1 Introduction

One of the most famous problems in classical mechanics is the Kepler problem. This is the problem of a point mass in a central force field of the form

$$\mathbf{F}(r) = \frac{-k}{r^2}\mathbf{e}_r. \tag{1}$$

A special thing about this problem is that there exists an extra conserved quantity besides the total energy and the angular momentum. This quantity is a vector called the *Runge-Lenz vector*. The Runge-Lenz vector $\mathbf{A}$ for a particle of mass m moving in a central force field $\mathbf{F} = -\frac{k}{r^2}\mathbf{e}_r$ is defined as

$$\mathbf{A} := \mathbf{p} \times \mathbf{L} - \frac{mk}{|\mathbf{r}|}\mathbf{r}. \tag{2}$$

Here $\mathbf{p}$ is the momentum of the particle, $\mathbf{L}$ is the angular momentum, m the mass and $\mathbf{r}$ the position vector of the particle. In the Kepler-problem the angular momentum and energy are conserved. One might then think that there exist seven conserved quantities. This is not the case, because the variables are not independent of each other. From Figure 1 one sees that the Runge-Lenz-vector lies in the plane of motion and thus $\mathbf{A}\cdot\mathbf{L} = 0$. Further on by taking the dot product $\mathbf{A} \cdot \mathbf{A}$ one obtains $A^2 = m^2k^2 + 2mEL^2$. From this one can see that there are only five independent constants of motion in the Kepler-problem [1]. The orbits in the Kepler-problem are conic-sections. A nice way to realize this is by using the Runge-Lenz-vector. By denoting θ as the angle between the position vector and the Runge-Lenz-vector one has

$$\mathbf{A} \cdot \mathbf{r} = Arcos\theta = \mathbf{r} \cdot \mathbf{p} \times \mathbf{L} - mkr = \mathbf{L} \cdot \mathbf{r} \times \mathbf{p} - mkr = L^2 - mkr. \tag{3}$$

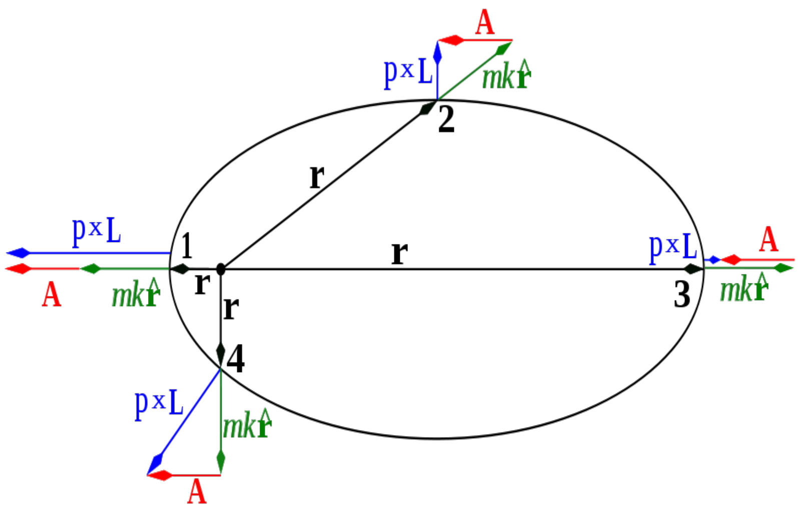

Figure 1: Illustration of how the Runge-Lenz-vector is oriented in the Kepler orbits [3].

Thus we can solve for r as

$$\frac{1}{r} = \frac{mk}{L^2}\left[1 + \frac{A}{mk} cos\theta\right], \tag{4}$$

which is the equation of a conic section with eccentricity $e = A/mk$ provided A is constant [3]. Thus we see that the conservation of the Runge-Lenz-vector actually is the reason that the orbits of the Kepler-problem are closed. The Runge-Lenz-vector can in principle be generalized to any central potential as we shall see in Section 2.2. However these generalized Runge-Lenz-vectors are often complicated functions and usually not expressible in closed form[3]. Since the conservation of the Runge-Lenz-vector implies closed orbits for the Kepler-problem one might expect that there exists some analogue of this derivation for the isotropic harmonic oscillator. This is indeed the case but we shall leave this question here [1].