order to derive that for all $x \in R$ there holds:

$$|R^\times x| \;=\; \sum_{Ry \leq Rx} |Ry|\, \mu(Ry, Rx).$$

Work: Without proof, we use the fact that $R^\times x = \{ux \mid u \in R^\times\}$ is indeed the set of all generators of the ideal Rx, which particularly implies $R^\times x = R^\times y$ if and only if $Rx = Ry$, for all $x, y \in R$.

(a) It is easily seen that the relation $x \sim y$ if and only if $R^\times y = R^\times x$ is an equivalence relation on R, where the equivalence class belonging to $x \in R$ is of the form $R^\times x$. For this reason we clearly have

$$\bigcup_{y \in Rx} R^\times y \;=\; Rx.$$

To make this a disjoint sum, we use the above preamble and then obtain

$$\dot{\bigcup_{Ry \leq Rx}} R^\times y \;=\; Rx.$$

(b) From **(a)**, we obtain

$$\sum_{Ry \leq Rx} |R^\times y| \;=\; |Rx|,$$

which yields the claim

$$|R^\times x| \;=\; \sum_{Ry \leq Rx} |Ry|\, \mu(Ry, Rx).$$

by Moebius inversion. Here, μ is in fact the Moebius function on the partially ordered set of all principal left ideals.

Problem 2: Let R be a finite ring, and let μ again denote the Moebius function on the poset of its left principal ideals. For arbitrary $\gamma > 0$, verify that the function $w : R \longrightarrow \mathbb{R}$ with

$$w(x) \;:=\; \gamma \left[1 - \frac{\mu(0, Rx)}{|R^\times x|}\right],$$

is indeed a (left) homogeneous weight, as it satisfies all of its defining criteria.

Work: We first observe that w is a real-valued function that maps 0 to $\gamma(1-1/1)=0$. As a second step, we verify that $w(ux)=w(x)$, because $Rux=Rx$ and $R^\times ux = R^\times x$, for all $u\in R^\times$ and $x\in R$. Finally, we compute

$$\begin{aligned}\sum_{y\in Rx} w(y) &= \sum_{Ry\leq Rx} w(y)\,|R^\times y| = \sum_{Ry\leq Rx}\gamma(|R^\times y| - \mu(0,Ry)) \\ &= \gamma\,|Rx| - \gamma\sum_{Ry\leq Rx}\mu(0,Ry) = \begin{cases}\gamma\,|Rx| & : \; x\neq 0,\\ 0 & : \; \text{otherwise.}\end{cases}\end{aligned}$$

Here we have used the fact, that

$$\sum_{Ry\leq Rx}\mu(0,Ry) = \begin{cases} 0 & : \; x\neq 0,\\ 1 & : \; \text{otherwise,}\end{cases}$$

which can be obtained easily from the properties of the Moebius function μ.

Problem 3:

(a) Compute the homogeneous weight of average 1 on $\mathbb{Z}_6$ and verify that this weight does not satisfy the triangle inequality.

(b) Compute the homogeneous weight w on $R:=\mathbb{Z}_2\times\mathbb{Z}_2$ and verify that $w(1,1)=0$, so that w is not strictly positive.

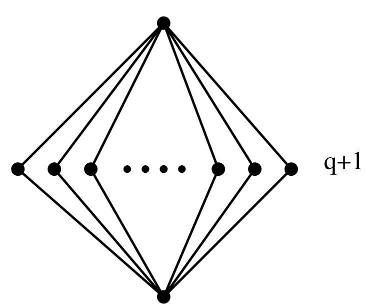
Work:

(a) We have $\gamma=1$ and observe $w(0)=0$ and $w(1)=w(5)=1-\frac{1}{2}=\frac{1}{2}$. Moreover, we find $w(3) = 1-\frac{-1}{1}=2$, and $w(2)=w(4)=1-\frac{-1}{2}=\frac{3}{2}$. The triangle inequality is violated because $w(2)=\frac{3}{2} > 1 = \frac{1}{2}+\frac{1}{2} = w(1)+w(1)$.

(b) For given $\gamma\in\mathbb{R}$ we have $w(0,0)=0$ and $w(1,0)=w(0,1)=\gamma(1-\frac{-1}{1})=2,\gamma$ whereas $w(1,1) = \gamma(1-\frac{1}{1})=0$. Hence this weight is not strictly positive.

Problem 4: Draw the Hasse diagram for the poset of all left ideals of $M_2(\mathbb{F}_q)$ and compute the homogeneous weight of average 1 for this ring.

Work: Let $R=M_2(\mathbb{F}_q)$, then the desired Hasse diagram of the lattice of all left ideals of R looks like the following:



Here we have $q+1$ simple ideals of size q^2, as the above illustration shows. For this reason, we find $\mu(0,0) = 1$, and $\mu(0,Rx)=-1$ for all these $q+1$ ideals, which eventually yields $\mu(0,R) = -1-(q+1)\cdot(-1) = q$. Moreover, we have $|R^\times| = (q^2-1)(q^2-q)$ and $|R^\times x| = q^2-1$ (why?). For given $\gamma\in\mathbb{R}$, we conclude

$$w(x) = \gamma\cdot\left\{\begin{array}{ccl} 0 & : & x=0,\\ 1+\frac{1}{q^2-1} & : & x\neq 0,\;\det(x)=0,\\ 1-\frac{q}{(q^2-1)(q^2-q)} & : & x \text{ invertible,}\end{array}\right\} = \gamma\cdot\left\{\begin{array}{ccl} 0 & : & x=0,\\ q^2-q & : & x\neq 0,\;\det(x)=0,\\ q^2-q-1 & : & x\text{ invertible,}\end{array}\right.$$

where we have tacitly rescaled γ in such a way, that the resulting expressions become integer-valued.