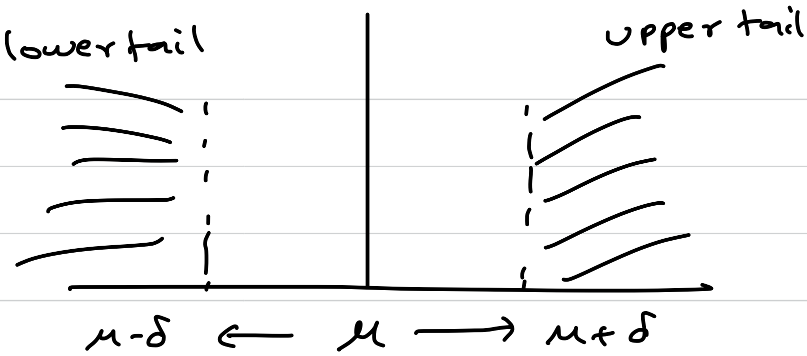
# Tail Bounds




## Chernoff

Let $X_1, X_2, \dots X_n$ be independent Poisson trials with $Pr[X_i = 0] = 1-p_i$ & $Pr[X_i = 1] = p_i$

Let $X = \sum_{i=1}^{n} X_i$ with $\mu = \mathbb{E}[X]$

1. For any $\delta > 0$

$$Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$$

2. For $0 < \delta < 1$

$$Pr[|X-\mu| \geq \delta\mu] \leq 2e^{-\delta^2\mu/3}$$

### Application 1 Balls into Bins

n balls are thrown u.a.r. into n bins

$X_1$ = # of balls in bin i

$$X_i = \sum_{j=1}^{n} X_{ij}$$

$$X_{ij} = \begin{cases} 1 & \text{if ball } j \text{ lands in bin } i \\ 0 & \text{o/w} \end{cases}$$

$$\mathbb{E}[X_{ij}] = \Pr[X_{ij} = 1] = \frac{1}{n}$$

$$\Rightarrow \quad \mathbb{E}[X_i] = 1$$

By Chernoff bound ①

$$\Pr[X_i \geq (1+\delta)] \leq \frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}$$

$$\Pr[X_i \geq \delta] \leq \frac{e^{\delta-1}}{\delta^{\delta}}$$

$$= \exp\left(\delta - 1 - \delta \log \delta\right)$$

$$\delta = c(\log n / \log\log n)$$

$$\Pr\left[X_i \geq O\left(\frac{\log n}{\log\log n}\right)\right] \leq \exp(c \log n)$$

$$\ll \frac{1}{n^c}$$

Application ② Coin tossing

(Section 4.2.2 of MU Book)

## Proof of Chernoff bounds

(Sections 4.1 & 4.2.1)

# Hoeffding Bound

(Section 4.5)

Hoeffding's bound extends the Chernoff bound technique to general random variables with a bounded range.

**Theorem 4.12 [Hoeffding Bound]:** *Let $X_1, \ldots, X_n$ be independent random variables such that for all $1 \le i \le n$, $\mathbf{E}[X_i] = \mu$ and $\Pr(a \le X_i \le b) = 1$. Then*

$$\Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \ge \epsilon\right) \le 2e^{-2n\epsilon^2/(b-a)^2}.$$

Application: Simple Random walk

$$X = \sum_{i=1}^{n} X_i \quad \text{where } X_i = \begin{cases} 1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases}$$

$$b = 1 \; ; \; a = -1 \; ; \; \mu = 0$$

$$\Pr\left[\left|\frac{X}{n}\right| \geq \varepsilon\right] \leq 2\exp\left(\frac{-2n\varepsilon^2}{4}\right)$$

$$\varepsilon = \sqrt{\frac{2\ln n}{n}}$$

$$\Pr\left[|X| \geq \sqrt{2n\ln n}\right] \leq 2\exp(-\ln n)$$

$$= \frac{2}{n}$$

This can be interpreted as Gambler's Ruin where the Gambler has no options.

When the gambler can interact / make choices based on past action, we get dependencies which Hoeffding bound cannot handle.

We can model this as a Martingale

See Section 13.1 of MU Book

## Azuma - Hoeffding Bound

**Theorem 13.6 [Azuma–Hoeffding Inequality]:** *Let $X_0, \ldots, X_n$ be a martingale such that*

$$B_k \leq X_k - X_{k-1} \leq B_k + d_k$$

*for some constants $d_k$ and for some random variables $B_k$ that may be functions of $X_0, X_1, \ldots, X_{k-1}$. Then, for all $t \geq 0$ and any $\lambda > 0$,*

$$\Pr(|X_t - X_0| \geq \lambda) \leq 2e^{-2\lambda^2/(\sum_{k=1}^{t} d_k^2)}.$$

Finally, we discussed :

Sections 13.5.1 , 13.5.3