7 Finite element methods for the Timoshenko beam problem
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7 Finite element methods for the Timoshenko beam problem

Contents

1. Strong and weak forms for Timoshenko beams
2. Finite element methods for Timoshenko beams

Learning outcome

A. *Understanding* of the basic properties of the Timoshenko beam problem and *ability* to derive the basic formulations related to the problem

B. *Basic knowledge and tools* for solving Timoshenko beam problems by finite element methods – with *locking free* elements, in particular

References

Lecture notes: chapter 9.2
Text book: chapters 5.4–5
7.0 Motivation for the Timoshenko beam element analysis

The relevance of beam structures – from rails to nano beams – has significantly grown due to new functional or smart materials spreading beams from civil engineering to new fields as space technology and biomechanics!
7.1 Strong and weak forms of Timoshenko beam elements

Let us consider a thin straight beam structure subject to such a loading that the deformation state of the beam can be modeled by the bending problem in a plane. The basic kinematical assumptions for dimension reduction of a thin or moderately thin beam, called Timoshenko beam (1921), i.e.,

(K1) normal fibres of the beam axis remain straight during the deformation
(K2) normal fibres of the beam axis do not stretch during the deformation
(K3) material points of the beam axis move in the vertical direction only
(K4) normal fibres of the beam axis remain as normals during the deformation
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come true if the displacements are presented as

\[ u(x, y) = -y \sin(\beta(x)) \approx -y\beta(x), \]
\[ v(x, y) = v(x, 0) - y(1 - \cos(\beta(x))) \approx v(x, 0) =: w(x), \]

with \( w \) denoting the deflection of the beam (central or neutral axis) and \( \beta \) denoting the rotation of the normal fibres of the axis – two variables both depending on the \( x \) coordinate only.
For linear deformations, the displacement field above implies the axial strain and transverse shear strain as

\[
\varepsilon_x(x, y) = -y\beta'(x), \quad \gamma_{xy}(x, y) = \gamma_{yx}(x, y) = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)(x, y) = (w' - \beta)(x).
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When defining the bending moment and shear force (through stresses)

$$M(x) := M_x(x) := \int_{A(x)} \sigma_x y dA, \quad Q(x) := Q_y(x) := \int_{A(x)} \tau_{xy} dA,$$

the energy balance of the principle of virtual work can be written in the form

$$0 = \delta W_{\text{int}} + \delta W_{\text{ext}} = -\int_V \sigma : \delta \varepsilon \ dV + \int_V b \cdot \delta u \ dV + \int_{S_t} t \cdot \delta u \ dS$$

$$= -\int_0^L \int_A \left(\sigma_x \delta \varepsilon_x + \tau_{xy} \delta \gamma_{xy}\right) dA dx + \int_{S_t} t_y \delta v dS; \quad b = 0, \ t = [0 \ t_y(x, z)]^T$$
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\[ = -\int_0^L \int_A (\sigma_x \delta \varepsilon_x + \tau_{xy} \delta \gamma_{xy}) dA dx + \int_{S_t} t_y \delta v dS; \quad b = 0, t = [0 \ t_y(x, z)]^T \]

\[ = \int_0^L \left( \int_A \sigma_x y \, dA \right)(\delta \beta)' \, dx - \int_0^L \left( \int_A \tau_{xy} \, dA \right)((\delta w)' - \delta \beta) \, dx + \int_0^L f \delta w \, dx \]
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$$= -\int_0^L \int_A (\sigma_x \delta \varepsilon_x + \tau_{xy} \delta \gamma_{xy}) \, dA \, dx + \int_{S_t} t_y \delta v \, dS; \quad b = 0, \; t = [0 \ t_y(x, z)]^T$$

$$= \int_0^L (\int_A \sigma_x y \, dA) (\delta \beta)' \, dx - \int_0^L (\int_A \tau_{xy} \, dA)((\delta w)' - \delta \beta) \, dx + \int_0^L f \delta w \, dx$$

$$= \int_0^L M(\delta \beta)' \, dx - \int_0^L Q((\delta w)' - \delta \beta) \, dx + \int_0^L f \delta w \, dx \quad \forall \delta w, \delta \beta,$$
where the beam is assumed to be subject to a vertical distributed surface loading \( t_y = t_y(x, y, z) \) acting on the upper and lower surfaces of the beam, defining a resultant loading

\[
f(x) := \int_{Z_t^+(x)} t_y(x, t/2, z) \, dz + \int_{Z_t^-(x)} t_y(x, -t/2, z) \, dz,
\]

where the integrals are taken along lines \( Z_t^+(x) \) and \( Z_t^-(x) \) in the \( z \) direction for each \( x \) on upper and lower surfaces \( S_t^+ \) and \( S_t^- \), respectively (for collecting the physical load from surfaces onto the beam axis).

**Remarks.** Other loading types could be considered as well.

For instance, distributed surface loadings in the direction of \( x \)-axis (see Home exercise 6.1), as well as body loads in the both directions, can be also taken into account.

Surface loadings on the end point faces of the beam at \( x = 0, L \) can be considered as well (cf. the given boundary resultants below).
Integration by parts in the term for the internal virtual work gives the form

\[ 0 = M \delta \beta|_0^L - \int_0^L M' \delta \beta \, dx - Q \delta w|_0^L + \int_0^L (Q' \delta w + Q \delta \beta) \, dx + \int_0^L f \delta w \, dx \quad \forall \delta w, \delta \beta, \]
7.1 Strong and weak forms of Timoshenko beams

Integration by parts in the term for the internal virtual work gives the form

\[ 0 = M \delta \beta \bigg|_0^L - \int_0^L M' \delta \beta \, dx - Q \delta w \bigg|_0^L + \int_0^L (Q' \delta w + Q \delta \beta) \, dx + \int_0^L f \delta w \, dx \quad \forall \delta w, \delta \beta, \]

implying the force balance and boundary conditions, i.e., the strong form, as

\[ -Q'(x) = f(x) \quad \forall x \in \Omega = (0, L) \quad \text{(T - Q)} \]
\[ M'(x) = Q(x) \quad \forall x \in \Omega = (0, L) \quad \text{(T - M)} \]
\[ M(0) = M_0 \quad \beta(0) = \beta_0 \]
\[ M(L) = M_L \quad \beta(L) = \beta_L \]
\[ Q(0) = Q_0 \quad w(0) = w_0 \]
\[ Q(L) = Q_L \quad w(L) = w_L. \]

**Remark.** It was assumed that the beam is subject to a vertical distributed surface loading in the direction of \( y \)-axis along the interval \((0, L)\); cf. (EB-M) in Chapter 6.1.
When taking into account the linearly elastic constitutive relations in the form

\[ \sigma_x(x, y) = (E \varepsilon_x)(x, y) = -y(E \beta')(x), \quad \tau_{xy}(x, y) = (G \gamma_{xy})(x) = (G(w' - \beta))(x) \]
7.1 Strong and weak forms of Timoshenko beams

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the moment and shear force are given in terms of deflection and rotation as

\[ M(x) = -(EI\beta')(x), \quad Q(x) = (GA(w' - \beta))(x), \quad \text{(T - QM)} \]
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the moment and shear force are given in terms of deflection and rotation as

\[ M(x) = -(EI\beta')(x), \quad Q(x) = (GA(w' - \beta))(x), \quad (T - QM) \]

and the strong form can be written as a displacement formulation as follows: For a given loading \( f : \Omega \rightarrow R \), find the deflection and rotation \( w, \beta : \overline{\Omega} \rightarrow R \) such that

\[ -(GA(w' - \beta))'(x) = f(x) \quad \forall x \in \Omega = (0, L) \quad (T - 1) \]
\[ -(EI\beta')'(x) - (GA(w' - \beta))(x) = 0 \quad \forall x \in \Omega = (0, L) \quad (T - 2) \]

\( (EI\beta')(0) = M_0 \quad \forall \beta(0) = \beta_0 \)
\( (EI\beta')(L) = M_L \quad \forall \beta(L) = \beta_L \)
\( (GA(w' - \beta))(0) = Q_0 \quad \forall w(0) = w_0 \)
\( (GA(w' - \beta))(L) = Q_L \quad \forall w(L) = w_L. \)
7.1 Strong and weak forms of Timoshenko beams

Model comparisons. The Timoshenko beam problem can be compared to the Euler–Bernoulli beam problem by writing \((T - 2), (T - 1)\) in the form

\[
(El\beta')''(x) = f(x) \quad \forall x \in \Omega = (0, L) \quad (T - \beta)
\]

\[
w'(x) = \beta(x) - \frac{(El\beta')'(x)}{(GA)(x)} \quad \forall x \in \Omega = (0, L), \quad (T - w)
\]

where the rotation can be solved first from \((T - \beta)\) and then the deflection follows from \((T - w)\) which also reveals the difference between the models since the Euler–Bernoulli model satisfies the condition \(w' = \beta\) due to the kinematical assumption (K4) of Chapter 6.1.
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On the other hand, since \(Q = M'\), equations \((T - w), (T - 2), (T - QM)\) finally give an equation resembling the Euler–Bernoulli equation (cf. \((EB - w)\)):

\[
(EIw''')''(x) = f(x) - \frac{d^2}{dx^2}(EI \frac{d}{dx} \frac{Q(x)}{(GA)(x)})(x) \quad \forall x \in \Omega = (0, L).
\]
7.1 Strong and weak forms of Timoshenko beams

Furthermore, for constant values of $EI$ and $GA$ with a sufficiently regular loading $f$, the equation above can be written in the form

$$(EIw'')''(x) = f(x) - \frac{EI}{GA} f''(x) \quad \forall x \in \Omega.$$
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$$(EIw'')''(x) = f(x) - \frac{EI}{GA} f''(x) \quad \forall x \in \Omega.$$ 

**Remark.** The *shear correction factor* $\kappa$ depending on the cross sectional shape is often included in the Timoshenko beam formulations above: $GA \leftrightarrow \kappa GA$. 


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**Remark.** If the second derivative of the loading vanishes (as it does for a constant load, for instance), the equation above coincides with the Euler–Bernoulli beam equation, which does not necessarily imply, however, that the solutions would coincide since there is still the rotation to be determined – from $(T - 1)$, for instance – and the boundary conditions of these two problems are not identical.
Furthermore, for constant values of $EI$ and $GA$ with a sufficiently regular loading $f$, the equation above can be written in the form

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Anyway, in light of the equation above, the Euler–Bernoulli beam model seems to be a good approximation to the Timoshenko beam model as far as it holds that

$$\frac{EI}{GA} \left| f''(x) \right| \ll 1 \quad \forall x \in \Omega = (0, L).$$
7.1 Strong and weak forms of Timoshenko beams

From \((T - w)\), instead, one can deduce that the Euler–Bernoulli beam problem is a good approximation to the Timoshenko beam problem as far as it holds that

\[
\left| \frac{(EI\beta')'(x)}{(GA)(x)} \right| << 1 \quad \forall x \in \Omega = (0, L).
\]
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From this condition, a corresponding inequality formulated in terms of the a priori problem parameters can be given:

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\frac{EI}{GAL^2} \ll 1.
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\]

From this condition, a corresponding inequality formulated in terms of the a priori problem parameters can be given:

\[
\frac{EI}{GAL^2} \ll 1.
\]

On the other hand, one can still propose another condition in the form

\[
\left| \frac{d^2}{dx^2} (EI \frac{d}{dx} \frac{Q(x)}{(GA)(x)}) (x) \right| \ll 1 \quad \forall x \in \Omega = (0, L).
\]

Altogether, the proximity of these two beam models is related to the material parameters of the beam as well as the regularity of the loading, or rotation and shear force.
7.1 Strong and weak forms of Timoshenko beams

The weak form of the problem is obtained from the virtual work expressions above or, as usual, by multiplying the strong form by test functions (variational functions), integrating over the domain and finally integrating by parts (cf. Chapter 6.1):

\[
0 = -\int_0^L [(EI\beta')' + GA(w' - \beta)]\hat{\beta} \, dx = -\left. E I \beta' \hat{\beta} \right|_0^L + \int_0^L [(EI\beta')\hat{\beta}' - GA(w' - \beta)\hat{\beta}] \, dx,
\]

\[
\int_0^L f\hat{w} \, dx = -\int_0^L (GA(w' - \beta))'\hat{w} \, dx = -\left. GA(w' - \beta)\hat{w} \right|_0^L + \int_0^L GA(w' - \beta)\hat{w}' \, dx,
\]

which are valid for all \((\hat{w}, \hat{\beta}) \in W \times V\).
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$$\int_0^L f\hat{w} \, dx = -\int_0^L (GA(w' - \beta))'\hat{w} \, dx = -GA(w' - \beta)\hat{w}\bigg|_0^L + \int_0^L GA(w' - \beta)\hat{w}' \, dx,$$

which are valid for all $$(\hat{w}, \hat{\beta}) \in W \times V$$. Adding all pieces together gives both the energy balance with respect to the variational space, or spaces,

$$\int_0^L EI\beta' \hat{\beta}' \, dx + \int_0^L GA(w' - \beta)(\hat{w}' - \hat{\beta}) \, dx = \int_0^L f\hat{w} \, dx \quad \forall (\hat{w}, \hat{\beta}) \in W \times V,$$

and the essential boundary conditions – for a cantilever beam, for instance, as

$$w(0) = 0 \quad \land \quad \beta(0) = 0; \quad \hat{w}(0) = 0 \quad \land \quad \hat{\beta}(0) = 0.$$
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\[ \int_0^L f\hat{w} \, dx = -\int_0^L (GA(w' - \beta))'\hat{w} \, dx = -GA(w' - \beta)\hat{w}\bigg|_0^L + \int_0^L GA(w' - \beta)\hat{w}' \, dx, \]

which are valid for all \((\hat{w}, \hat{\beta}) \in W \times V\). Adding all pieces together gives both the energy balance with respect to the variational space, or spaces,

\[ \int_0^L EI\beta' \hat{\beta}' \, dx + \int_0^L GA(w' - \beta)(\hat{w}' - \hat{\beta}) \, dx = \int_0^L f\hat{w} \, dx \quad \forall (\hat{w}, \hat{\beta}) \in W \times V, \]

and the essential boundary conditions – for a cantilever beam, for instance, as

\[ w(0) = 0 \quad \land \quad \beta(0) = 0; \quad \hat{w}(0) = 0 \quad \land \quad \hat{\beta}(0) = 0. \]

In addition, the trial and test function spaces are determined by the weak form, as usual – although in this case the space considered is a combination of two spaces due to two different variables – deflection and rotation – present in the formulation.
7.1 Strong and weak forms of Timoshenko beams

**Weak form of the Timoshenko beam problem:** Let us consider a cantilever beam subject to a distributed load \( f \in L^2(\Omega) \), \( \Omega = (0, L) \). Find \( w \in W, \beta \in V \) such that

\[
a(w, \beta; \hat{w}, \hat{\beta}) = l(\hat{w}) \quad \forall \hat{w} \in W, \forall \hat{\beta} \in V,
\]

with the bilinear form, load functional and function spaces

\[
a(w, \beta; \hat{w}, \hat{\beta}) = \int_{\Omega} EI \beta' \hat{\beta}' d\Omega + \int_{\Omega} GA(w' - \beta)(\hat{w}' - \hat{\beta}) d\Omega,
\]

\[
l(\hat{w}) = \int_{\Omega} f\hat{w} d\Omega,
\]

\[
W = \{ v \in H^1(\Omega) \mid v(0) = 0 \},
\]

\[
V = \{ \eta \in H^1(\Omega) \mid \eta(0) = 0 \}.
\]
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$$a(w, \beta; \hat{w}, \hat{\beta}) = l(\hat{w}) \quad \forall \hat{w} \in W, \forall \hat{\beta} \in V,$$

with the bilinear form, load functional and function spaces

$$a(w, \beta; \hat{w}, \hat{\beta}) = \int_\Omega EI \beta' \hat{\beta}' d\Omega + \int_\Omega GA(w' - \beta)(\hat{w}' - \hat{\beta}) d\Omega,$$

$$l(\hat{w}) = \int_\Omega f \hat{w} d\Omega,$$

$$W = \{ v \in H^1(\Omega) \mid v(0) = 0 \},$$

$$V = \{ \eta \in H^1(\Omega) \mid \eta(0) = 0 \}.$$

**Remark.** The *energy norm* of the problem – now a norm including two variables – is defined as

$$\| (v, \eta) \|^2_a := a(v, \eta; v, \eta) = \int_\Omega EI (\eta')^2 d\Omega + \int_\Omega GA(v' - \eta)^2 d\Omega.$$
Show that the bilinear form of the Timoshenko beam problem is elliptic and continuous with respect to the $H^1(\Omega)$ norm:

\[(i) \quad a(v, \eta; v', \eta') = \int_{\Omega} EI \eta' \eta' d\Omega + \int_{\Omega} GA(v' - \eta)(v' - \eta)d\Omega \geq \ldots \]

\[\geq \alpha \left( \| v \|_1^2 + \| \eta \|_1^2 \right) \quad \forall (v, \eta) \in W \times V, \]

\[(ii) \quad a(v, \eta; \hat{v}, \hat{\eta}) = \int_{\Omega} EI \eta' \hat{\eta'} d\Omega + \int_{\Omega} GA(v' - \eta)(\hat{v'} - \hat{\eta})d\Omega \leq \ldots \]

\[\leq C \left( \| v \|_1 + \| \eta \|_1 \right) \left( \| \hat{v} \|_1 + \| \hat{\eta} \|_1 \right) \quad \forall (v, \eta), (\hat{v}, \hat{\eta}) \in W \times V. \]

For which type of values of the cross sectional quantities $EI$ and $GA$ the quotient $C/\alpha$ appearing in the corresponding error estimates will be large/small?
Standard form finite element method for the Timoshenko beam problem: Let the distributed load of a cantilever beam be \( f \in L^2(\Omega), \Omega = (0, L) \). Find the deflection and rotation approximations \( w_h \in W_h \subset W, \beta_h \in V_h \subset V \) such that

\[
a(w_h, \beta_h; \hat{w}, \hat{\beta}) = l(\hat{w}) \quad \forall \hat{w} \in W_h, \forall \hat{\beta} \in V_h,
\]

with the bilinear form, load functional and function spaces

\[
a(v, \eta; \hat{w}, \hat{\beta}) = \int_{\Omega} EI \eta' \hat{\beta}' d\Omega + \int_{\Omega} GA(v' - \eta)(\hat{w}' - \hat{\beta})d\Omega,
\]

\[
l(\hat{w}) = \int_{\Omega} f\hat{w} d\Omega,
\]

\[
W_h = \{ v \in C(\overline{\Omega}) \mid v(0) = 0, v|_K \in P_k(K) \},
\]

\[
V_h = \{ \eta \in C(\overline{\Omega}) \mid \eta(0) = 0, \eta|_K \in P_k(K) \}. \quad x = 0 \quad x = L
\]
7.2 Finite element methods for Timoshenko beams

Standard form finite element method for the Timoshenko beam problem: Let the distributed load of a cantilever beam be \( f \in L^2(\Omega), \ \Omega = (0, L) \). Find the deflection and rotation approximations \( w_h \in W_h \subset W, \ \beta_h \in V_h \subset V \) such that

\[
a(w_h, \beta_h; \hat{w}, \hat{\beta}) = l(\hat{w}) \quad \forall \hat{w} \in W_h, \ \forall \hat{\beta} \in V_h,
\]

with the bilinear form, load functional and function spaces

\[
a(v, \eta; \hat{w}, \hat{\beta}) = \int_{\Omega} EI \eta' \hat{\beta}' \, d\Omega + \int_{\Omega} GA(v' - \eta)(\hat{w}' - \hat{\beta}) \, d\Omega,
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W_h = \{ \, v \in C(\overline{\Omega}) \mid v(0) = 0, \ v_{|K} \in P_k(K) \, \},
\]

\[
V_h = \{ \, \eta \in C(\overline{\Omega}) \mid \eta(0) = 0, \ \eta_{|K} \in P_k(K) \, \}.
\]

Remark. This standard form finite element method is far from optimal: for thin beams, the approximation converges extremely slowly to the exact solution due to so called shear locking phenomena which will be clarified in detail below.
Example for a **shear locking** Timoshenko beam element approximation: Finite element approximation of the deflection compared to the exact solution of the deflection for a thin \((H/L = 1/100)\) clamped beam.

The standard – locking – method presented above is compared below to a **shear reduction method** presented later on – which has been proven to be locking free.
Shear locking can be revealed by deriving a standard error estimate in the form

$$\| w - w_h \|_1 + \| \beta - \beta_h \|_1 \leq \frac{C}{\alpha} (\| w - v \|_1 + \| \beta - \eta \|_1) \quad \forall (v, \eta) \in W_h \times V_h$$

$$\leq \frac{C}{\alpha} c h^k (| w |_{k+1} + | \beta |_{k+1}),$$

which shows once again that coercivity and ellipticity constants are present in the error estimate affecting the accuracy of the approximation provided by the method.
7.2 Finite element methods for Timoshenko beams

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Let us consider the ratio of these constants in detail; in the case of a quadrangular cross section, for simplicity (cf. Break exercise 7):

$$EI = \frac{EBH^3}{12}, \quad GA = \frac{EBH}{2(1+\nu)}$$

$$\Rightarrow C \sim \max(EI, GA) = GA, \quad \alpha \sim \min(EI, GA) = EI, \quad \text{for} \ H < 1 \ll L$$
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\[
\Rightarrow \frac{C}{\alpha} \sim \frac{\max(EI, GA)}{\min(EI, GA)} = \frac{GA}{EI} = \frac{6}{(1+\nu)H^2} \rightarrow \infty, \quad \text{if } H \rightarrow 0.
\]
The nature of the problem, and hence the error estimate as well, depends on the dimensions of the beam (the thickness parameter), which can be clearly seen by dividing the weak form of the problem by the bending stiffness giving the form

\[ \int_0^L \beta' \hat{\beta}' dx + \int_0^L \frac{GA}{EI} (w' - \beta)(\hat{w}' - \hat{\beta}) dx = \int_0^L \frac{f}{EI} \hat{w} dx =: \int_0^L g\hat{w} dx \quad \forall \hat{w} \in W, \forall \hat{\beta} \in V. \]
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This shows that in the thickness limit (in practice, for thin beams) the coefficient of the shear term in the formulation above blows up,

\[ \frac{GA}{EI} = \frac{6}{(1 + \nu)H^2} \rightarrow \infty, \text{ for } H \rightarrow 0, \]

which implies that the whole shear term blows up unless the rest of the integrand is able to balance the behaviour by converging to zero as follows:

\[ (w' - \beta)(\hat{w}' - \hat{\beta}) \rightarrow 0 \quad \forall \hat{w} \in W, \forall \hat{\beta} \in V \]

\[ \Rightarrow w' - \beta \rightarrow 0 \iff w' \rightarrow \beta. \]

**Remark.** The limit case \( w' = \beta \) comes true in the Euler–Bernoulli beam problem which is actually known to be the limit of the Timoshenko beam problem – as described by the argumentation above.
7.2 Finite element methods for Timoshenko beams

In the thickness limit (for all thin beams, in practice), the finite element approximation strives for satisfying the Euler–Bernoulli condition as well:

\[ w_h' - \beta_h \to 0 \iff w_h' \to \beta_h. \]

This leads to locking, however, which can be seen in computations as an extremely low convergence towards the exact solution – the thinner the beam, the lower the convergence rate. In the limit case, for a clamped beam with linear elements, the approximation locks as fully as possible: \( w_h = 0 = \beta_h. \)
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**Remark.** The locking phenomena is not limited to the linear elements only. However, for low order elements the locking effect takes a leading role – and this effect can be easily understood by considering the approximations of the deflection and rotation:

\[ w_h|_K, \beta_h|_K \in P_1(K) \implies (w_h|_K)' \in P_0(K), \]
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\[ w_{h|K}, \beta_{h|K} \in P_1(K) \implies (w_{h|K})' \in P_0(K), \]

and now the condition \( w_h' \to \beta_h \) means that the deflection derivative, a piecewise constant function, should coincide with the piecewise linear continuous rotation, enforcing the both variables to be global constants (zero, for clamped boundaries).
7.2 Finite element methods for Timoshenko beams

Locking free *reduced integration* finite element method for the Timoshenko beam problem: Let the distributed loading of a cantilever beam be given as $f \in L^2(\Omega)$, $\Omega = (0, L)$. Find $w_h \in W_h \subset W$, $\beta_h \in V_h \subset V$ such that

$$a(w_h, \beta_h; \hat{w}, \hat{\beta}) = l(\hat{w}) \quad \forall \hat{w} \in W_h, \forall \hat{\beta} \in V_h,$$

with the bilinear form, load functional and variational spaces

$$a(v, \eta; \hat{w}, \hat{\beta}) = \int_{\Omega} EI \eta' \hat{\beta}' d\Omega + \int_{\Omega} GA P_h (v' - \eta) P_h (\hat{\omega}' - \hat{\beta}) d\Omega,$$

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$$a(v, \eta; \hat{w}, \hat{\beta}) = \int_{\Omega} EI \eta' \hat{\beta}' d\Omega + \int_{\Omega} GA P_h (\nu' - \eta) P_h (\hat{\nu}' - \hat{\beta}) d\Omega,$$

$$l(\hat{w}) = \int_{\Omega} f \hat{w} d\Omega,$$

$$W_h = \{ v \in C(\overline{\Omega}) \mid v(0) = 0, \nu|_K \in P_k(\Omega) \},$$

$$V_h = \{ \eta \in C(\overline{\Omega}) \mid \eta(0) = 0, \eta|_K \in P_k(\Omega) \}.$$

The only difference compared to the standard method is the reduction operator, the $L^2$ projection of the shear term to a lower order polynomial space:

$$P_h : L^2(\Omega) \to S_h, \quad (P_h v, r) = (v, r) \quad \forall r \in S_h = \{ r \in L^2(\Omega) \mid r|_K \in P_{k-1}(K) \}.$$
The term *(selective)* reduced integration originates from the fact that the projection operation in the bilinear form can be accomplished in practice by using a lower order integration rule for the numerical integration of the shear term as follows:

The highest polynomial order of the shape functions in the shear term is $k$, and hence the highest polynomial order of the whole integrand is $2k$,

$$w_h |_K, \beta_h |_K \in P_k(K) \implies (w_h' - \beta_h) |_K \in P_k(K) \implies (w_h' - \beta_h)(\hat{w}' - \hat{\beta}) |_K \in P_{2k}(K),$$
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which would need $k+1$ *Gauss integration points* for accurate integration since $k$ points *Gauss quadrature* integrates accurately polynomials of order $2k-1$, while $k+1$ points is enough for accurate integration of polynomials of order $2k+1$. 
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$$w_{h|K}, \beta_{h|K} \in P_k(K) \Rightarrow (w_{h} - \beta_{h})|_{K} \in P_k(K) \Rightarrow (w_{h} - \beta_{h})(\hat{w}' - \hat{\beta})|_{K} \in P_{2k}(K),$$

which would need $k+1$ \textit{Gauss integration points} for accurate integration since $k$ points \textit{Gauss quadrature} integrates accurately polynomials of order $2k-1$, while $k+1$ points is enough for accurate integration of polynomials of order $2k+1$. In reduced integration scheme, the shear term is summed up over $k$ points instead of $k+1$:

$$\int_{\Omega} GA P_h (w_{h} - \beta_{h}) P_h (\hat{w}' - \hat{\beta}) d\Omega = \sum_{i_G=1}^{k} (GA) (x_{i_G}) (w_{h} - \beta_{h}) (x_{i_G}) (\hat{w}' - \hat{\beta}) (x_{i_G}) \bar{w}_{i_G}.$$  

\textbf{Remark.} This quadrature of $k$ points integrates accurately also the bending term of the bilinear form since the shape function derivatives therein are of order $k-1$, and hence the whole integrand is of order $2k-2 (< 2k-1)$. 

7.2 Finite element methods for Timoshenko beams
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In the standard form Timoshenko beam element method, the finite element approximation of the shear force is computed from the deflection and rotation approximations by simply using the definition of the shear force:

\[ Q_h = GA(w_h' - \beta_h). \]

In the reduced integration method, instead, the projection operator is applied:

\[ Q_h = GAP_h(w_h' - \beta_h) \in S_h. \]
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**Remark.** The *projection operator* above, and also in the bilinear form, is applied essentially to the shape functions related to the rotation since the derivative of the deflection approximation is of order \( k - 1 \) and hence belongs to the projection space:

\[ w_h|_K \in P_k(K) \Rightarrow w_h'|_K \in P_{k-1}(K) \Rightarrow w_h' \in S_h. \]
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$$\implies Q_h = GAP_h(w_h' - \beta_h) = GA(w_h' - P_h\beta_h),$$

$$\int_{\Omega} GA P_h(v' - \eta) P_h(\hat{w}' - \hat{\beta}) d\Omega = \int_{\Omega} GA (v' - P_h\eta)(\hat{w}' - P_h\hat{\beta}) d\Omega.$$


A simple example of applying the $L^2$ projection: In order to calculate the $L^2$ projection of a linear polynomial, say, a polynomial defined locally on element $K$ as

$$v(x) = a + bx \in P_1(K),$$

the definition of the projection above has to be applied, with $k = 1$, in its local form (on element $K$), i.e.,

$$P_h : L^2(\Omega) \rightarrow S_h, \quad (P_h v, r) = (v, r) \quad \forall r \in S_h = \{ r \in L^2(\Omega) \mid r_{|K} \in P_{k-1}(K) \}$$

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$$\Rightarrow \quad P_h v \in P_0(K), \quad (P_h v, r) = (v, r) \quad \forall r \in P_0(K).$$

Now, "for all $r$ ..." simply means "for all constants". On the other hand, $P_h v$ itself is simply a constant, say $c$. Hence, one can finally write the integral condition in a concrete form and obtain the projected polynomial which is now just the constant $c$:

$$\int_K (P_h v) \ r \ dx = \int_K v \ r \ dx \quad \forall r \quad \Leftrightarrow \quad \int_K c \ r \ dx = \int_K (a + bx) \ r \ dx \quad \forall r$$

$$\Leftrightarrow \quad c\int_K dx = \int_K (a + bx) \ dx \quad \Rightarrow \quad P_h v = c = \int_K (a + bx) \ dx / \int_K dx$$
Reduced integration is the only nonstandard feature of the method derived above. Hence, the typical Lagrange shape functions can be used as a basis for obtaining a continuous finite element approximation. Continuous approximation for the deflection and rotation with linear elements implies, for instance, that

- the moment approximation is piecewise constant and discontinuous
- the shear force approximation is piecewise constant and discontinuous.
7.2 Finite element methods for Timoshenko beams

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— the shear force approximation is piecewise constant and discontinuous.

Remark. Regarding the system equations, one should note that for \( n \) nodes, for instance, two variables results \( 2n \) unknowns:

\[
\begin{align*}
    w_h(x) &= \sum_{i=1}^{n} \phi_i(x)d_{w,i}, \quad \beta_h(x) = \sum_{i=1}^{n} \phi_i(x)d_{\beta,i}, \quad d_{w,i} = w_h(x_i), \quad d_{\beta,i} = \beta_h(x_i).
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\]

The same is true for each element: using quadratic elements implies 3 nodes per element and hence 6 unknowns per element (3 for the deflection, 3 for the rotation).
In terms of local shape functions, the local finite element approximations can be written in the form

\[
\begin{align*}
    w_h(x)|_{K^{(e)}} &= \sum_{i=1}^{n^{(e)}} N_i(\xi(x)) w_i^{(e)} = \begin{bmatrix} N_1(\xi(x)) & \cdots & N_{n^{(e)}}(\xi(x)) \end{bmatrix} \begin{bmatrix} w_1^{(e)} \\ \vdots \\ w_{n^{(e)}}^{(e)} \end{bmatrix} \\
    &=: N^T d_w^{(e)},
\end{align*}
\]

and

\[
\begin{align*}
    \beta_h(x)|_{K^{(e)}} &= \sum_{i=1}^{n^{(e)}} N_i(\xi(x)) \beta_i^{(e)} = \begin{bmatrix} N_1(\xi(x)) & \cdots & N_{n^{(e)}}(\xi(x)) \end{bmatrix} \begin{bmatrix} \beta_1^{(e)} \\ \vdots \\ \beta_{n^{(e)}}^{(e)} \end{bmatrix} \\
    &=: N^T d_\beta^{(e)},
\end{align*}
\]
In the shear part of the bilinear form, the deflection and rotation are coupled, which implies the cross terms into the local stiffness matrix:

\[
K^{(e)} = \begin{bmatrix}
K_{ww}^{(e)} & K_{w\beta}^{(e)} \\
K_{\beta w}^{(e)} & K_{\beta\beta}^{(e)}
\end{bmatrix}, \quad F^{(e)} = \begin{bmatrix} F_w^{(e)} \\ F_{\beta}^{(e)} \end{bmatrix}, \quad d^{(e)} = \begin{bmatrix} d_w^{(e)} \\ d_{\beta}^{(e)} \end{bmatrix}.
\]

In terms of local shape functions, the stiffness matrix terms can be written as

\[
K_{ww}^{(e)} = \int_{K_{\text{ref}}} |J_{K^{(e)}}|^{-1} GA N' N'^T d\xi,
\]

\[
K_{w\beta}^{(e)} = -\int_{K_{\text{ref}}} GA N' P_h N'^T d\xi = K_{\beta w}^{(e)T},
\]

\[
K_{\beta\beta}^{(e)} = \int_{K_{\text{ref}}} |J_{K^{(e)}}|^{-1} EI N' N'^T d\xi + \int_{K_{\text{ref}}} |J_{K^{(e)}}| GAP_h N P_h N'^T d\xi,
\]

\[
F_w^{(e)} = \int_{K_{\text{ref}}} |J_{K^{(e)}}| f N'^T d\xi, \quad K^{(e)} = (x_i, x_{i+1}).
\]
7.2 Finite element methods for Timoshenko beams

With local degrees of freedoms given as \( \mathbf{d}^{(e)} = \begin{bmatrix} w_1^{(e)} & \beta_1^{(e)} & \cdots & w_n^{(e)} & \beta_n^{(e)} \end{bmatrix}^T \), we get

\[
\mathbf{K}^{(e)} = \begin{bmatrix}
K_{w_1^{(e)}w_1^{(e)}}^{(e)} & K_{w_1^{(e)}\beta_1^{(e)}}^{(e)} & \cdots & K_{w_1^{(e)}w_n^{(e)}}^{(e)} & K_{w_1^{(e)}\beta_n^{(e)}}^{(e)} \\
K_{\beta_1^{(e)}w_1^{(e)}}^{(e)} & K_{\beta_1^{(e)}\beta_1^{(e)}}^{(e)} & \cdots & K_{\beta_1^{(e)}w_n^{(e)}}^{(e)} & K_{\beta_1^{(e)}\beta_n^{(e)}}^{(e)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
K_{w_n^{(e)}w_1^{(e)}}^{(e)} & K_{w_n^{(e)}\beta_1^{(e)}}^{(e)} & \cdots & K_{w_n^{(e)}w_n^{(e)}}^{(e)} & K_{w_n^{(e)}\beta_n^{(e)}}^{(e)} \\
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\end{bmatrix},
\]

\[
\mathbf{F}^{(e)} = \begin{bmatrix} F_{w_1}^{(e)} \\ \vdots \\ F_{w_n}^{(e)} \\ F_{\beta_1}^{(e)} \\ \vdots \\ F_{\beta_n}^{(e)} \end{bmatrix}.
\]
7.2 Finite element methods for Timoshenko beams

The element stiffness matrix and force vector can be written in a compact form, as usual, by using the bilinear form and load functional; for instance, as

$$K_{w_i^{(e)} w_j^{(e)}}^{(e)} = K_{w w}^{(e)}[i, j] = a(\phi_i, 0; \phi_j, 0)|_{K^{(e)}} = \int_{K^{(e)}} G A \phi_i' \phi_j' \, dx,$$

or alternatively with local shape functions as above.
The element stiffness matrix and force vector can be written in a compact form, as usual, by using the bilinear form and load functional; for instance, as

\[ K^{(e)}_{w_i^{(e)}w_j^{(e)}} = K_{ww}[i, j] = a(\phi_i, 0; \phi_j, 0)|_{K^{(e)}} = \int_{K^{(e)}} GA\phi_i'\phi_j'\,dx, \]

or alternatively with local shape functions as above.

**Remark.** The stiffness matrix is essentially composed of products of shape functions and their first derivatives which can be easily computed by standard means, especially for low order elements as depicted below for a two elements case.
Error analysis for the reduced integration Timoshenko beam element does not follow the standard procedure: the problem is analyzed in the *mixed from* (cf. Chapter 7.X) which results an error estimate for the shear force as well, as written down below. Ellipticity condition or *stability condition* will be proved in the mixed form by utilizing so called *Babuska–Brezzi condition* which is often called the *inf–sup condition*. 
7.2 Finite element methods for Timoshenko beams

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After all, the error estimate of the method is optimal (with respect to the polynomial order and the regularity of the exact solution):

\[
\| w - w_h \|_1 + \| \beta - \beta_h \|_1 + \| Q - Q_h \|_0 \leq c_k h^k (\| w \|_{k+1} + \| \beta \|_{k+1} + \| Q \|_k).
\]

With certain additional assumptions, an \(L^2(\Omega)\) error estimate can be derived:

\[
\| w - w_h \|_0 + \| \beta - \beta_h \|_0 \leq c h^{k+1} (\| w \|_{k+1} + \| \beta \|_{k+1} + \| Q \|_k).
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With certain additional assumptions, an $L^2(\Omega)$ error estimate can be derived:

\[ \| w - w_h \|_0 + \| \beta - \beta_h \|_0 \leq c h^{k+1} (\| w \|_{k+1} + \| \beta \|_{k+1} + \| Q \|_k). \]

Remark. The most accurate point values are obtained with this method in the end points and integrations points of each element, i.e., at Gauss–Lobatto points – this holds true for $k > 1$, which is described in the following example.
Example for the accuracy of the reduced integration Timoshenko beam finite element method (2002): For a thin \((H/L = 1/100)\) clamped beam with the loading \(f(x) = e^{3x} - 1\).

Solid lines for the convergence of the finite element deflection approximation; in the \(H^1\) norm for the element degrees \(k = 1, 2, 3\). Circles for the superconvergent points; with two elements for the element degrees \(k = 2, 3, 4, 5\).
7.X Mixed formulation

Reduced integration finite element method for the Timoshenko beam problem is analyzed in so called *mixed form* in which the shear force $Q = GA(w' - \beta)$ is taken as a new independent variable. The weak form of the problem is then of the following form: For a given loading $f \in L^2(\Omega), \ \Omega = (0, L)$, find $w \in W, \ \beta \in V, \ Q \in S \subset L^2(\Omega)$ such that

\[
\int_0^L EI\beta' \hat{\beta}'dx + \int_0^L Q(\hat{w}' - \hat{\beta})dx = \int_0^L f\hat{w}dx \quad \forall \hat{w} \in W, \ \forall \hat{\beta} \in V,
\]

\[
\int_0^L Q \hat{Q}dx = \int_0^L \hat{Q}GA(w' - \beta)dx \quad \forall \hat{Q} \in S.
\]

**Remark.** In the corresponding finite element formulation (*mixed method*), the finite element subspaces for the deflection and rotation remain the same, while the shear force approximation is a subspace of $L^2(\Omega)$:

\[
S_h = \{ r \in L^2(\Omega) \mid r|_K \in P_{k-1}(K) \}.
\]

The projection operator does not (explicitly) appear in the mixed method.
(i) Derive the weak form of the mixed formulation 7.X by starting from the strong form (T - 1), (T - 2) of the Timoshenko beam problem together with the definition of the shear force in (T - QM).

(ii) Formulate the corresponding mixed finite element method.
QUESTIONS?

ANSWERS”

LECTURE BREAK!