Probabilistic Method

Used to prove existence of combinatorial structures.

Set up a probabilistic experiment & let $\mathcal{E}$ be the event corresponding to the structure.

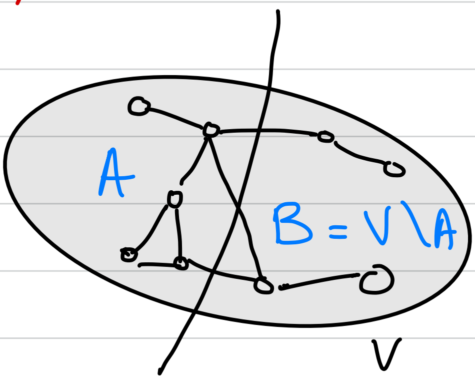
$$\Pr[\mathcal{E}] > 0 \implies \text{the structure exists!}$$

Many ways to show non-zero probabilities

$\mu = \mathbb{E}[X]$ then $\Pr[X \geq \mu] > 0$ and
$\Pr[X \leq \mu] > 0$

Why? Otherwise $\mathbb{E}[X] \neq \mu$

Finding a Large Cut



Show that for any graph G there exists a cut (A, B) of size $\geq m/2$

($m =$ number of edges in G,
size of cut is the number of crossing edges
$\{u, v\}$: $u \in A, v \in B$ or $u \in B, v \in A$)

Probabilistic experiment: put each vertex independently in A w.p. $1/2$ & in B w.p. $1/2$

$X =$ size of cut (A, B)

$$X = \sum_{e \in E} X_e \quad \text{where}$$

$$X_e = \begin{cases} 1 & \text{if } e \text{ crosses } (A,B) \\ 0 & \text{o/w} \end{cases}$$

X_e's are pairwise independent but not 3-wise independent!

$$\Pr[X_e = 1] = 1/2$$

$$\Rightarrow \quad E[X] = m/2$$

$$\Rightarrow \quad \Pr[X \geq m/2] > 0$$

$\Rightarrow$ there exists a cut of size $\geq m/2$ in any graph G.

How do we efficiently find a large cut?

Let $p = \Pr[X \geq m/2]$

$$E[X] = \sum_{x=0}^{m} x \cdot \Pr[X = x]$$

$$\frac{m}{2} = \sum_{x=0}^{m/2-1} x \cdot \Pr[X = x] + \sum_{x=m/2}^{m} x \cdot \Pr[X = x]$$

$$\leq \left(\frac{m}{2} - 1\right) \Pr\left[X < \frac{m}{2}\right] + m \cdot \Pr\left[X \geq \frac{m}{2}\right]$$

$$\frac{m}{2} \leq \left(\frac{m}{2} - 1\right)(1-p) + m \cdot p$$

$$\Rightarrow \frac{\cancel{m}}{\cancel{2}} \leq \frac{\cancel{m}}{\cancel{2}} - 1 + \frac{mp}{2} + p$$

$$0 \leq -1 + p\left(\frac{m}{2} + 1\right)$$

$$p \geq \frac{1}{\left(\frac{m}{2}+1\right)} = \Pr[\text{Success}]$$

Therefore, we have a Las Vegas algorithm with run time $\sim \text{Geom}(p)$

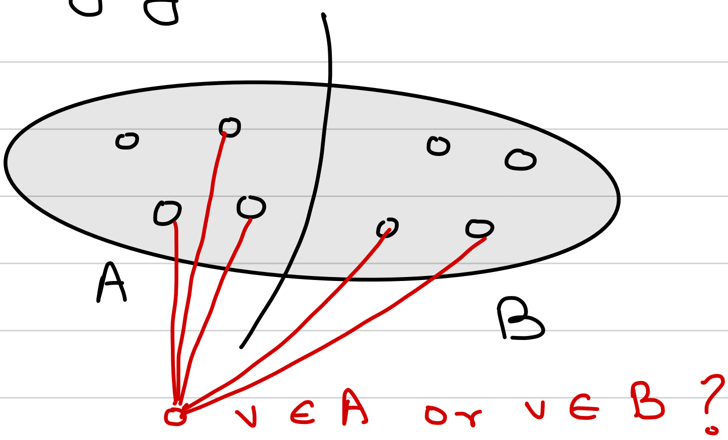
$$\Rightarrow \text{Expected run time} = \frac{1}{p} \leq \frac{m}{2} + 1$$

We can also make it deterministic!

$$\mathbb{E}[X] = \frac{1}{2}\left(\mathbb{E}[X \mid v \in A] + \mathbb{E}[X \mid v \in B]\right)$$

$$\Rightarrow \mathbb{E}[X] \leq \max\left(\mathbb{E}[X \mid v \in A], \mathbb{E}[X \mid v \in B]\right)$$

This also holds if we condition on other vertices belonging to A or B.



The choice only depends on the edges between v & already partioned vertices.

If v has more edges to A then put it in B
otherwise put it in A

Linear time deterministic algorithm that finds a cut of size $\geq m/2$, by above analysis.

Finding a Large Independent Set

Every graph with average degree d contains an independent set of size $\geq n/2d$

$$d = 2m/n \Rightarrow n/2d = n^2/4m$$

For any graph G:

1. Sample vertices with probability $1/d$
2. From the sampled graph, remove each edge along with one end point.

The remaining vertices form an independent set

X = # of vertices sampled in 1
Y = # of vertices removed in 2

Independent set size = $X - Y$

$\mathbb{E}[X] = n/d$

What is $\mathbb{E}[Y]$?

Y = # of edges with both end points sampled.

$Y = \sum_{e \in E} Y_e$ where for $e = \{u, v\}$

$$Y_e = \begin{cases} 1 & \text{if } u \ \& \ v \text{ are sampled} \\ 0 & \text{otherwise.} \end{cases}$$

$\mathbb{E}[Y_e] = Pr[Y_e = 1] = 1/d^2$

$\Rightarrow \mathbb{E}[Y] = m/d^2 = \left(\frac{m}{d}\right) \cdot \left(\frac{n}{2m}\right)$

$\Rightarrow \mathbb{E}[Y] = n/2d$

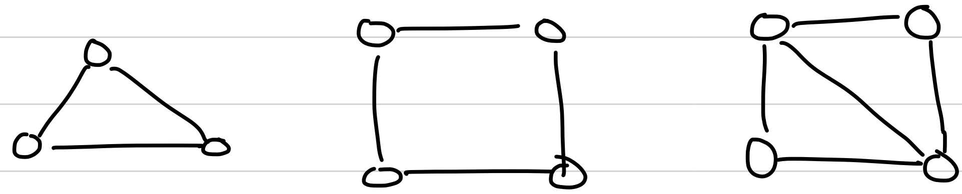
$\Rightarrow \mathbb{E}[\text{size}] = \dfrac{n}{2d}$

$\Rightarrow Pr[\text{size} \geq n/2d] > 0$

$\Rightarrow$ There exists an independent set of size $\geq n/2d$.

Dense Graphs with Large Girth:

Girth of a graph is the length of the smallest cycle.



3 4 3

Larger the girth, fewer the # of edges

For any const. k, $\exists$ an n-vertex graph G that

1. has girth $\geq k$ and
2. $\Omega(n^{1+1/k})$ edges.

(For all n large enough)

$G(n,p)$: n vertices & each edge is present independently w.p. p

we set $p = \frac{n^{1/k}}{n}$;

Let X = # edges in $G(n,p)$

$$\mathbb{E}[X] = \binom{n}{2} \cdot p = \frac{n-1}{2} \cdot n^{1/k}$$

To guarantee that $G(n,p)$ has girth $\geq k$, we remove one edge from all cycles of length $k-1$ or less.

Let Y = # of edges removed.

Y_i = # of cycles of length i

$$Y = \sum_{i=3}^{k-1} Y_i$$

(Because we remove one edge for every short cycle)

Y_i can be further decomposed.

$$Y_i = \sum_{c \in C_i} Y_{i,c} \; ; \; Y_{i,c} = \begin{cases} 1 & \text{if cycle } c \text{ exists in } G \sim G(n,p) \\ 0 & \text{o/w} \end{cases}$$

$E[Y_{i,c}] = p^i$ (each edge needs to exist)

How many cycles of length i?

$$|C_i| \leq \underbrace{\binom{n}{i}}_{\text{choose } i \text{ distinct vertices}} \cdot \underbrace{(i-1)!}_{\text{\# ways to arrange } i \text{ vertices in a cycle}} \cdot \underbrace{\frac{1}{2}}_{\text{a cycle is the same in reverse}}$$

$$\leq \frac{n!}{(n-i)!} \cdot \frac{1}{2} \leq \frac{1}{2} \cdot n^i$$

Each cycle exists with probability p^i

$$\therefore E[Y] \leq \sum_{i=3}^{k-1} \frac{1}{2} \cdot n^i \cdot p^i$$

$$= \sum_{i=3}^{k-1} \frac{1}{2} n^{i/k} \leq k \cdot n^{(k-1)/k}$$

$$\Rightarrow \quad \mathbb{E}[\#\text{ edges}] = \mathbb{E}[X - Y]$$

$$\geq \binom{n-1}{2} n^{1/k} - k \cdot n^{(k-1)/k}$$

$$\geq \frac{n^{1+1/k}}{4} \quad \left(\begin{array}{c}\text{for } n \\ \text{large} \\ \text{enough}\end{array}\right)$$