Lovasz Local Lemma (LLL)

Bad Events : $E_1, E_2, \ldots, E_n$

$$\Pr[E_i] \leq p \quad \forall\, i = 1..n$$

What is probability of avoiding them all?

$$\Pr\left[\bigcap_{i=1}^{n} \overline{E_i}\right] = ?$$

Say E_i's are mutually independent

$$\Pr\left[\bigcap_{i=1}^{n} \overline{E_i}\right] = \prod_{i=1}^{n} \Pr[\overline{E_i}] \geq (1-p)^n$$

But what if we have dependencies?

$$\Pr\left[\bigcap_{i=1}^{n} \overline{E_i}\right] = 1 - \Pr\left[\bigcup_{i=1}^{n} E_i\right]$$

$$\geq 1 - \sum_{i=1}^{n} \Pr[E_i]$$

$$\geq 1 - n \cdot p$$

If we only have "few" dependencies can we interpolate between the two?

<u>Dependency Graph:</u> Nodes are $\{1, \ldots n\}$ corresponding to the bad events $E_1, \ldots, E_n$.

$$(i) \text{———} (j) \iff E_i \;\&\; E_j \text{ are dependent.}$$

More formally: E_i is mutually independent of the events

$$\bar{N}(i) = \{ E_j \mid (i,j) \notin E \}$$

Degree of the dependency graph is the maximum degree of any vertex.

$\boxed{\text{LLL}}$ (symmetric version)

Let $E_1, E_2, \ldots E_n$ be bad events such that

1. for all i, $\Pr[E_i] \leq p$
2. degree of dependency graph $\leq d$
3. $4pd \leq 1$

Then $\Pr[\bigcap_{i=1}^{n} \bar{E}_i] > 0$

Application: $\boxed{\text{Edge-Disjoint Paths}}$

n pairs of users that want to communicate with each other

Underlying communication network connects each pair using a set of paths.

Pair i can choose one path from set F_i of m paths.

<u>Catch</u>: paths in F_i can share edges with paths in F_j (at most k)

We want to find the condition where it is possible for each pair to choose edge disjoint paths.

Say each pair chooses a path in F_j uniformly at random.

$\boxed{\text{Bad events:}}$ $E_{i,j}$ where pairs i & j choose a path with common edges.

$$\Pr[E_{i,j}] \leq \frac{k}{m} (= p)$$

$\boxed{\text{Dependency Graph:}}$ $E_{i,j}$ is mutually independent of all events $E_{p,q}$ such that $p \notin \{i,j\}$ $q \notin \{i,j\}$

$E_{i,j}$ has an edge to $E_{p,q}$ iff either $p = i,j$ or $q = i,j$

Degree of dependency graph is $\leq 2n$ $(= d)$

LLL condition:

$$4dp \le 1$$

$$\frac{8nk}{m} \le 1 \Rightarrow \underbrace{m}_{\#\text{ paths between pair } i} \ge 8 \underbrace{n}_{\#\text{ pairs}} \underbrace{k}_{\#\text{ of paths in } F_i \text{ not disjoint with } F_j}$$

So if there are at least $8nk$ paths between each pair, we can tolerate k "shared" paths between each pair i, j.

<u>Depressing news</u>: $m > nk$ suffices by a simple counting argument.

<u>Application</u>: $\boxed{\text{Sinkless Orientations}}$

Want to orient edges of a d-reg. graph such that we do not have sinks (i.e. nodes with no outgoing edges)

<u>Probabilistic experiment</u>: Orient each edge independently in one of the two directions

Bad Events: a node v has all edges incoming...

$$\Pr[E_v] = \left(\tfrac{1}{2}\right)^d = p$$

Dependency Graph: input graph...

LLL condition: $4pd \le 1$

$$4 \cdot \frac{1}{2^d} \cdot d \le 1$$

$d = 1$: $4pd = 2$
$d = 2$: $4pd = 2$
$d = 3$: $4pd = 3/2$
$d = 4$: $4pd = 1 \le 1$
$d = 5$: $4pd = 5/8 < 1$

So if $d \geq 4$ then a sinkless orientation always exists!

We will see other applications in tutorial 4 and in week 5

Next we prove the LLL. See Section 6.7 of the MU textbook.