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Partial Differential Equations

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Contents

Partial differential equations are not only extremely important in applications of mathematics in physical, geometric and probabilistic phenomena, but they also are of theoretic interest.

1 Introduction

These notes are meant to be an elementary introduction to partial differential equations (PDEs) for undergraduate students in mathematics, the natural sciences and engineering. They assume only advanced multidimensional differential calculus including partial derivatives, integrals and the Gauss-Green formulas. The sections denoted by * consist of additional material, which is essential in understanding the rest of the material, but can omitted or glanced through quickly in the first reading.

A partial differential equation is an equation involving an unknown function of two ore more variables and its partial derivatives. Although PDEs are generalizations of ordinary differential equations (ODEs), for most PDE problems it is not possible to write down explicit formulas for solutions that are common in the ODE theory. This means that there is greater emphasis on qualitative features. There is no general method to solve PDEs, however, some methods have turned out to be more useful than other. We study special cases, in which explicit solutions and representation formulas are available, and focus on features that are present in more general situations. Qualitative aspects are also important in numerical solutions of PDE. Without existence, uniqueness and stability results numerical methods may give inaccurate or completely wrong solutions.

Let $x \in \Omega$, where Ω is an open subset of $\mathbb{R}^n$ and $t \in \mathbb{R}$. In these notes we study

(1) Laplace's equation

$$\Delta u = f, \quad u = u(x), \quad \Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2},$$

(2) the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = f, \quad u = u(x,t),$$

(3) and the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f \quad u = u(x,t).$$

Here we have set all physical constants equal to one. Physically, solutions of Laplace's equation correspond to steady states or equilibria for time evolutions in heat distribution or wave motion, with f corresponding to external driving forces such as heat sources or wave generators. A solution $u = u(x)$ to Laplace's equation gives, for example, the temperature at the point $x \in \Omega$ and a solution $u = u(x,t)$ to the heat equation gives the temperature at the point $x \in \Omega$ at the moment of time t. A solution $u = u(x,t)$ to the wave equation gives the displacement of a body at the point $x \in \Omega$ at the moment of time t. We shall later discuss the physical interpretation of these PDEs in more detail. If $f = 0$ (the function, which is identically zero), the PDE is called homogeneous, otherwise it is said to be inhomogeneous. All homogeneous versions of the PDEs above are linear, which means that any linear combination of solutions is a solution. More precisely, if u_1 and u_2 are solutions, then $au_1 + bu_2$, $a, b \in \mathbb{R}$, is a solution of the corresponding equation as well.

By solving a PDE we mean that we find all functions u satisfying the PDE in a class of functions, which possibly satisfy certain auxiliary conditions. A PDE typically has many solutions, but there may be only one solution satisfying specific boundary or initial value conditions. These conditions are motivated by the physics and describe the physical state at a given moment or/and on the boundary of the domain. For Laplace's equation we can describe, for example, the temperature on the boundary $\partial\Omega$. For the heat equation we can, in addition, describe the initial temperature and for the wave equation the initial velocity at a given moment of time. By finding a solution to a PDE we mean that we obtain explicit representation formulas for solutions or deduce general properties that hold true for all solutions. A PDE problem is well posed, if

(1) (existence) the problem has a solution,
(2) (uniqueness) there exists only one solution and
(3) (stability) the solution depends continuously on the data given in the problem.

These are all desirable features when we talk about solving a PDE. The last condition is particularly important in physical problems, since we would like that our (unique) solution changes little when the conditions specifying the problem change little.

There is at least one more important aspect in solving PDE. We have not yet specified what does it mean that a function actually is a solution to a PDE. We shall consider classical solutions, which means that all partial derivatives which appear in the PDE exist and are continuous. In this case, we can verify by a direct computation that a function solves the PDE. However, the PDE can be so strong that it forces the solution to be smoother than assumed in the beginning. A PDE may also have physically relevant weak solutions with less regularity than classical solutions, consider for example a saw tooth wave. These questions are studied in regularity theory for PDEs.

The PDEs above are examples of the three most common types of linear equations: Laplace's equation is elliptic, the heat equation is parabolic and the wave equation is hyperbolic, although general classification is somewhat useless since it does not give any method to solve the PDEs. There are many other PDE that arise from physical problems. Let us consider, for example, Maxwell's equations. Let $\Omega \subset \mathbb{R}^3$ be an open set and $\Omega \times \mathbb{R}$ be the corresponding space-time cylinder. Maxwell's equations are

$$\begin{cases} \operatorname{div} E = \dfrac{\rho}{\epsilon_0}, \\ \operatorname{div} B = 0, \\ \operatorname{curl} E = -\dfrac{\partial B}{\partial t}, \\ \operatorname{curl} B = \mu_0\Big(J + \epsilon_0 \dfrac{\partial E}{\partial t}\Big), \end{cases}$$

where E is the electric field and B is the magnetic field (which both are maps form $\Omega \times \mathbb{R} \to \mathbb{R}^3$) corresponding to a charge density ρ and a current density J (which are functions from $\Omega \times \mathbb{R} \to \mathbb{R}$ and $\Omega \times \mathbb{R} \to \mathbb{R}^3$ correspondingly). Here ϵ_0 and μ_0 are positive physical constants called the permittivity and permeability of free space, respectively. Recall that the divergence of a vector field $E = (E_1, E_2, E_3)$ is

$$\operatorname{div} E = \nabla \cdot E = \sum_{i=1}^{3} \frac{\partial E_i}{\partial x_i}$$

and the curl of E is

$$\operatorname{curl} E = \nabla \times E = \left(\frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3}, \frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1}, \frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} \right).$$

In order to understand Maxwell's equations physically, it is instructive to consider an integral version of the PDE. By integrating the first two Maxwell's equations over a subdomain $D \subset \Omega$ and using the Gauss-Green theorem we have

$$\int_{\partial D} E \cdot \nu \, dS = \int_D \operatorname{div} E \, dx = \int_D \frac{\rho}{\epsilon_0} \, dx$$

and

$$\int_{\partial D} B \cdot \nu \, dS = \int_D \operatorname{div} B \, dx = 0,$$

where ν is the unit outer normal of ∂D. Let S be a surface in Ω with boundary given by an oriented curve C. For the last two equations the Stokes theorem gives

$$\int_C E \cdot dS = \int_S \operatorname{curl} E \cdot \nu \, dS = -\int_S \frac{\partial B}{\partial t} \cdot \nu \, dS$$

and

$$\int_C B \cdot dS = \int_S \operatorname{curl} E \cdot \nu \, dx = \mu_0 \int_S \left(J + \epsilon_0 \frac{\partial E}{\partial t} \right) \cdot \nu \, dS.$$

Observe that these equations hold for every subdomain D and surface S in Ω. It is also possible to go back to the differential version of Maxwell's equations by using the fact that if $f, g \in C(\mathbb{R}^3)$ and, for example,

$$\int_D f(x) \, dx = \int_D g(x) \, dx$$

for every $D \subset \mathbb{R}^3$, then $f(x) = g(x)$ for every $x \in \mathbb{R}^3$.

If there are no charges or currents in Maxwell's equations, we have

$$\begin{cases} \operatorname{div} E = 0, \\ \operatorname{div} B = 0, \\ \operatorname{curl} E = -\dfrac{\partial B}{\partial t}, \\ \operatorname{curl} B = c^{-2} \dfrac{\partial E}{\partial t}, \quad c = \dfrac{1}{\sqrt{\mu_0 \epsilon_0}}. \end{cases}$$

Since (exercise)

$$\operatorname{curl}(\operatorname{curl} E) = \nabla(\operatorname{div} E) - \operatorname{div}(\nabla E),$$

where the divergence is taken componentwise, that is, $\operatorname{div}(\nabla E) = (\operatorname{div} \nabla E_1, \operatorname{div} \nabla E_2, \operatorname{div} \nabla E_3)$, for every $E : \Omega \to \mathbb{R}^3$ with $E \in C^2(\mathbb{R}^3)$ we have

$$\begin{aligned} -\Delta E = -\operatorname{div} \nabla E &= \underbrace{\nabla(\operatorname{div} E)}_{=0} - \operatorname{div}(\nabla E) \\ &= \operatorname{curl}(\operatorname{curl} E) = -\operatorname{curl}\left(\frac{\partial B}{\partial t} \right) \\ &= -\frac{\partial}{\partial t}(\operatorname{curl} B) = -\frac{\partial}{\partial t}\left(c^{-2} \frac{\partial E}{\partial t} \right) \end{aligned}$$

and thus

$$c^2 \Delta E = \frac{\partial^2 E}{\partial t^2}.$$

That is, each component of $E = (E_1, E_2.E_3)$ satisfies the wave equation with the speed of waves c. Similarly, B satisfies the same wave equation. These are the electromagnetic waves.

Another special case of Maxwell's equations is electrostatistics. In this case there is no current and the field is independent of the time t. Then we have $\operatorname{curl} E = 0$, which implies that E is a gradient of a function (in a simply connected domain Ω). Thus $E = -\nabla V$, where V is called the electrostatic potential. Then

$$\operatorname{div} E = -\operatorname{div}(\nabla V) = -\Delta V$$

so that

$$\Delta V = -\frac{\rho}{\epsilon_0}.$$

That is, V is a solution to inhomogeneous Laplace's equation, called Poisson's equation. Note that V is defined only up to an additive constant, which does not affect the negative gradient E.

Fourier series is a series representation of a function defined on a bounded interval on the real axis as trigonometric polynomials. The function does not have to be smooth, but the convergence of a Fourier series is a delicate issue. However, the Fourier series gives the best square approximation of the function and it has many other elegant and useful properties. It also converges pointwise, if the function is smooth enough. Solutions to several problems in partial differential equations, including the Laplace operator, the heat operator and the wave operator, can be obtained using Fourier series and convolutions.

2 Fourier series and PDEs

Historically the study of the motion of a vibrating string fixed at its end points, and later the heat flow in a one-dimensional rod, lead to the development of the Fourier series and Fourier analysis. These physical phenomena are modeled by PDEs and, as we shall see, these problems can be solved using the Fourier series. Fourier claimed that for an arbitrary function

$$S_n f(t) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt} = \sum_{j=-n}^{n} \widehat{f}(j)(\cos(jt) + i\sin(jt)) \to f(t) \quad \text{as} \quad n \to \infty,$$

where

$$\widehat{f}(j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ijt}\, dt, \quad j \in \mathbb{Z}.$$

In other words, any function defined on a bounded interval on the real axis, in this case $[-\pi, \pi]$, can be represented as a Fourier series

$$f(t) = \sum_{j=-\infty}^{\infty} \widehat{f}(j) e^{ijt}.$$

This is somewhat analogous to Taylor series in the sense that it gives a method to express a given function as an infinite sum of the elementary functions

$$e_j(t) = e^{ijt}, \quad j \in \mathbb{Z}.$$

One of the advantages of the Fourier series is that it applies to functions that are not necessarily smooth, for example, functions $f \in L^2([-\pi, \pi])$. As we shall see, the convergence of the Fourier series is a delicate issue and it depends on in which sense the limit is taken. Fourier analytic methods play an important role in solving linear PDEs and they have many applications in several branches of mathematics. A useful property of the functions $e_j(t)$ from the PDE point of view is that each basis vector is an eigenfunction of the derivative operator in the sense that

$$e_j'(t) = ij e_j(t), \quad j \in \mathbb{Z}.$$

We shall start by taking a more careful look at the Fourier series. The Fourier series apply only for periodic functions. This is not a serious restriction, as we shall see.

2.1 Periodic functions*

We are mainly interested in real valued functions, but complex numbers are useful not only in Fourier analysis but also in PDEs. We say that a function $f:\mathbb{R}\to\mathbb{C}$ is 2π-periodic if for every $t\in\mathbb{R}$ we have

$$f(t+2\pi)=f(t). \tag{2.1}$$

More generally, a function $f:\mathbb{R}\to\mathbb{C}$ is called T-periodic , $T\in\mathbb{R}$, $T\neq 0$, if

$$f(t+T)=f(t) \tag{2.2}$$

for every $t\in\mathbb{R}$. Observe, that the period T is not unique. If f is T-periodic, then it is also nT-periodic for every $n=1,2,\dots$. The smallest positive value of T (if it exists) for which (2.2) holds is called the fundamental period. We shall consider functions f on $[-\pi,\pi]$ with $f(-\pi)=f(\pi)$, and assume that they are 2π-periodic by extending f periodically to the whole $\mathbb{R}$. In order to study a 2π-periodic function f it is enough to do so on any interval of length 2π. For this course we mainly work with the basic interval $[-\pi,\pi]$, but we could choose any other interval of length 2π as well.

THE MORAL: Every function $f:[a,b]\to\mathbb{C}$ defined on an interval with finite endpoints can be extended to a periodic function to the whole $\mathbb{R}$. Thus it is not too restrictive to consider periodic functions.

Remark 2.1. There is a natural connection between 2π-periodic functions on $\mathbb{R}$ and functions on the unit circle. A point on the unit circle is of the form $e^{i\theta}$, where θ is a real number that is unique up to integer multiples of 2π. If F is a function on the unit circle, then we may define for each real number θ

$$f(\theta)=F(e^{i\theta}),$$

and observe that with this definition, the function f is 2π-periodic. Thus 2π-periodic functions on $\mathbb{R}$ and functions on any interval of length 2π that take on the same value at its end points are the same mathematical objects.

Examples 2.2:

(1) The function $f:\mathbb{R}\to\mathbb{C}$, $f(t)=e^{ijt}$, $j\in\mathbb{Z}$, is 2π-periodic, since

$$f(t+2\pi)=e^{ij(t+2\pi)}=e^{ijt}\underbrace{e^{i2\pi j}}_{=1}=f(t)$$

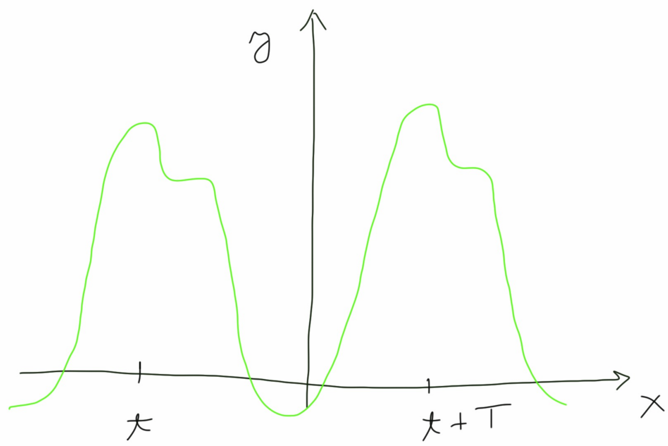


Figure 2.1: A graph of a periodic function.

for every t, since by Euler's formula

$$e^{i2\pi j} = \cos(2\pi j) + i\sin(2\pi j) = 1, \quad j \in \mathbb{Z}.$$

However, 2π is not the fundamental period of f. In the same way as above we can show that f is $\frac{2\pi}{|j|}$-periodic for $j \neq 0$. The fundamental period of f is $\frac{2\pi}{|j|}$ for $j \neq 0$.

(2) Let $L > 0$. The functions

$$f : \mathbb{R} \to \mathbb{R}, f(t) = \sin\left(\frac{j\pi t}{L}\right) \quad \text{and} \quad g : \mathbb{R} \to \mathbb{R}, g(t) = \cos\left(\frac{j\pi t}{L}\right), \quad j = 1, 2, \dots,$$

are $\frac{2L}{j}$-periodic.

If f and g are T-periodic functions with a common period T, then their product fg and linear combination $af + bg$, $a, b \in \mathbb{C}$, are also T-periodic. To prove the latter statement, let $F(t) = af(t) + bg(t)$. Then

$$F(t+T) = af(t+T) + bg(t+T) = af(t) + bg(t) = F(t).$$

The former statement is left as an exercise.

Lemma 2.3. Let $f : \mathbb{R} \to \mathbb{C}$ be a T-periodic function for some $T > 0$. Then for every $a \in \mathbb{R}$ we have

$$\int_0^T f(t)\,dt = \int_a^{a+T} f(t)\,dt.$$

THE MORAL: The integrals of a 2π-periodic function over intervals of length 2π coincide. In other words, the integral is independent of the interval.

Proof. If a is of the form kT for some integer k, then

$$\int_a^{a+T} f(t)\,dt = \int_{kT}^{(k+1)T} f(t)\,dt.$$

By changing variables $s = t - kT$ we have

$$\int_a^{a+T} f(t)\,dt = \int_0^T f(s+kT)\,ds = \int_0^T f(s)\,ds$$

since f is T-periodic and $f(s) = f(s+T) = \cdots = f(s+kT)$ for every $s \in \mathbb{R}$, $k \in \mathbb{Z}$.

Now if a is not of the form kT there exists a unique k such that

$$kT \leqslant a < (k+1)T.$$

This is because the intervals $[kT,(k+1)T)$ partition the real line. Thus

$$\int_a^{a+T} f(t)\,dt = \int_{kT}^{(k+1)T} f(t)\,dt - \int_{kT}^{a} f(t)\,dt + \int_{(k+1)T}^{a+T} f(t)\,dt \tag{2.3}$$

where observe that $a+T > kT+T = (k+1)T$. By the case $a = kT$ already considered we have

$$\int_{kT}^{(k+1)T} f(t)\,dt = \int_0^T f(t)\,dt.$$

For the last term in (2.3) we change variables $s = t - T$ and get

$$\int_{(k+1)T}^{a+T} f(t)\,dt = \int_{kT}^{a} f(s+T)\,ds = \int_{kT}^{a} f(s)\,ds$$

by the periodicity of f. This shows that the last two terms in (2.3) cancel each other. This proves the claim. □

2.2 The L^p space on $[-\pi,\pi]$*

To be able to consider functions that are not necessarily smooth, we develop the theory of L^p spaces. The most important spaces are L^1 and L^2, which are needed in the definition and properties of Fourier series.

Definition 2.4. Let $1 \leqslant p < \infty$. A function $f : [-\pi,\pi] \to \mathbb{C}$ belongs to $L^p([-\pi,\pi])$, if

$$\|f\|_{L^p([-\pi,\pi])} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^p\,dt\right)^{\frac{1}{p}} < \infty.$$

The number $\|f\|_{L^p([-\pi,\pi])}$ is called the L^p-norm of f.

THE MORAL: Instead of of the absolute value of the function, a power of the absolute value of the function is required to be integrable. Geometrically this means that the area of the graph of $|f|^p$ is finite. If $p = 2$, when we talk about square integrable functions. In particular, functions belonging to $L^p([-\pi,\pi])$ do not have to be continuous or smooth. The only requirement is that the integral above makes sense and is finite.

Remark 2.5. Note that

$$\|f\|_{L^p([-\pi,\pi])} < \infty \Longleftrightarrow \int_{-\pi}^{\pi} |f(t)|^p \, dt < \infty.$$

The factor $\frac{1}{2\pi}$ and the power $\frac{1}{p}$ are not more than normalising parameters. For example, if $f : [-\pi,\pi] \to \mathbb{R}$, $f(t) = 1$, then

$$\|f\|_{L^p([-\pi,\pi])} = 1 \quad \text{and} \quad \|af\|_{L^p([-\pi,\pi])} = |a|, \quad a \in \mathbb{R}.$$

This shows that the definition is compatible with constant functions and scalings.

Examples 2.6:

(1) **Claim**: $C([-\pi,\pi]) \subset L^2([-\pi,\pi])$.

Reason.

$$\int_{-\pi}^{\pi} |f(t)|^2 dt \leqslant 2\pi(\max_{t\in[-\pi,\pi]} |f(t))|)^2 < \infty.$$

∎

The reverse inclusion is not true. For example, $f : [-\pi,\pi] \to \mathbb{R}$,

$$f(t) = \begin{cases} 0, & t \in [-\pi, 0), \\ 1, & t \in [0,\pi], \end{cases}$$

is not continuous, but $f \in L^2([-\pi,\pi])$. Thus $L^2([-\pi,\pi])$ is not a subset of $C([-\pi,\pi])$.

(2) Let $f : [-\pi,\pi] \to \mathbb{R}$,

$$f(t) = \begin{cases} |t|^{-\frac{1}{4}}, & t \neq 0, \\ 0, & t = 0. \end{cases}$$

Then

$$\int_{-\pi}^{\pi} |f(t)|^2 \, dt = \int_{-\pi}^{\pi} \frac{1}{\sqrt{|t|}} \, dt = 2\int_{0}^{\pi} \frac{1}{\sqrt{|t|}} \, dt = 2\Big|_{0}^{\pi} 2\sqrt{|t|} = 4\sqrt{\pi} < \infty.$$

Thus $f \in L^2([-\pi,\pi))$ and

$$\|f\|_{L^2([-\pi,\pi])} = \left(\frac{4\sqrt{\pi}}{2\pi}\right)^{\frac{1}{2}} = \sqrt{2}\pi^{-\frac{1}{4}}.$$

(3) Let $f:[-\pi,\pi]\to\mathbb{R}$,

$$f(x)=\begin{cases}\frac{1}{\sqrt{|t|}}, & t\neq 0,\\ 0, & t=0.\end{cases}$$

Then

$$\int_{-\pi}^{\pi}|f(t)|^2\,dt=\int_{-\pi}^{\pi}\frac{1}{|t|}\,dt=\infty.$$

Thus $f\notin L^2([-\pi,\pi])$. Observe, that $f\in L^1([-\pi,\pi])$ so that, in general, $L^1([-\pi,\pi])$ is not contained in $L^2([-\pi,\pi])$.

THE MORAL: Both functions in (2) and (3) have a singularity at $t=0$. Whether the function belongs to $L^2([-\pi,\pi])$ depends on how fast the function blows up near $t=0$.

Next we consider vector space properties. Indeed, $L^2([-\pi,\pi])$ is a complex vector space with the natural addition and multiplication operations

$$(f+g)(t)=f(t)+g(t)\quad\text{and}\quad(af)(t)=af(t),\quad a\in\mathbb{C}.$$

Note that vectors (or elements) in $L^2([-\pi,\pi])$ are functions. We define an inner product of $f,g\in L^2([-\pi,\pi])$ by

$$\begin{aligned}\langle f,g\rangle&=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\overline{g(t)}\,dt\\&=\frac{1}{2\pi}\left(\int_{-\pi}^{\pi}\operatorname{Re}(f(t)\overline{g(t)})\,dt+i\int_{-\pi}^{\pi}\operatorname{Im}(f(t)\overline{g(t)})\,dt\right).\end{aligned}$$

Here $\overline{z}=x-iy\in\mathbb{C}$ is the complex conjugate of $z=x+iy\in\mathbb{C}$, where $x,y\in\mathbb{R}$ and i is the imaginary unit.

THE MORAL: An inner product gives a notion of an angle between vectors and orthogonality is the same way as for the standard Euclidean inner product we have $\langle x,y\rangle=|x||y|\cos\alpha$, where α is the angle between x and y. There are many ways to define inner products depending on the applications. We shall focus on the standard inner product in $L^2([-\pi,\pi])$, but several results hold true for other inner products as well.

Example 2.7. Let $e_j:[-\pi,\pi]\to\mathbb{C}$, $e_j(t)=e^{ijt}=\cos(jt)+i\sin(jt)$, $j\in\mathbb{Z}$ (Euler's formula). Then $e_j\in C([-\pi,\pi])$ and consequently $e_j\in L^2([-\pi,\pi])$ with

$$\|e_j\|_{L^2([-\pi,\pi])}=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{|e^{ijt}|^2}_{=1}\,dt\right)^{\frac12}=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}1\,dt\right)^{\frac12}=1,\quad j=1,2,\dots.$$

The inner product of two such functions is

$$\begin{aligned}\langle e_j,e_k\rangle&=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ijt}\overline{e^{ikt}}\,dt=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ijt}e^{-ikt}\,dt\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(j-k)t}\,dt=\Big|_{-\pi}^{\pi}\frac{1}{2\pi}\frac{e^{i(j-k)t}}{i(j-k)}=0\end{aligned}$$

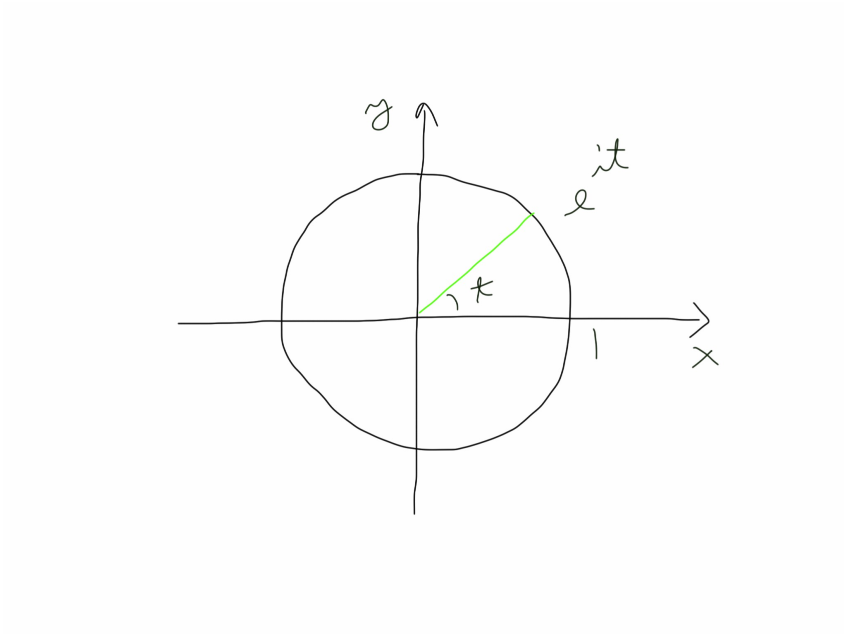


Figure 2.2: Polar coordinates.

provided $j \neq k$. On the other hand if $j = k$ we have $e^{ijt}\overline{e^{ijt}} = |e^0|^2 = 1$ so that $\langle e_j, e_j \rangle = 1$. This shows that the set $\{e_j\}_{j\in\mathbb{Z}}$ is an orthonormal set in $L^2([-\pi,\pi])$ and we summarize this as

$$\langle e_j, e_k \rangle = \begin{cases} 0, & j \neq k, \\ 1, & j = k. \end{cases}$$

Sometimes this is denoted as $\langle e_j, e_k \rangle = \delta_{jk}$, where δ_{ik} is Kronecker's delta.

Remark 2.8. The inner product in $L^2([-\pi,\pi])$ satisfies the following properties:

(1) $\langle f, f \rangle = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\overline{f(t)}\,dt = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} |f(t)|^2\,dt \geqslant 0.$

(2) $\langle f, f \rangle = 0$ if and only if $f = 0$ in $L^2([-\pi,\pi])$, that is, $\|f\|_{L^2([-\pi,\pi])} = 0$.

(3) $\langle f, g \rangle = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \overline{\overline{f(t)}g(t)}\,dt = \dfrac{1}{2\pi}\overline{\displaystyle\int_{-\pi}^{\pi} \overline{f(t)}g(t)\,dt} = \overline{\langle g, f \rangle}.$

(4) $\langle af, g \rangle = a\langle f, g \rangle$, $a \in \mathbb{C}$,

(5) $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$.

Properties (1)–(5) in Remark 2.8 can be taken as the definition of an abstract inner product $\langle x, y \rangle$, $x, y \in H$ on a complex vector space H. If $x, y \in H$ and $\langle x, y \rangle = 0$, we say that x and y are orthogonal. Observe that this definition is symmetric: If x, y are orthogonal then y, x are orthogonal. Let $\|\cdot\|$ be the norm induced by an inner product of H, that is,

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}}, \quad x \in H.$$

Moreover, for every $x, y \in H$ with $\langle x, y\rangle = 0$ (x, y are orthogonal) we have

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

This is the Pythagorean theorem, see (2.5) (exercise).

THE MORAL: A norm is a length of a vector.

Examples 2.9:

(1) $\langle x, y\rangle = \sum_{j=1}^n x_j y_j$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ is an inner product in the real vector space $\mathbb{R}^n$. Moreover

$$\|x\| = \langle x, x\rangle^{\frac{1}{2}} = \sqrt{\sum_{j=1}^n x_j^2}$$

is the Euclidean norm in $\mathbb{R}^n$.

(2) $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$, $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_n)$ is an inner product in the complex vector space $\mathbb{C}^n$. Here $\overline{w}$ is the complex conjugate of w. Moreover

$$\|z\| = \langle z, z\rangle^{\frac{1}{2}} = \sqrt{\sum_{j=1}^n z_j \overline{z}_j} = \sqrt{\sum_{j=1}^n |z_j|^2}$$

is a norm in $\mathbb{C}^n$.

The L^2-norm is induced by the standard L^2-inner product, since

$$\|f\|_{L^2([-\pi,\pi])} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2\,dt\right)^{\frac{1}{2}} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{f(t)}\,dt\right)^{\frac{1}{2}} = \langle f, f\rangle^{\frac{1}{2}}.$$

Here we used the fact that $z\overline{z} = |z|^2$, $z \in \mathbb{C}$.

Remark 2.10. The norm $\|\cdot\|_{L^2([-\pi,\pi])}$ satisfies the following properties:

(1) $\|f\|_{L^2([-\pi,\pi])} \geqslant 0$ for every $f \in L^2([-\pi,\pi])$.

(2) $\|f\|_{L^2([-\pi,\pi])} = 0$ if and only if $f = 0$ in $L^2([-\pi,\pi])$.
WARNING: This does not imply that $f(t) = 0$ for every $t \in [-\pi,\pi]$. In fact, it implies that $f(t) = 0$ for almost every $t \in [-\pi,\pi]$ with respect to the one-dimensional (Lebesgue) measure.
AGREEMENT: $f = g$ in $L^2([-\pi,\pi])$ if and only if

$$\|f-g\|_{L^2([-\pi,\pi])} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)-g(t)|^2\,dt\right)^{\frac{1}{2}} = 0.$$

(3) $\|af\|_{L^2([-\pi,\pi])} = |a|\|f\|_{L^2([-\pi,\pi])}$ for every $a \in \mathbb{C}$ and $f \in L^2([-\pi,\pi])$.

(4) The triangle inequality

$$\|f+g\|_{L^2([-\pi,\pi])} \leqslant \|f\|_{L^2([-\pi,\pi])} + \|g\|_{L^2([-\pi,\pi])}$$

holds for every $f, g \in L^2([-\pi,\pi])$, see Remark 2.14 below.

Properties (1)–(5) in Remark 2.10 can be taken as the definition of an abstract norm $\|\cdot\|$ in a vector space.

We shall prove the following Cauchy-Schwarz inequality with the general properties of an inner product.

Lemma 2.11 (Cauchy-Schwartz inequality). Let H be an inner product space. For every $x, y \in H$, we have

$$|\langle x,y\rangle| \leqslant \langle x,x\rangle^{\frac{1}{2}} \langle y,y\rangle^{\frac{1}{2}}.$$

Proof. Denote by $\|\cdot\|$ the norm defined by the inner product of H, that is, $\|x\| = \langle x,x\rangle^{\frac{1}{2}}$, $x \in H$. If $y=0$ it is clear that the Cauchy-Schwarz holds with equality. So let us assume that $y \neq 0$. We set

$$z = x - \frac{\langle x,y\rangle}{\langle y,y\rangle} y.$$

Then

$$\langle z,y\rangle = \langle x,y\rangle - \left\langle \frac{\langle x,y\rangle}{\langle y,y\rangle} y, y \right\rangle = 0.$$

Thus vectors z and y are orthogonal. Observe, that $\frac{\langle x,y\rangle}{\langle y,y\rangle} y$ is the projection of x to y. Since

$$x = \frac{\langle x,y\rangle}{\langle y,y\rangle} y + z$$

we can use the Pythagorean theorem to obtain

$$\|x\|^2 = \frac{\langle x,y\rangle^2}{\langle y,y\rangle^2}\|y\|^2 + \|z\|^2 = \frac{\langle x,y\rangle^2}{\|y\|^2} + \|z\|^2 \geqslant \frac{\langle x,y\rangle^2}{\|y\|^2}.$$

This proves the claim. □

Remarks 2.12:

(1) The Cauchy-Schwarz inequality in $L^2([-\pi,\pi])$ reads

$$|\langle f,g\rangle| \leqslant \|f\|_{L^2([-\pi,\pi])}\|g\|_{L^2([-\pi,\pi])}.$$

This implies

$$\left|\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt\right| \leqslant \left(\int_{-\pi}^{\pi} |f(t)|^2\,dt\right)^{\frac{1}{2}} \left(\int_{-\pi}^{\pi} |g(t)|^2\,dt\right)^{\frac{1}{2}}$$

and

$$\|fg\|_{L^1([-\pi,\pi])} \leqslant \|f\|_{L^2([-\pi,\pi])}\|g\|_{L^2([-\pi,\pi])}.$$

These special cases of Hölder's inequality are very useful inequalities for integrals.

(2) By replacing $f(t)$ with $|f(t)|$ and choosing $g(t)=1$, we may conclude that $L^2([-\pi,\pi]) \subset L^1([-\pi,\pi])$. We saw in Example 2.6 (3) that the converse inclusion does not hold, in general.

Lemma 2.13. If H is a space with inner product then $\|x\| = \langle x,x\rangle^{\frac{1}{2}}$, $x \in H$, is a norm in H.

THE MORAL: In particular, this means that a norm induced by an inner product satisfies the triangle inequality.

Proof. All other properties of a norm are easily verified except maybe for the triangle inequality. To prove this, we observe that

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle = \langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,y\rangle \\
&= \langle x,x\rangle + \langle y,y\rangle + \overline{\langle x,y\rangle} + \langle x,y\rangle \\
&= \|x\|^2 + \|y\|^2 + 2\operatorname{Re}\langle x,y\rangle.
\end{aligned}$$

Now the Cauchy-Schwarz inequality implies

$$2|\operatorname{Re}\langle x,y\rangle| \leqslant 2|\langle x,y\rangle| \leqslant 2\|x\|\,\|y\|,$$

from which we conclude that

$$\|x+y\|^2 \leqslant \|x\|^2 + \|y\|^2 + 2\|x\|\,\|y\| = (\|x\|+\|y\|)^2. \qquad \square$$

Remark 2.14. The triangle inequality in $L^2([-\pi,\pi])$ reads

$$\|f+g\|_{L^2([-\pi,\pi])} \leqslant \|f\|_{L^2([-\pi,\pi])} + \|g\|_{L^2([-\pi,\pi])}.$$

This implies

$$\left(\int_{-\pi}^{\pi} |f(t)+g(t)|^2\,dt\right)^{\frac{1}{2}} \leqslant \left(\int_{-\pi}^{\pi} |f(t)|^2\,dt\right)^{\frac{1}{2}} + \left(\int_{-\pi}^{\pi} |g(t)|^2\,dt\right)^{\frac{1}{2}}.$$

2.3 The Fourier series*

We begin with the definition of the Fourier series.

Definition 2.15 (Fourier series). Let $f \in L^1([-\pi,\pi])$. The nth partial sum of a Fourier series is

$$S_n f(t) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt}, \quad n = 0,1,2,\ldots,$$

where

$$\widehat{f}(j) = \langle f, e_j\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-ijt}\,dt, \quad j \in \mathbb{Z},$$

is the jth Fourier coefficient of f. Here $e_j : \mathbb{R} \to \mathbb{C}$, $e_j(t) = e^{ijt}$, $j \in \mathbb{Z}$. The Fourier series of f is the limit of the partial sums $S_n f$ as $n \to \infty$, provided the limit exists in some reasonable sense. In this case we may write

$$f(t) = \lim_{n\to\infty} S_n f(t) = \lim_{n\to\infty} \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt} = \sum_{j=-\infty}^{\infty} \widehat{f}(j) e^{ijt}.$$

THE MORAL: This is a series approximation of a function. The point is that this approximation also applies to functions which do not have to be smooth as in the case of Taylor series, for example. At least in the definition, it is enough to assume that $f \in L^1([-\pi,\pi])$. As we shall see, the space $L^2([-\pi,\pi])$ is needed to understand the Fourier coefficients and the convergence of the Fourier series.

Remarks 2.16:

(1) In the convergence of the Fourier series we always consider symmetric partial sums, where the indices run from $-n$ to n.

(2) By the Cauchy-Schwarz inequality, see Lemma 2.11, we have

$$|\widehat{f}(j)| = |\langle f, e_j\rangle| \leqslant \|f\|_{L^2([-\pi,\pi])} \underbrace{\|e_j\|_{L^2([-\pi,\pi])}}_{=1} = \|f\|_{L^2([-\pi,\pi])} < \infty, \quad j \in \mathbb{Z}.$$

This means that the Fourier coefficients are well defined and finite also if $f \in L^2([-\pi,\pi])$.

(3) Since e_j, $j \in \mathbb{Z}$, is 2π-periodic, the partial sum $S_n f(t)$, $n = 0,1,2,\dots$, of a Fourier series is 2π-periodic. Consequently the pointwise limit

$$f(t) = \lim_{n\to\infty} S_n f(t)$$

is 2π-periodic, whenever it exists.

THE MORAL: If the Fourier series converges pointwise, the sum is 2π-periodic. In this sense we can only approximate 2π-periodic functions by the Fourier series.

Example 2.17. Let $f : [-\pi,\pi] \to \mathbb{R}$, $f(t) = t$. Then $f \in L^1([-\pi,\pi])$, since

$$\int_{-\pi}^{\pi} |f(t)|\,dt = \int_{-\pi}^{0} -t\,dt + \int_0^{\pi} t\,dt = -\Big|_{-\pi}^{0} \frac{t^2}{2} + \Big|_0^{\pi} \frac{t^2}{2} = \pi^2 < \infty.$$

The Fourier coefficients $f(j)$, $j \neq 0$, can be calculated by integration by parts as

$$\begin{aligned}\widehat{f}(j) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} t e^{-ijt}\,dt = \frac{1}{2\pi}\Big|_{-\pi}^{\pi} \frac{te^{-ijt}}{-ij} - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-ijt}}{-ij}\,dt \\ &= \frac{1}{2\pi}\left(\frac{\pi e^{-ij\pi}}{-ij} - \frac{-\pi e^{ij\pi}}{-ij}\right) - \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-ijt}}{-ij}\,dt}_{=0} = \frac{\cos((j+1)\pi)}{ij}.\end{aligned}$$

On the other hand,

$$\widehat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} t \underbrace{e^0}_{=1}\,dt = 0.$$

Thus

$$\widehat{f}(j) = \begin{cases} 0, & j = 0, \\ \frac{\cos((j+1)\pi)}{ij}, & j \neq 0 \end{cases}$$

and

$$S_n f(t) = \sum_{j=1}^{n}\left(\frac{\cos((j+1)\pi)}{ij} e^{ijt} - \frac{\cos((-j+1)\pi)}{ij} e^{-ijt}\right) = 2\sum_{j=1}^{n} \frac{\cos((j+1)\pi)}{j}\sin(jt).$$

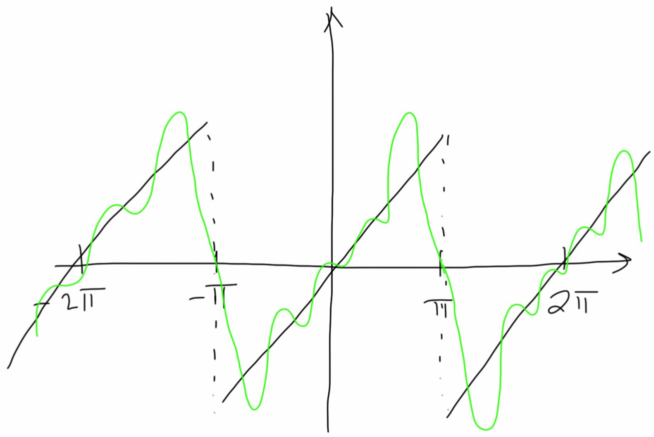


Figure 2.3: The Fourier series approximation of the saw tooth function.

Remark 2.18. We make two observations related to the previous example.

(1) The function f, extended as 2π-periodic function to $\mathbb{R}$, is not continuous at the points $t = \pm k\pi$, $k \in \mathbb{Z}$. At a point of discontinuity, for example at $t = \pi$, we have

$$S_n f(\pi) = 0 = \frac{1}{2}\left(\lim_{t\to\pi^-} f(t) + \lim_{t\to\pi} f(t)\right).$$

The sum of the Fourier series at a point of a jump discontinuity is the average of the limits from the both sides. This ia a general property of the Fourier series. Moreover, there is Gibb's phenomenon

$$\lim_{n\to\infty} \big(\max_{t\in[-\pi,\pi])} S_n(t)\big) \approx 1,179.$$

This means that the absolute error made in the approximation is about 18% independent of the degree of the approximation. In particular, the error does not go to zero as $n \to \infty$. This is an unexpected phenomenon.

(2) We have $|\widehat{f}(j)| \leqslant \frac{2}{j}$ for $j \neq 0$ while $\widehat{f}(0) = 0$. It follows that $\widehat{f}(j) \to 0$ as $|j| \to \infty$. This kind of decay property of the Fourier coefficients holds for every function $f \in L^1([-\pi,\pi])$ or $f \in L^2([-\pi,\pi])$, see Remark 2.26 (2).

Example 2.19. Let $f : [-\pi,\pi] \to \mathbb{R}$,

$$f(t) = \begin{cases} -1, & t \in [-\pi, 0), \\ 1, & t \in [0,\pi]. \end{cases}$$

It is clear that the function $f \in L^1([-\pi,\pi])$ so that we can calculate the Fourier coefficients $\widehat{f}(j)$, $j \neq 0$, as

$$\begin{aligned}
\widehat{f}(j) &= \frac{1}{2\pi}\left(-\int_{-\pi}^{0} e^{-ijt}\,dt + \int_{0}^{\pi} e^{-ijt}\,dt\right) = \frac{1}{2\pi}\left(-\Big|_{-\pi}^{0} \frac{e^{-ijt}}{-ij} + \Big|_{0}^{\pi} \frac{e^{-ijt}}{-ij}\right)\\
&= \frac{1}{2\pi i j}\left(e^{0} - e^{ij\pi} - (e^{-ij\pi} - e^{0})\right) = \frac{1}{2\pi i j}(1-\cos(j\pi)-\cos(j\pi)+1)\\
&= \frac{1}{\pi i j}(1-\cos(j\pi)) = \frac{i}{\pi j}(\cos(j\pi)-1) = \begin{cases} 0, & j \text{ even},\\ -\frac{2i}{\pi j}, & j \text{ odd}.\end{cases}
\end{aligned}$$

For $j = 0$ we have

$$\widehat{f}(0) = \frac{1}{2\pi}\left(-\int_{-\pi}^{0} 1\,dt + \int_{0}^{\pi} 1\,dt\right) = 0.$$

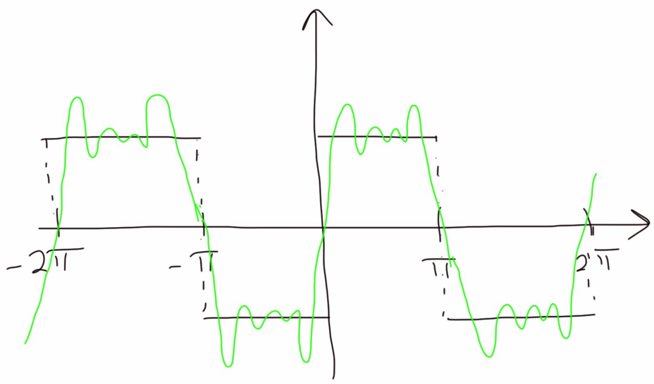


Figure 2.4: The Fourier series approximation of the sign function.

Note that at the points of jump discontinuity we have

$$S_n f(0) = 0 = \frac{1}{2}\left(\lim_{t\to 0^-} f(t) + \lim_{t\to 0} f(t)\right).$$

We collect some easy properties of the Fourier coefficients in the following proposition.

Lemma 2.20. Let $f, g \in L^1([-\pi,\pi])$ and $a, b \in \mathbb{C}$.

(1) (Linearity) $\widehat{af+bg}(j) = a\widehat{f}(j) + b\widehat{g}(j)$, $j \in \mathbb{Z}$.

(2) (Boundedness) $|\widehat{f}(j)| \leqslant \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} |f(t)|\,dt = \|f\|_{L^1([-\pi,\pi])}$, $j \in \mathbb{Z}$.

(3) $\widehat{f}(0) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\,dt$.

(4) (Conjugation) $\widehat{\overline{f}}(j) = \overline{\widehat{f}(-j)}$, where $\overline{f}$ is the complex conjugate of f.

(5) (Reflection) $\widehat{f(-t)}(j) = \widehat{f}(-j)$, $j \in \mathbb{Z}$.

(6) (Shift) $\widehat{f(t+s)}(j) = e^{ijs}\widehat{f}(j)$, $j \in \mathbb{Z}$, for a fixed s.

(7) (Modulation) $\widehat{e^{ikt}f(t)}(j) = \widehat{f}(j-k)$, $j \in \mathbb{Z}$, for a fixed $k \in \mathbb{Z}$.

Proof. $\boxed{(1)}$ Property (i) is an immediate consequence of linearity of the integral.

$\boxed{(2)}$ $|\widehat{f}(j)| = \dfrac{1}{2\pi}\left|\displaystyle\int_{-\pi}^{\pi} f(t)e^{-ijt}\,dt\right| \leqslant \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} |f(t)e^{-ijt}|\,dt = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} |f(t)|\,dt$.

$\boxed{(3)}$ $\widehat{f}(0) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\underbrace{e^{-i0t}}_{=1}\,dt = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\,dt$.

$\boxed{(6)}$ A change of variables $u = t+s$ gives

$$\widehat{f(t+s)}(j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t+s)e^{-ijt}\,dt = \frac{1}{2\pi}\int_{-\pi+s}^{\pi+s} f(u)e^{-ij(u-s)}\,du.$$

Now using Lemma 2.3, and the fact that the function $f(u)e^{-ij(u-s)}$ is 2π-periodic, we have

$$\frac{1}{2\pi}\int_{-\pi+s}^{\pi+s} f(u)e^{-ij(u-s)}\,du = e^{ijs}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)e^{-iju}\,du = e^{ijs}\widehat{f}(j).$$

Other claims are left as exercises. □

2.4 The best square approximation*

It is very instructive to consider Fourier series in terms of projections.

Lemma 2.21. The projection of the vector $f \in L^2([-\pi,\pi])$ to a subspace spanned by $\{e_j\}_{j=-n}^{n}$ is

$$S_nf(t) = \sum_{j=-n}^{n} \langle f, e_j\rangle e_j(t) = \sum_{j=-n}^{n} \widehat{f}(j)e^{ijt}, \quad n = 0,1,2,\dots,$$

where

$$\widehat{f}(j) = \langle f, e_j\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ijt}\,dt.$$

THE MORAL: Let $x = (x_1,\dots,x_n) \in \mathbb{R}^n$. The projection of x to the subspace spanned by the first k standard basis vectors e_j, $j = 1,\dots,k$, is $\sum_{j=1}^{k}\langle x_j, e_j\rangle e_j$. The previous lemma tells that the same holds true in $L^2([-\pi,\pi])$.

Proof.

$$\begin{aligned}\langle f - S_n f, e_k\rangle &= \left\langle f - \sum_{j=-n}^{n} \widehat{f}(j) e_j, e_k \right\rangle \\ &= \langle f, e_k\rangle - \sum_{j=-n}^{n} \widehat{f}(j)\langle e_j, e_k\rangle \\ &= \widehat{f}(k) - \widehat{f}(k) = 0, \quad k = -n, \dots, n,\end{aligned}$$

implies

$$\left\langle f - S_n f, \sum_{j=-n}^{n} a_j e_j \right\rangle = \sum_{j=-n}^{n} \overline{a_j} \langle f - S_n f, e_j\rangle = 0, \tag{2.4}$$

for every $a_j \in \mathbb{C}$, $j = -n, \dots, n$. Since any vector belonging to the subspace spanned by $\{e_j\}_{j=-n}^{n}$ is a linear combination $\sum_{j=-n}^{n} a_j e_j$, this means that $f - S_n f$ is orthogonal to the subspace spanned by $\{e_j\}_{j=-n}^{n}$. $\square$

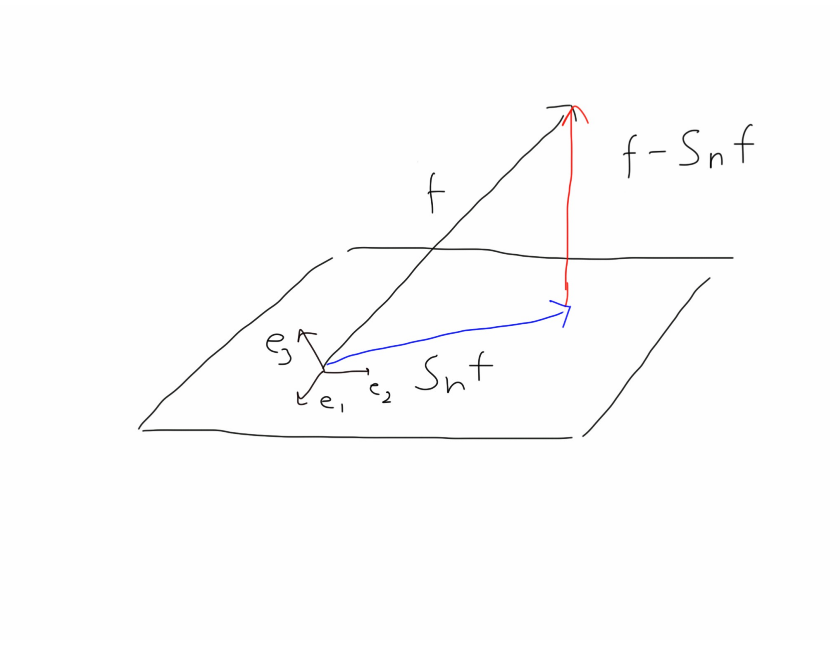


Figure 2.5: The least square approximation.

In particular, this implies that $f = S_n f + (f - S_n f)$, where $S_n f$ and $f - S_n f$ are orthogonal. From this we have

$$
\begin{aligned}
\|f\|^2_{L^2([-\pi,\pi])} &= \|f - S_nf + S_nf\|^2_{L^2([-\pi,\pi])} \\
&= \langle (f - S_nf) + S_nf, (f - S_nf) + S_nf \rangle \\
&= \langle f - S_nf, f - S_nf \rangle + \underbrace{\langle f - S_nf, S_nf \rangle}_{=0} \\
&\qquad + \underbrace{\langle S_nf, f - S_nf \rangle}_{=0} + \langle S_nf, S_nf \rangle \\
&= \|f - S_nf\|^2_{L^2([-\pi,\pi])} + \|S_nf\|^2_{L^2([-\pi,\pi])}.
\end{aligned}
\tag{2.5}
$$

This is the Pythagorean theorem in $L^2([-\pi,\pi])$.

Since $\{e_j\}_{j\in\mathbb{Z}}$ is an orthonormal set in $L^2([-\pi,\pi])$, we obtain

$$
\begin{aligned}
\left\| \sum_{j=-n}^{n} \widehat{f}(j) e_j \right\|^2_{L^2([-\pi,\pi])} &= \left\langle \sum_{j=-n}^{n} \widehat{f}(j) e_j, \sum_{k=-n}^{n} \widehat{f}(k) e_k \right\rangle \\
&= \sum_{j=-n}^{n} \sum_{k=-n}^{n} \widehat{f}(j) \overline{\widehat{f}(k)} \langle e_j, e_k \rangle = \sum_{j=-n}^{n} |\widehat{f}(j)|^2.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|f\|^2_{L^2([-\pi,\pi])} &= \|f - S_nf\|^2_{L^2([-\pi,\pi])} + \|S_nf\|^2_{L^2([-\pi,\pi])} \\
&= \|f - S_nf\|^2_{L^2([-\pi,\pi])} + \sum_{j=-n}^{n} |\widehat{f}(j)|^2.
\end{aligned}
\tag{2.6}
$$

THE MORAL: Note that $f = S_nf + (f - S_nf)$, where S_nf is the Fourier series approximation of f and $f - S_nf$ is the error made in the approximation. The partial sums S_nf approximate f with the mean square error $\|f - S_nf\|_{L^2([-\pi,\pi])}$.

TERMINOLOGY:

$$
\sum_{j=-n}^{n} a_j e_j = \sum_{j=-n}^{n} a_j e^{ijt} = \sum_{j=-n}^{n} a_j (e^{it})^j = \sum_{j=-n}^{n} a_j z^j,
$$

where $z = e^{it}$, $a_j \in \mathbb{C}$, is called a trigonometric polynomial of degree n.

Example 2.22. Trigonometric polynomials are different for the standard polynomials. For example,

$$
\cos(jt) = \frac{e^{ijt} + e^{-ijt}}{2} \quad \text{and} \quad \sin(jt) = \frac{e^{ijt} - e^{-ijt}}{2i}, \quad j \in \mathbb{Z},
$$

are trigonometric polynomials.

Theorem 2.23 (Theorem of best square approximation). If $f \in L^2([-\pi,\pi])$, then

$$
\|f - S_nf\|_{L^2([-\pi,\pi])} \leqslant \left\| f - \sum_{j=-n}^{n} a_j e_j \right\|_{L^2([-\pi,\pi])}
$$

for every $a_j \in \mathbb{C}$, $j = -n, \dots, n$.

THE MORAL: The partial sum $S_n f$ of a Fourier series gives the best L^2-approximation for the function $f \in L^2([-\pi,\pi])$ among all trigonometric polynomials of degree n.

Proof. Clearly

$$f - \sum_{j=-n}^{n} a_j e_j = \left(f - \sum_{j=-n}^{n} \widehat{f}(j) e_j \right) + \sum_{j=-n}^{n} (\widehat{f}(j) - a_j) e_j,$$

where

$$\left\langle f - \sum_{j=-n}^{n} \widehat{f}(j) e_j, \sum_{j=-n}^{n} (\widehat{f}(j) - a_j) e_j \right\rangle = 0,$$

since $\sum_{j=-n}^{n} (\widehat{f}(j) - a_j) e_j$ belongs to the subspace spanned by $\{e_j\}_{j=-n}^{n}$, see (2.4). The Pythagorean theorem implies

$$\begin{aligned} \left\| f - \sum_{j=-n}^{n} a_j e_j \right\|^2_{L^2([-\pi,\pi])} &= \left\| f - \sum_{j=-n}^{n} \widehat{f}(j) e_j \right\|^2_{L^2([-\pi,\pi])} + \underbrace{\left\| \sum_{j=-n}^{n} (\widehat{f}(j) - a_j) e_j \right\|^2_{L^2([-\pi,\pi])}}_{\geqslant 0} \\ &\geqslant \left\| f - \sum_{j=-n}^{n} \widehat{f}(j) e_j \right\|^2_{L^2([-\pi,\pi])}. \end{aligned}$$

$\square$

Remark 2.24. Equality occurs in the previous theorem if and only if we have equalities throughout in the proof of the theorem. This implies that the equality occurs if and only if

$$\left\| \sum_{j=-n}^{n} a_j e_j - S_n f \right\|_{L^2([-\pi,\pi])} = 0,$$

that is, $S_n f = \sum_{j=-n}^{n} a_j e_j$ in $L^2([-\pi,\pi])$.

Let $f \in L^2([-\pi,\pi])$. By (2.6) we have

$$\|f\|^2_{L^2([-\pi,\pi])} = \underbrace{\|f - S_n f\|^2_{L^2([-\pi,\pi])}}_{\geqslant 0} + \sum_{j=-n}^{n} |\widehat{f}(j)|^2 \geqslant \sum_{j=-n}^{n} |\widehat{f}(j)|^2, \quad n = 1, 2, \dots.$$

It follows that

$$\sum_{j=-\infty}^{\infty} |\widehat{f}(j)|^2 = \lim_{n\to\infty} \sum_{j=-n}^{n} |\widehat{f}(j)|^2 \leqslant \|f\|^2_{L^2([-\pi,\pi])}. \tag{2.7}$$

This is called Bessel's inequality. Equality occurs in (2.7) if and only if

$$\lim_{n\to\infty} \|f - S_n f\|^2_{L^2([-\pi,\pi])} = 0,$$

in which case we have Parseval's identity

$$\|f\|^2_{L^2([-\pi,\pi])} = \sum_{j=-\infty}^{\infty} |\widehat{f}(j)|^2. \tag{2.8}$$

THE MORAL: Parseval's identity is the Pythagorean theorem with infinitely many coefficients in the sense that the Fourier coefficients give the coordinates of a function in $L^2([-\pi,\pi])$.

Parseval's identity is equivalent with the convergence of the partial sums of the Fourier series in the L^2-sense, which is the content of the following result.

Theorem 2.25. Let $f \in L^2([-\pi,\pi])$. Then

$$\lim_{n\to\infty} \|f - S_n f\|_{L^2([-\pi,\pi])} = 0.$$

THE MORAL: The partial sums $S_n f$ approximate $f \in L^2([-\pi,\pi])$ so that the mean square error $\|f - S_n f\|_{L^2([-\pi,\pi])}$ goes to zero. This means that the Fourier series always converges in $L^2([-\pi,\pi])$. In other words, every function in $L^2([-\pi,\pi])$ can be represented as a Fourier series.

WARNING: The partial sums of the Fourier series of a L^2-function are only claimed to converge in the L^2-norm. This mode of convergence is rather weak. In particular, it does not follow in general that $f(t) = \lim_{n\to\infty} S_n f(t)$ pointwise for every $t \in [-\pi,\pi]$, see Example 2.17.

Proof. Let $f \in L^2([-\pi,\pi])$ and $\varepsilon > 0$. By density of the trigonometric polynomials in $L^2([-\pi,\pi])$, there exists a trigonometric polynomial g of some degree m such that $\|f - g\|_{L^2([-\pi,\pi])} < \varepsilon$, but the proof of this density result is out of the scope of this course. Combining this with the best approximation Theorem 2.23, for $n \geqslant m$ we have

$$\|f - S_n f\|_{L^2([-\pi,\pi])} \leqslant \|f - g\|_{L^2([-\pi,\pi])} < \varepsilon.$$

Here we use the fact that since g is a trigonometric polynomial of degree m and $n \geqslant m$, we may consider g as a trigonometric polynomial of order n with the interpretation that some of the coefficients are zero which proves the claim. □

Remarks 2.26:

(1) Theorem 2.25 implies that $\{e_j\}_{j=-\infty}^{\infty}$ is an orthonormal basis for the space $L^2([-\pi,\pi])$ in the sense that

$$\lim_{n\to\infty} \left\| \sum_{j=-n}^{n} \widehat{f}(j) e_j - f \right\|_{L^2([-\pi,\pi])} = 0$$

for every $f \in L^2([-\pi,\pi])$. This means that

$$f = \lim_{n\to\infty} \sum_{j=-n}^{n} \widehat{f}(j) e_j = \sum_{j=-\infty}^{\infty} \widehat{f}(j) e_j$$

in $L^2([-\pi,\pi])$. In this sense every function in $L^2([-\pi,\pi])$ can be represented as a Fourier series. Since there are infinitely many vectors in the basis, the space $L^2([-\pi,\pi])$ is infinite dimensional.

THE MORAL: The Fourier coefficients $\widehat{f}(j)$, $j \in \mathbb{Z}$, are the coordinates of the function $f \in L^2([-\pi,\pi])$ with respect to the orthonormal basis $\{e_j\}_{j=-\infty}^{\infty}$ in a similar way as $x = \sum_{i=1}^n \langle x, e_i \rangle e_j = \sum_{j=1}^n x_i e_j = (x_1,\dots,x_n)$ is the coordinate representation of $x \in \mathbb{R}^n$ with respect to the standard basis $\{e_j\}_{j=1}^n$.

(2) **Claim:** If $f \in L^2([-\pi,\pi])$, then $\widehat{f}(j) \to 0$ as $|j| \to \infty$.

Reason. By Parseval's identity (2.8)

$$\sum_{j=-\infty}^{\infty} |\widehat{f}(j)|^2 = \|f\|^2_{L^2([-\pi,\pi])} < \infty.$$

This implies that the series above converges. Thus $|\widehat{f}(j)|^2 \to 0$ and $\widehat{f}(j) \to 0$ as $|j| \to \infty$. ■

This claim holds also for $f \in L^1([-\pi,\pi])$, but this is out of the scope of this course.

Parseval's identity (2.8) implies a uniqueness result for the Fourier series.

Corollary 2.27 (Uniqueness). Let $f, g \in L^2([-\pi,\pi])$ such that $\widehat{f}(j) = \widehat{g}(j)$ for all $j \in \mathbb{Z}$. Then $f = g$ in $L^2([-\pi,\pi])$.

THE MORAL: All Fourier coefficients of two functions coincide if and only if the functions are same. Hence a function is uniquely determined by its Fourier coefficients.

Proof. By Parseval's identity (2.8) we have

$$\|f-g\|^2_{L^2([-\pi,\pi])} = \sum_{j=-\infty}^{\infty} |\widehat{(f-g)}(j)|^2 = \sum_{j=-\infty}^{\infty} |\widehat{f}(j) - \widehat{g}(j)|^2 = 0.$$

This implies that $f = g$ in $L^2([-\pi,\pi])$. □

Remarks 2.28:

(1) $\widehat{f}(j) = 0$ for every $j \in \mathbb{Z}$ if and only if $f = 0$ in $L^2([-\pi,\pi])$.

(2) Since the definition of the Fourier series required integration, for example, two functions which differ only at finitely many points have the same Fourier series. This shows that the equality does not hold pointwise without additional assumptions.

(3) If $f, g \in C([-\pi,\pi])$, then we can conclude that $f(x) = g(x)$ for every $x \in [-\pi,\pi]$.

2.5 The Fourier series on a general interval*

Theory for Fourier series can be developed for functions defined on other intervals than $[-\pi,\pi]$ in the same way as above. Let $f:[a,b]\to\mathbb{C}$ be a function and assume that $f\in L^2([a,b])$. Then the nth partial sum of a Fourier series of f on $[a,b]$ is

$$S_n f(t)=\sum_{j=-n}^{n}\langle f,e_j\rangle e_j=\sum_{j=-n}^{n}\widehat{f}(j)e^{\frac{2\pi ijt}{b-a}},\quad n=0,1,2,\dots,$$

where the Fourier coefficients are

$$\widehat{f}(j)=\frac{1}{b-a}\int_a^b f(t)e^{\frac{-2\pi ijt}{b-a}}\,dt,\quad j\in\mathbb{Z}.$$

This follows from a change of variables. For example, if f is defined on $[-\pi,\pi]$, then $g(x)=f(2\pi x-\pi)$ defined on $[0,1]$ and a change of variables shows that jth Fourier coefficient of f equals jth Fourier coefficient of g.

THE MORAL: The interval $[-\pi,\pi]$ does not play any special role in the Fourier theory, but we shall mainly consider this case.

2.6 The real form of the Fourier series*

Now we describe a different way of writing the Fourier series of a function.

Theorem 2.29. Let $f\in L^1([-\pi,\pi))$. The nth partial sum of a Fourier series can be written as

$$S_n f(t)=\frac{a_0}{2}+\sum_{j=1}^{n}(a_j\cos(jt)+b_j\sin(jt)),$$

where

$$a_j=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos(jt)\,dt,\quad j=0,1,2,\dots$$

and

$$b_j=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin(jt)\,dt,\quad j=1,2,\dots.$$

This is called the real form of the Fourier series of f. The coefficients a_j are called the Fourier cosine coefficients of f and b_j are called the Fourier sine coefficients of f. The corresponding series are called the Fourier cosine and sine series of f correspondingly.

Conversely, any trigonometric series of the form

$$\frac{a_0}{2}+\sum_{j=1}^{n}(a_j\cos(jt)+b_j\sin(jt)) \tag{2.9}$$

can be written in the complex form

$$\sum_{j=-n}^{n} c_j e^{ijt}, \tag{2.10}$$

where

$$c_j = \begin{cases} \frac{a_j - ib_j}{2}, & j = 1,2,\dots,n \\ \frac{a_0}{2}, & j = 0, \\ \frac{a_{-j}+ib_{-j}}{2}, & j = -1,-2,\dots,n. \end{cases}$$

T H E M O R A L : If f is real valued, then the Fourier cosine and sine series consist of real numbers. However, the use of the complex form is preferable since it contains the same information as the cosine and sine coefficients in one complex Fourier coefficient. Moreover, properties of the Fourier coefficients take an elegant and easy-to-remember form in the complex notation. The real form is only for those who are afraid of complex numbers.

Proof. Since $e^{ijt} = \cos(jt) + i\sin(jt)$ we have

$$S_n f(t) = \widehat{f}(0) + \sum_{j\neq 0} \widehat{f}(j)\cos(jt) + i\sum_{j\neq 0} \widehat{f}(j)\sin(jt).$$

For the first sum we have

$$\begin{aligned} \sum_{j\neq 0} \widehat{f}(j)\cos(jt) &= \sum_{j=1}^{n} \widehat{f}(j)\cos(jt) + \sum_{j=-1}^{-n} \widehat{f}(j)\cos(jt) \\ &= \sum_{j=1}^{n} \widehat{f}(j)\cos(jt) + \sum_{j=1}^{n} \widehat{f}(-j)\cos(-jt) \\ &= \sum_{j=1}^{n} (\widehat{f}(j) + \widehat{f}(-j))\cos(jt) \end{aligned}$$

and for the second sum

$$\begin{aligned} i\sum_{j\neq 0} \widehat{f}(j)\sin(nt) &= i\sum_{j=1}^{n} \widehat{f}(j)\sin(jt) + i\sum_{j=-1}^{-n} \widehat{f}(j)\sin(jt) \\ &= i\sum_{j=1}^{n} \widehat{f}(j)\sin(jt) + i\sum_{j=1}^{n} \widehat{f}(-j)\sin(-jt) \\ &= \sum_{j=1}^{n} i(\widehat{f}(j) - \widehat{f}(-j))\sin(jt). \end{aligned}$$

By the identities

$$\frac{e^{it} + e^{-it}}{2} = \cos t \quad \text{and} \quad \frac{e^{it} - e^{-it}}{2i} = \sin t, \tag{2.11}$$

we have

$$\begin{aligned}
a_0 &= 2\widehat{f}(0) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt,\\
a_j &= \widehat{f}(j)+\widehat{f}(-j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)(e^{-ijt}+e^{ijt})dt\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)2\cos(jt)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(jt)dt,\\
b_j &= i(\widehat{f}(j)-\widehat{f}(-j)) = \frac{i}{2\pi}\int_{-\pi}^{\pi} f(t)(e^{-ijt}-e^{ijt})dt\\
&= \frac{i}{2\pi}\int_{-\pi}^{\pi} f(t)(-2i)\sin(jt)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(jt)dt.
\end{aligned}$$

Now consider the real trigonometric series as in (2.9). Using (2.11) again we have

$$\begin{aligned}
&\frac{a_0}{2}+\sum_{j=1}^{n}(a_j\cos(jt)+b_j\sin(jt))\\
&\quad= \frac{a_0}{2}+\sum_{j=1}^{n}a_j\frac{e^{ijt}+e^{-ijt}}{2}+\sum_{j=1}^{n}b_j\frac{e^{ijt}-e^{-ijt}}{2i}\\
&\quad= \frac{a_0}{2}+\sum_{j=1}^{n}\frac{1}{2}(a_j-ib_j)e^{ijt}+\sum_{j=-1}^{-n}\frac{1}{2}(a_{-j}+ib_{-j})e^{ijt}. \qquad \square
\end{aligned}$$

Remarks 2.30:

(1) A function f is even, if $f(-t)=f(t)$ for every t and odd if $f(-t)=-f(t)$ for every t. For a 2π-periodic odd function all Fourier cosine coefficents a_j, $j=1,2,\dots$, in (2.9) are zero. Similarly, for a 2π-periodic even function all Fourier sine coefficents b_j, $j=1,2,\dots$, in (2.9) are zero.

(2) In applications we are interested in representing a function $f(t)$ defined on the bounded interval $0<t<L$ by a Fourier series. There are several ways to extend f as a periodic function to $\mathbb{R}$. The even periodic extension of f is defined by

$$f(t)=\begin{cases} f(t), & 0<t<L,\\ f(-t), & -L<t<0,\end{cases}$$

and $f(t)=f(t+2L)$ otherwise. Similarly, the odd periodic extension of f is defined by

$$f(t)=\begin{cases} f(t), & 0<t<L,\\ -f(-t), & -L<t<0,\end{cases}$$

and $f(t)=f(t+2L)$ otherwise. We do not worry about the definition of extensions at the points $0,\pm L,\pm 2L,\dots$, since that does not affect the Fourier coefficients. The cosine coefficients of the odd periodic extension are zero and the sine coefficients of the even periodic extension are zero. Thus we obtain two Fourier expansions

$$f(t)=\frac{a_0}{2}+\sum_{j=1}^{\infty}a_j\cos(jt),$$

with

$$a_j = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{j\pi t}{L}\right)dt, \quad j = 0,1,2,\dots$$

and

$$f(t) = \sum_{j=1}^{\infty} b_j \sin(jt),$$

with

$$b_j = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{j\pi t}{L}\right)dt, \quad j = 1,2,\dots.$$

Note carefully, that both cosine and sine series represent the same function. Thus a function can be represented only by its sine or cosine series.

2.7 The Fourier series and differentiation*

The smoothness of the functions affects to the decay of the Fourier coefficients. In general, the smoother the function, the faster the decay of the Fourier coefficients.

Lemma 2.31. Assume that $f \in C^1(\mathbb{R})$ is a 2π-periodic function. Then

$$\widehat{f'}(j) = ij\widehat{f}(j), \quad j \in \mathbb{Z}.$$

THE MORAL: The Fourier series can be differentiated termwise and differentiation becomes multiplication on the Fourier side,

$$f(t) = \sum_{j=-\infty}^{\infty} \widehat{f}(j)e^{ijt} \Longrightarrow f'(t) = \sum_{j=-\infty}^{\infty} ij\widehat{f}(j)e^{ijt}.$$

This is a very useful property in the PDE theory.

Proof. An integration by parts gives

$$\begin{aligned}
\widehat{f'}(j) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(t)e^{-ijt}\,dt \\
&= \frac{1}{2\pi}\left(\Big|_{-\pi}^{\pi} f(t)e^{-ijt} - \int_{-\pi}^{\pi} f(t)(-ije^{-ijt})\,dt\right) \\
&= \frac{1}{2\pi}\left(f(\pi)e^{-ij\pi} - \underbrace{f(-\pi)}_{=f(\pi)}e^{ij\pi} + ij\int_{-\pi}^{\pi} f(t)e^{-ijt}\,dt\right) \\
&= \frac{f(\pi)}{2\pi}\underbrace{\left(e^{-ij\pi} - e^{ij\pi}\right)}_{=0} + \frac{ij}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ijt}\,dt \\
&= ij\widehat{f}(j).
\end{aligned}$$

□

Remark 2.32. The lemma above implies

$$\widehat{f}(j) = \frac{\widehat{f'}(j)}{ij}, \quad j \neq 0,$$

and consequently

$$|\widehat{f}(j)| = \frac{|\widehat{f'}(j)|}{|j|} \leqslant \frac{1}{|j|}\frac{1}{2\pi}\int_{-\pi}^{\pi}|f'(t)|\,dt \leqslant \frac{1}{|j|}\underbrace{\max_{t\in[-\pi,\pi]}|f'(t)|}_{<\infty} \to 0$$

as $|j|\to\infty$. This means that $\widehat{f}(j)\to 0$ with speed $\frac{1}{|j|}$ as $|j|\to\infty$. We can iterate this procedure if f has higher order derivatives and obtain a faster decay.

T H E M O R A L : Â By Remark 2.26 (2), we see that for every function $f\in L^2([\pi,\pi])$ (or $f\in L^1([\pi,\pi])$) we have $\widehat{f}(j)\to 0$ as $|j|\to\infty$. In other words, the Fourier coefficients of an L^2 function always converge to zero, but there is no estimate for the speed of convergence. If the function is smoother, then the Fourier coefficients converge to zero faster with an estimate for the speed.

2.8 The Dirichlet kernel*

Next we introduce the Dirichlet kernel, which turns out to be a very important object related to the convergence of Fourier series. Let $f:\mathbb{R}\to\mathbb{C}$ be a continuous 2π-periodic function. The partial sums of the Fourier series of f can be written as

$$\begin{aligned} S_nf(t) &= \sum_{j=-n}^{n}\widehat{f}(j)e^{ijt} = \sum_{j=-n}^{n}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}f(s)e^{-ijs}\,ds\right)e^{ijt} \\ &= \frac{1}{2\pi}\sum_{j=-n}^{n}\int_{-\pi}^{\pi}f(s)e^{ij(t-s)}\,ds = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(s)\left(\sum_{j=-n}^{n}e^{ij(t-s)}\right)ds. \end{aligned}$$

Now we define the Dirichlet kernel as

$$D_n(t) = \sum_{j=-n}^{n}e^{ijt}, \quad n=0,1,2,\ldots.$$

With this definition we have the formula

$$S_nf(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(s)D_n(t-s)\,ds, \quad t\in[-\pi,\pi], \quad n=0,1,2,\ldots. \tag{2.12}$$

T H E M O R A L : This is an integral representation of the partial sum of the Fourier series. In fact, (2.12) is a convolution of f with the Dirichlet kernel as we shall see soon.

Before discussing further this formula let us take a closer look at the Dirichlet kernel.

Lemma 2.33.

$$D_nf(t) = \sum_{j=-n}^{n}e^{ijt} = \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin\left(\frac{1}{2}t\right)}, \quad t\neq 0, \quad n=0,1,2,\ldots.$$

Furthermore, the Dirichlet kernel is a 2π-periodic function that satisfies

$$D_n(0) = 2n+1 \quad \text{and} \quad D_n(\pi) = (-1)^n, \quad n=0,1,2,\ldots.$$

THE MORAL: The Dirichlet kernel (2.12) can be computed explicitely.

Proof. Recall the following formula for a geometric series.

Claim: If $a \in \mathbb{C} \setminus \{1\}$ and $n = 0, 1, 2, \ldots$, then

$$S = \sum_{j=-n}^{n} a^j = \frac{a^{-n} - a^{n+1}}{1-a}.$$

Reason. Clearly

$$\begin{aligned} S &= a^{-n} + a^{-n+1} + \cdots + a^{-1} + 1 + a + \cdots + a^{n-1} + a^n \\ \implies -aS &= -a^{-n+1} - a^{-n+2} - \cdots - 1 - a - a^2 - \cdots - a^n - a^{n+1}. \end{aligned}$$

By summing these up termwise we obtain

$$(1-a)S = a^{-n} - a^{n+1} \Rightarrow S = \frac{a^{-n} - a^{n+1}}{1-a}$$

as required. ■

Setting $a = e^{it}$ in the previous formula we get for $t \neq 0$ that

$$\begin{aligned} D_n(t) &= \sum_{j=-n}^{n} (e^{it})^j = \frac{e^{-int} - e^{i(n+1)t}}{1 - e^{it}} = \frac{e^{-i\frac{t}{2}}\left(e^{-int} - e^{i(n+1)t}\right)}{e^{-i\frac{t}{2}}(1 - e^{it})} \\ &= \frac{e^{-i(n+\frac{1}{2})t} - e^{i(n+\frac{1}{2})t}}{e^{-i\frac{t}{2}} - e^{i\frac{t}{2}}} = \frac{-2i\sin\left((n+\frac{1}{2})t\right)}{-2i\sin\left(\frac{t}{2}\right)} = \frac{\sin\left((n+\frac{1}{2})t\right)}{\sin\left(\frac{t}{2}\right)}. \end{aligned}$$

In the equality before the last we have used the latter one of the trigonometric identities in (2.11).

For $t = 0$ the value $D_n(0)$ can be calculated directly since

$$D_n(0) = \sum_{j=-n}^{n} 1 = 2n + 1.$$

This can be recovered by the general formula as well by taking the limit as $t \to 0$. On the other hand, for $t = \pi$ we have

$$D_n(\pi) = \frac{\sin\left(\left(n+\frac{1}{2}\right)\pi\right)}{\sin\left(\frac{\pi}{2}\right)} = \sin\left(\pi n + \frac{\pi}{2}\right) = \cos(\pi n) = (-1)^n$$

as claimed. □

The calculation of the integral of the Dirichlet kernel will turn out to be important in some of the applications we shall discuss.

Lemma 2.34.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t)\,dt = 1, \quad n = 0, 1, 2, \ldots.$$

THE MORAL: The total mass of the Dirichlet kernel is one.

Proof. A direct calculation gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t)\,dt = \sum_{j=-n}^{n} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ijt}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i0t}\,dt = 1.$$

□

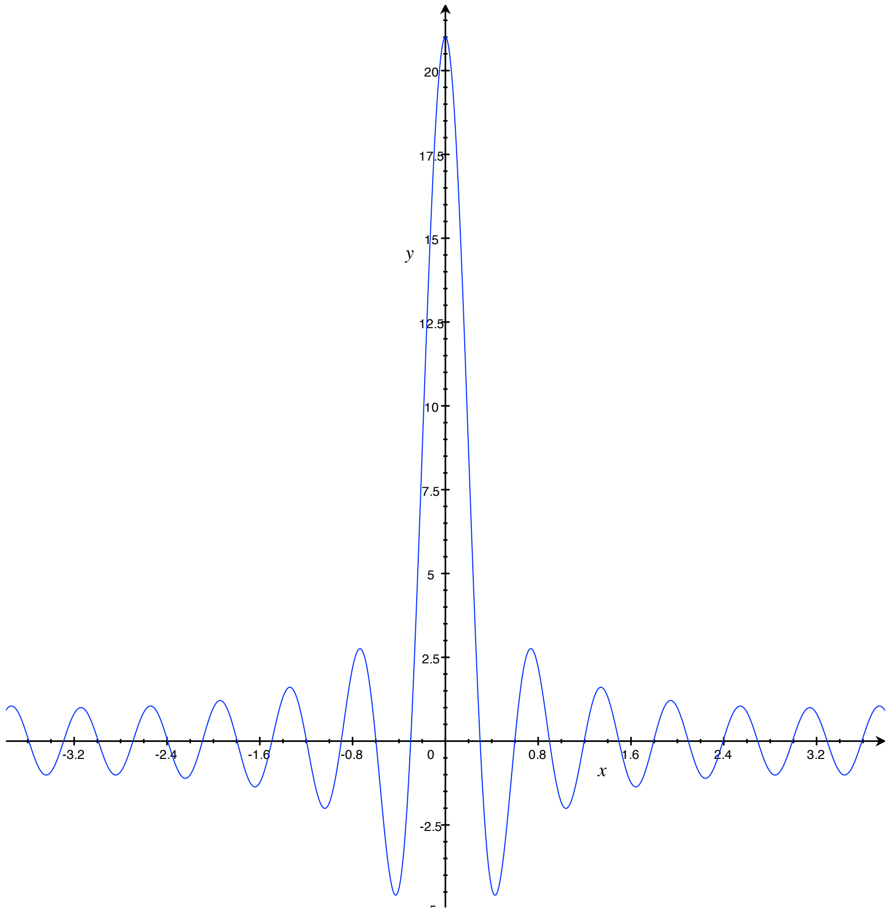


Figure 2.6: The graph of the Dirichlet kernel for $N = 10$.

2.9 Convolutions*

The notion of convolution plays a fundamental role in Fourier analysis and PDEs. Let $f, g \in C(\mathbb{R})$ be 2π-periodic. The convolution of f and g on $[-\pi, \pi]$ is the function $f * g$ defined as

$$(f * g)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)g(s)\,ds = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)g(t-s)\,ds, \quad t \in [-\pi, \pi].$$

The second equality follows by a change of variables and the fact that f and g are 2π-periodic. Indeed, setting $u = t - s$ in the defining integral we have

$$\int_{-\pi}^{\pi} f(t-s)g(s)\,ds = -\int_{t+\pi}^{t-\pi} f(u)g(t-u)\,du = \int_{-\pi}^{\pi} f(u)g(t-u)\,du$$

by Lemma 2.3. To apply this lemma we used the fact that for a fixed t the function $F(u) = f(u)g(t-u)$ is 2π-periodic, if f, g are 2π-periodic. Indeed

$$F(u+2\pi) = f(u+2\pi)g((t-u)-2\pi) = f(u)g(t-u) = F(u).$$

Furthermore, the function $f*g$ is itself 2π-periodic, provided that f, g are 2π-periodic, since

$$\begin{aligned}(f*g)(t+2\pi) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f((t+2\pi)-s)g(s)\,ds\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{f((t-s)+2\pi)}_{=f(t-s)}g(s)\,ds = (f*g)(t).\end{aligned}$$

If $g=1$, then $f*g$ is constant and equal to $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\,ds$. Observe, that this is the integral average of f over $[-\pi,\pi]$.

THE MORAL: Convolutions can be considered as weighted averages of the functions.

Next we shall show that the convolution of f and g is well defined also if $f,g\in L^1([-\pi,\pi])$ or $L^2([-\pi,\pi])$. If $f,g\in L^1([-\pi,\pi])$, then

$$\begin{aligned}\|f*g\|_{L^1([-\pi,\pi])} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)g(s)\,ds\right| dt\\ &\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t-s)|\,dt\right)|g(s)|\,ds\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)|\,dt\right)|g(s)|\,ds\\ &= \|f\|_{L^1([-\pi,\pi])}\|g\|_{L^1([-\pi,\pi])}<\infty.\end{aligned}$$

For $f,g\in L^2([-\pi,\pi])$, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|(f*g)(t)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)g(s)\,ds\right|\\ &\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t-s)||g(s)|\,ds\\ &\leqslant \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t-s)|^2\,ds\right)^{\frac{1}{2}}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(s)|^2\,ds\right)^{\frac{1}{2}}\\ &= \|f\|_{L^2([-\pi,\pi])}\|g\|_{L^2([-\pi,\pi])}<\infty, \quad t\in[-\pi,\pi].\end{aligned}$$

Thus we obtain

$$\sup_{t\in[-\pi,\pi]}|(f*g)(t)| \leqslant \|f\|_{L^2([-\pi,\pi])}\|g\|_{L^2([-\pi,\pi])}.$$

We gather the previous observations and other properties of the convolution in the following lemma.

Lemma 2.35. Let $f,g,h:\mathbb{R}\to\mathbb{C}$ be 2π-periodic functions.

(1) The function $f*g$ is 2π-periodic.

(2) If $f,g\in L^1([-\pi,\pi])$ then $f*g$ is well defined and

$$\|f*g\|_{L^1([-\pi,\pi])} \leqslant \|f\|_{L^1([-\pi,\pi])}\|g\|_{L^1([-\pi,\pi])}.$$

(3) If $f, g \in L^2([-\pi,\pi])$ then $f * g$ is well defined and bounded with

$$\sup_{t\in[-\pi,\pi]} |(f*g)(t)| \leqslant \|f\|_{L^2([-\pi,\pi])}\|g\|_{L^2([-\pi,\pi])}.$$

(4) $f * g = g * f$.

(5) $f * (g * h) = (f * g) * h$.

(6) $f * (g + h) = f * g + f * h$.

(7) $a(f * g) = (af) * g = f * (ag)$, $a \in \mathbb{C}$.

(8) $\widehat{(f * g)}(j) = \widehat{f}(j)\widehat{g}(j)$, $j \in \mathbb{Z}$.

Proof. $\boxed{(8)}$:

$$\begin{aligned}
\widehat{(f*g)}(j) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(f*g)(t)e^{-ijt}\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)g(t-s)\,ds\right)e^{-ijt}\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)e^{-ijs}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(t-s)e^{-ij(t-s)}\,dt\right)ds \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)e^{-ijs}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-ijt}\,dt\right)ds = \widehat{f}(j)\widehat{g}(j).
\end{aligned}$$

In the third equality we changed the order of integration and on the fourth equality we used the fact that the integrand is 2π-periodic and Lemma 2.3.

Other claims are left as exercises. □

Remark 2.36. The Dirichlet formula (2.12) can be written as a convolution

$$S_n f(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)D_n(t-s)\,ds = (D_n * f)(t), \quad t \in [-\pi,\pi], \quad n = 0,1,2,\ldots.$$

Thus $S_n f$ is a certain average of f with the weight D_n.

THE MORAL: Many useful quantities in Fourier analysis and PDEs can be written as convolutions.

2.10 A local result for the convergence of Fourier series*

We now turn to the issue of pointwise convergence of Fourier series under additional assumptions on the function.

Theorem 2.37 (Local convergence of Fourier series). Let $f \in C([-\pi,\pi])$ be a 2π-periodic function which is differentiable at some point $t_0 \in [-\pi,\pi]$. Then

$$\lim_{n\to\infty} S_n f(t_0) = f(t_0).$$

THE MORAL: The Fourier series of a smooth function converges pointwise everywhere.

Proof. We define the function

$$F(t)=\begin{cases}\frac{f(t_0-t)-f(t_0)}{t}, & \text{if} \quad t\neq 0 \quad \text{and} \quad |t|<\pi,\\ -f'(t_0), & \text{if} \quad t=0.\end{cases}$$

Since f is differentiable at t_0 and continuous everywhere we conclude that $F\in C([-\pi,\pi])$ and thus F is bounded on $[-\pi,\pi]$. We have

$$\begin{aligned}S_nf(t_0)-f(t_0)&=(D_n*f)(t_0)-f(t_0)\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}D_n(t)f(t_0-t)\,dt-\underbrace{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}D_n(t)\,dt\right)}_{=1}f(t_0)\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(t_0-t)-f(t_0))D_n(t)\,dt\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}tF(t)D_n(t)\,dt\end{aligned}$$

by the definition of F. Remembering the formula for the Dirichlet kernel and using standard trigonometric identities we have

$$\begin{aligned}tD_n(t)&=t\frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin\left(\frac{t}{2}\right)}\\&=t\frac{\sin(nt)\cos\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)}+t\frac{\cos(nt)\sin\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)}\\&=t\frac{\sin(nt)\cos\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)}+t\cos(nt),\quad t\neq 0.\end{aligned}$$

Thus

$$S_nf(t_0)-f(t_0)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{t}{\sin\left(\frac{t}{2}\right)}\sin(nt)\cos\left(\frac{t}{2}\right)F(t)\,dt+\frac{1}{2\pi}\int_{-\pi}^{\pi}F(t)t\cos(nt)\,dt.$$

Now the function

$$\psi(t)=\frac{t\cos(t/2)}{\sin(t/2)}F(t)$$

is continuous on $[-\pi,\pi]$ and thus in $L^2([-\pi,\pi])$. Thus

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\psi(t)\sin(nt)\,dt\right|\leqslant|\widehat{\psi}(n)|\to 0\quad\text{as}\quad n\to\infty$$

by the Riemann-Lebesgue lemma. Similarly for the second term we see that the function $\phi(t)=tF(t)$ is continuous and thus $\phi\in L^2([-\pi,\pi])$. Again by the Riemann-Lebesgue lemma we conclude

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(t)\cos(nt)\,dt\right|\leqslant|\widehat{\phi}(n)|\to 0\quad\text{as}\quad n\to\infty.$$

The last two estimates complete the proof. □

Remarks 2.38:

(1) It is enough to assume that f is Lipschitz continuous, that is,

$$|f(t)-f(s)| \leqslant L|t-s| \quad \text{for all} \quad t,s \in [-\pi,\pi].$$

In this case

$$\sup_{t\in[-\pi,\pi]} |S_n f(t)-f(t)| \to 0 \quad \text{as} \quad n\to\infty.$$

This means that $S_n f \to f$ uniformly in $[-\pi,\pi]$ as $n\to\infty$.

(2) The theorem shows that convergence of the Fourier series at a given point depends only on the behaviour of the function in an arbitrarily small neighbourhood of the point. This is somewhat unexpected, since the Fourier coefficients are defined as integrals over the whole interval $[-\pi,\pi]$.

(3) If f is twice continuously differentiable, denoted $f \in C^2(\mathbb{R})$, then the Fourier series converges absolutely and uniformly to f. This can be seen by Remark 2.32. There are even stronger results. It can be shown, for example, that the Fourier series of f converges uniformly, assuming only that $f \in C^1(\mathbb{R})$, that is, f is continuously differentiable.

Remark 2.39. We list here some further results related to the pointwise convergence of the Fourier series.

(1) Kolmogorov showed in 1926 that there exists an integrable function $f \in L^1([-\pi,\pi])$ whose Fourier series diverges at every point.

(2) There is a continuous function on $[-\pi,\pi]$ such that the Fourier series diverges on a countable, dense set of points in $[-\pi,\pi]$. A dense subset of an interval $[-\pi,\pi]$ is a set which contains sequences that approximate every point in $[-\pi,\pi]$. For example, the rational numbers in $[-\pi,\pi]$ form a dense subset of $[-\pi,\pi]$.

(3) Furthermore, Carleson in 1966 proved the (very deep) theorem that the Fourier series of every function in $L^2([-\pi,\pi])$ converge pointwise to the function almost everywhere. This holds, in particular, for continuous functions on $[-\pi,\pi]$.

2.11 The Laplace equation in the unit disc

We shall consider the Laplace equation, which models heat distribution when the system has reached thermal equilibrium and the temperature does not change in time. The Laplace equation occurs also in other branches of mathematical physics. For example, in considering electrostatic fields, the electric potential in a dielectric medium containing no electric charges satisfies the Laplace equation. Similarly, the potential of a particle in free space acted only by the gravitational forces satisfies the same equation. Morover, the real and imaginary parts of a complex analytic function are solutions to the Laplace equation.

We begin with considering the two-dimensional case. Let

$$\Omega = \{(x,y) \in \mathbb{R}^2 : (x^2+y^2)^{\frac{1}{2}} < 1\}$$

be the unit disc in $\mathbb{R}^2$ and assume that $g \in C(\partial\Omega)$ is a continuous function on the boundary

$$\partial\Omega = \{(x,y) \in \mathbb{R}^2 : (x^2+y^2)^{\frac{1}{2}} = 1\}.$$

The problem is to find $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} \Delta u(x,y) = \dfrac{\partial^2 u}{\partial x^2}(x,y) + \dfrac{\partial^2 u}{\partial y^2}(x,y) = 0, & (x,y) \in \Omega, \\ u(x,y) = g(x,y), & (x,y) \in \partial\Omega. \end{cases} \tag{2.13}$$

This is called the Dirichlet problem for the Laplace equation in the unit disc.

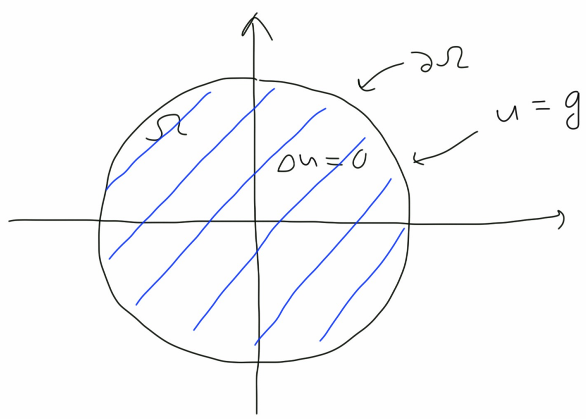


Figure 2.7: The Dirichlet problem in the unit disc.

T H E M O R A L : For a given boundary function g we want to find a solution of the Laplace equation $\Delta u = 0$ inside the domain Ω such that it attains boundary values g on the boundary $\partial\Omega$. Physically this means that the temperature on the boundary $\partial\Omega$ of the disc is given by g and that the system has reached thermal equilibrium. In this model the temperature distribution in Ω is given by a solution of the Dirichlet problem for the Laplace equation $\Delta u = 0$. This is a time independent problem, where all physical constants are set to be one.

We will solve this Dirichlet problem using Fourier series together with a technique called separation of variables. This means that we look for solutions of the form

$$u(x, y) = A(x)B(y),$$

where the dependencies on x and y are separated. The problem is now reduced to finding the functions $A(x)$ and $B(x)$. The directions of the coordinate axes play a special role in this approach and this would work, for example, for the Dirichlet problem in a rectangular domain.

For the unit disc we switch to polar coordinates. More precisely, any point in the plane can be uniquely determined by its distance from the origin r and the angle θ that the line segment from the origin to the point forms with the x-axis, that is,

$$(x, y) = (r\cos\theta, r\sin\theta), \quad (x, y) \in \mathbb{R}^2, \quad 0 \leqslant r < \infty, \quad -\pi \leqslant \theta < \pi,$$

where $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$. In polar coordinates, we have

$$\Omega = \{(r, \theta) : 0 \leqslant r < 1, -\pi \leqslant \theta < \pi\} \quad \text{and} \quad \partial\Omega = \{(1, \theta) : -\pi \leqslant \theta < \pi\}.$$

Note that the unit disc becomes a rectangular set in polar coordinates and this is compatible with separation of variables.

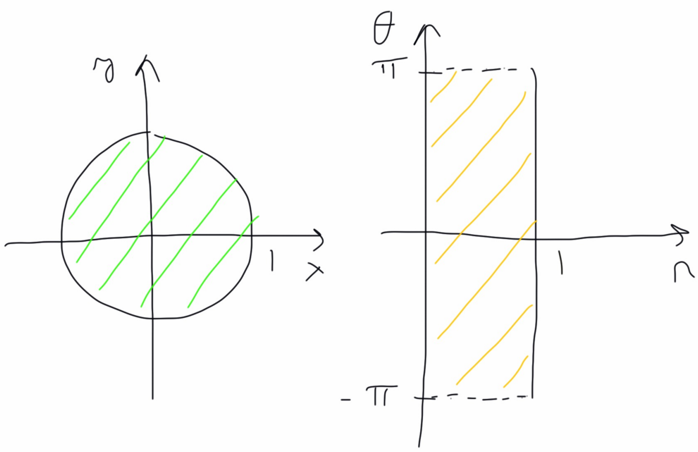


Figure 2.8: Transition to polar coordinates.

The next goal os to express the Laplace equation in polar coordinates as well. This is done in the following lemma.

Lemma 2.40. The two-dimensional Laplace operator in polar coordinates is

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \qquad 0 < r < \infty, \qquad -\pi \leqslant \theta < \pi.$$

Proof. Remember that $x = r\cos\theta$ and $y = r\sin\theta$. We use the chain rule to express the r and θ derivatives in terms of the x and y derivatives. This gives

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y},\\
\frac{\partial^2 u}{\partial r^2} &= \cos\theta\frac{\partial}{\partial x}\left(\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}\right) + \sin\theta\frac{\partial}{\partial y}\left(\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}\right)\\
&= \cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x \partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y},\\
\frac{\partial^2 u}{\partial \theta^2} &= -r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial u}{\partial y} - r\sin\theta\frac{\partial}{\partial \theta}\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial}{\partial \theta}\frac{\partial u}{\partial y}\\
&= -r\frac{\partial u}{\partial r} - r\sin\theta\left(-r\sin\theta\frac{\partial^2 u}{\partial x^2} + r\cos\theta\frac{\partial^2 u}{\partial x \partial y}\right)\\
&\quad + r\cos\theta\left(-r\sin\theta\frac{\partial^2 u}{\partial x \partial y} + r\cos\theta\frac{\partial^2 u}{\partial y^2}\right)\\
&= -r\frac{\partial u}{\partial r} + r^2\left(\sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x \partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2}\right).
\end{aligned}$$

Thus

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

as desired. ☐

We return to the Dirichlet problem (2.13) in the unit disc. By switching to polar coordinates we obtain

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < r < 1, \quad -\pi \leqslant \theta < \pi,\\ u(1,\theta) = g(\theta), \quad -\pi \leqslant \theta < \pi, \end{cases} \tag{2.14}$$

for $u = u(r,\theta)$.

THE MORAL: This is the polar form of the Dirichlet problem (2.13) in the unit disc. Observe that the domain becomes rectangular in polar coordinates. Solution of (2.14) will give a solution of (2.13) in polar coordinates.

We shall derive a solution to (2.14) in four steps.

Step 1 (Separation of variables): We look for a product solution of the form

$$u(r,\theta) = A(\theta)B(r)$$

for this problem. Here $A(\theta)$ is a function of θ alone and $B(r)$ is a function of r alone. By inserting this into the PDE in (2.14) and multiplying by r^2, we obtain

$$r^2A(\theta)B''(r)+rA(\theta)B'(r)+B(r)A''(\theta)=0 \Longleftrightarrow \frac{r^2B''(r)+rB'(r)}{B(r)}=-\frac{A''(\theta)}{A(\theta)}.$$

Observe that the variables have been separated in the sense that the left-hand side depends only on r and the right-hand side only on θ. This can happen only if both sides are equal to a constant, say equal to λ. This is called the separation constant.

Reason. Denote

$$\lambda(r,\theta)=\frac{r^2B''(r)+rB'(r)}{B(r)}=-\frac{A''(\theta)}{A(\theta)}$$

for every r and θ in the appropriate intervals. Then

$$\frac{\partial\lambda}{\partial r}(r,\theta)=0 \quad\text{and}\quad \frac{\partial\lambda}{\partial\theta}(r,\theta)=0$$

and thus $\lambda(r,\theta)$ is a constant function. Another way to see this is to observe that the term including B does not depend on θ and the term including A does not depend on r. Thus we may conclude that λ is independent of both variables. ■

Thus

$$\frac{r^2B''(r)+rB'(r)}{B(r)}=\lambda=-\frac{A''(\theta)}{A(\theta)}$$

for every r and θ. Consequently, we may rewrite the separated equations as two ordinary differential equations (ODEs)

$$\begin{cases} A''(\theta)+\lambda A(\theta)=0, \\ r^2B''(r)+rB'(r)-\lambda B(r)=0. \end{cases} \tag{2.15}$$

THE MORAL: The PDE has been reduced to a system two ODEs.

As we shall see, not all values of the separation constant λ give nontrivial solutions to these ODEs. However, these are simple second order ODEs with constant coefficients and we can solve them explicitely.

Step 2 (Solution to the separated equations): Now we take into account the boundary data

$$u(1,\theta)=A(\theta)B(1)=g(\theta).$$

Since g is defined on the circle it can be identified with a 2π-periodic function and thus A has to be 2π-periodic as well.

The question is that for which values of λ we have nontrivial solutions to the ODE

$$A''(\theta)+\lambda A(\theta)=0 \Longleftrightarrow -A''(\theta)=\lambda A(\theta).$$

In other words, we are interested in eigenvalues λ and eigenfunctions A of the second derivative operator $-\frac{\partial^2}{\partial\theta^2}$. We are mainly interested in real valued solutions, but sometimes it is useful to consider complex valued solutions as well. In principle, then the eigenvalues can be complex numbers. However, we begin with showing that all eigenvalues λ of the problem above are real.

$\boxed{\lambda \in \mathbb{R}}$ Let $\lambda \in \mathbb{C}$ be an eigenvalue and A the corresponding complex valued eigenfunction. Then $-A'' = \lambda A$ and by taking the complex conjugate of this equation we obtain $-\overline{A}'' = \overline{\lambda}\overline{A}$. By the chain rule, we have

$$-A''\overline{A} + A\overline{A}'' = (-A'\overline{A} + A\overline{A}')'.$$

We integrate to get

$$\begin{aligned}\int_{-\pi}^{\pi} (-A''\overline{A} + A\overline{A}'')\,d\theta &= \int_{-\pi}^{\pi} (-A'\overline{A} + A\overline{A}')'\,d\theta = \Big|_{-\pi}^{\pi} (-A'\overline{A} + A\overline{A}') \\ &= -A'(\pi)\overline{A}(\pi) + A(\pi)\overline{A}'(\pi) - (-A'(-\pi)\overline{A}(-\pi) + A(-\pi)\overline{A}'(-\pi)).\end{aligned}$$

Since A is periodic, we have $A(\pi) = A(-\pi)$ and $A'(\pi) = A'(-\pi)$, from which we conclude that the right-hand side of the previous display is zero. Thus

$$\begin{aligned}(\lambda - \overline{\lambda}) \int_{-\pi}^{\pi} |A|^2\,d\theta &= (\lambda - \overline{\lambda}) \int_{-\pi}^{\pi} A\overline{A}\,d\theta = \int_{-\pi}^{\pi} (\lambda A\overline{A} - A\overline{\lambda}\overline{A})\,d\theta \\ &= \int_{-\pi}^{\pi} (-A''\overline{A} + A\overline{A}'')\,d\theta = 0.\end{aligned}$$

Since A is not identically zero, we have $\int_{-\pi}^{\pi} |A|^2\,d\theta > 0$. Thus $\lambda - \overline{\lambda} = 0$, which implies that λ is a real number.

We divide the analysis into three cases depending on the value of the separation constant.

$\boxed{\lambda < 0}$ Then $\lambda = -\mu^2$ for some $\mu > 0$ and the first ODE above becomes $A'' - \mu^2 A = 0$. This is a second order linear ODE with the general solution

$$A(\theta) = c_1 e^{\mu\theta} + c_2 e^{-\mu\theta}.$$

In this case the only possible 2π-periodic solution is $u = 0$ which corresponds to $c_1 = c_2 = 0$. Thus the case $\lambda < 0$ gives only trivial solutions.

$\boxed{\lambda = 0}$ The equation for A reduces to $A'' = 0$ with the general solution

$$A(\theta) = c_1\theta + c_2.$$

In order for this function to be 2π-periodic ,we must have $c_1 = 0$ and thus $A(\theta) = c_2$ is constant. For $\lambda = 0$, the ODE for B becomes $rB'' + B' = 0$. The general solution of this equation is $B(r) = d_1 + d_2 \ln r$ and thus we get

$$u(r,\theta) = A(\theta)B(r) = c_2(d_1 + d_2 \ln r).$$

This solution becomes unbounded as $r \to 0$ which is contrary to any physical intuition that the temperatures should remain bounded (and also not compatible with the hypothesis that our solution should be continuous and bounded in the interior of the disc). Thus $\lambda = 0$ gives nonphysical solutions, which are excluded.

$\boxed{\lambda > 0}$ Then $\lambda = \mu^2$, $\mu \neq 0$, so the ODE for A becomes $A'' + \mu^2 A = 0$. The general complex solution of this equation is

$$A(\theta) = c_1 e^{i\mu\theta} + c_2 e^{-i\mu\theta}.$$

(We could consider the real solutions $A(\theta) = c_1 \sin(\mu\theta) + c_2 \cos(\mu\theta)$, but the complex notation in more convenient for the Fourier series.) For this solution to be 2π-periodic we need $\mu = \sqrt{\lambda}$ to be an integer. Thus we have $\lambda = \mu^2 = j^2$, $j \in \mathbb{Z} \setminus \{0\}$ as the case $j = 0$ has already been considered. These are the eigenvalues and eigenfunctions for our problem.

Now the ODE for B is

$$r^2 B'' + rB' - j^2 B = 0.$$

It can be shown that the general solution of this equation, called the Euler equation, is

$$B(r) = d_1 r^j + d_2 r^{-j}, \quad j = 1, 2, \ldots.$$

Again, since $\mu > 0$, the term with r^{-j} blows up as $r \to 0$ which contradicts the continuity of u as well as the physical intuition. Thus we only include the solution $B(r) = d_1 r^j$. Thus $\lambda > 0$ gives solutions of the form

$$u(r,\theta) = A(\theta)B(r) = r^{|j|} e^{ij\theta}, \quad j \in \mathbb{Z}.$$

Observe that these functions are special solutions of the Laplace equation in polar coordinates with boundary values $u(1,\theta) = e^{ij\theta}$, $j \in \mathbb{Z}$.

Step 3 (Fourier series solution of the entire problem): To solve the original Dirichlet problem we try to express a general solution as a linear combination of the special solutions in such a way that the boundary condition is satisfied. Since the Laplace operator is a linear, any linear combination of solutions is again a solution. We conclude that the general solution should be given in the form

$$u(r,\theta) = \sum_{j=-\infty}^{\infty} a_j r^{|j|} e^{ij\theta}, \quad 0 \leqslant r < 1, \quad -\pi \leqslant \theta < \pi. \tag{2.16}$$

This representation should be compatible with the boundary data g when $r = 1$. This means that

$$u(1,\theta) = \sum_{j=-\infty}^{\infty} a_j e^{ij\theta} = g(\theta).$$

We are already familiar with this question for the Fourier series. The question is that can we determine coefficients a_j so that the series representation above holds? If $g \in L^2([-\pi,\pi))$, then by Thoerem 2.25 we see that this is possible, at least if the equality above is interpreted in L^2-sense. If $g \in C^1(\mathbb{R})$, then the discussion in Section 2.10 shows that the series above converges even uniformly on $[-\pi,\pi]$ and the coefficients a_j will be the Fourier coefficients of g, that is,

$$a_j = \widehat{g}(j), \quad j \in \mathbb{Z}.$$

T H E M O R A L : The solution of the Dirichlet problem for the Laplace equation in the unit disc is given by the Fourier series of the boundary function.

There are several nontrivial points related to the formula above that remain to be discussed:

(1) Does the series in (2.16) converge?

(2) Is the function given by (2.16) really a solution to the PDE?

(3) Does the solution given by (2.16) attain the correct boundary values?

(4) Are there other solutions than given by (2.16)?

A direct computation suggests that the obtained series is a solution to the Laplace equation and by substituting $r=1$ we see that the solution has the desired boundary values. On a formal level this is correct, but a special attention has to be paid to switch the order of the limit and the infinite series. We return to this question in Section 2.14. Uniqueness will be discussed in Chapter 4.

Step 4 (Explicit representation formula): By subsituting the definition of the Fourier coefficients of g in (2.16) we obtain

$$\begin{aligned} u(r,\theta) &= \sum_{j=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-ijt}\,dt \right) r^{|j|} e^{ij\theta} \\ &= \lim_{n\to\infty} \sum_{j=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-ijt}\,dt \right) r^{|j|} e^{ij\theta} \\ &= \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \left(\sum_{j=-n}^{n} r^{|j|} e^{ij(\theta-t)} \right) dt. \end{aligned}$$

The finite sum inside the integral is bounded by

$$\left| \sum_{j=-n}^{n} r^{|j|} e^{ij(\theta-t)} \right| \leqslant \sum_{j=-\infty}^{\infty} r^{|j|} < \infty, \quad 0 \leqslant r < 1,$$

and thus the sequence of the finite partial sums is uniformly bounded by a constant. By switching the order of the limit and the integral, we have

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \left(\lim_{n\to\infty} \sum_{j=-n}^{n} r^{|j|} e^{ij(\theta-t)} \right) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \left(\sum_{j=-\infty}^{\infty} r^{|j|} e^{ij(\theta-t)} \right) dt. \end{aligned}$$

This can be justified by the dominated convergence theorem, which is out of the scope of this course. We define the Poisson kernel for the disc to be the 2π-periodic function

$$P_r(\theta) = P(r,\theta) = \sum_{j=-\infty}^{\infty} r^{|j|} e^{ij\theta}. \tag{2.17}$$

By using convolutions from Section 2.9 this means that the solution to the Dirichlet problem can be written as

$$u(r,\theta) = (g * P_r)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t) P_r(\theta - t)\,dt.$$

THE MORAL: This suggests that the solution of the Dirichlet problem for the Laplace equation in the unit disc is a convolution of the boundary data with the Poisson kernel. This is an integral representation of the solution.

From this we see that the solution is well defined whenever the convolution of g and P_r is well defined. The functions u that satisfy the Laplace equation in an open domain Ω are called harmonic in Ω. Let us look at properties of the Poisson kernel in more detail.

Lemma 2.41.

$$P_r(\theta) = P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}, \quad 0 \leqslant r < 1, \quad -\pi \leqslant \theta < \pi.$$

Proof.

$$\begin{aligned}
\sum_{j=-n}^{n} r^{|j|} e^{ij\theta} &= 1 + \sum_{j=1}^{n} r^j e^{ij\theta} + \sum_{j=1}^{n} r^j e^{-ij\theta} \\
&= 1 + \sum_{j=1}^{n} (re^{i\theta})^j + \sum_{j=1}^{n} (re^{-i\theta})^j \\
&= 1 + \frac{(re^{i\theta})^{n+1} - re^{i\theta}}{-1 + re^{i\theta}} + \frac{(re^{-i\theta})^{n+1} - re^{-i\theta}}{re^{-i\theta} - 1}.
\end{aligned}$$

Letting $n \to \infty$, and remembering that $r < 1$, the right-hand side of the display above tends to

$$\begin{aligned}
1 + \frac{-re^{i\theta}}{re^{i\theta}-1} + \frac{re^{-i\theta}}{1-re^{-i\theta}} &= 1 + \frac{-re^{i\theta} - re^{-i\theta} + 2r^2}{(re^{i\theta}-1)(1-re^{-i\theta})} = 1 + 2\frac{r^2 - r\cos\theta}{re^{i\theta} - r^2 - 1 + re^{-i\theta}} \\
&= 1 + 2\frac{r\cos\theta - r^2}{r^2 + 1 - 2r\cos\theta} = \frac{1-r^2}{1-2r\cos\theta + r^2},
\end{aligned}$$

which is the desired formula. □

Remark 2.42. It is obvious that the Poisson kernel is a non-negative function in the disc, since

$$P(r,\theta) = \frac{1-r^2}{\sin^2\theta + (\cos\theta - r)^2} \geqslant 0.$$

Furthermore

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = 1$$

for every $r \in (0,1)$. The integral of $P_r(\theta)$ can be calculated by integrating (2.17) term by term (exercise). Observe, that

$$P(r,0) = \frac{1-r^2}{1-2r+r^2}, \quad 0 \leqslant r < 1,$$

and consequently $\lim_{r\to 1} P(r,0) = \infty$. However, $\lim_{r\to 1} P(r,\theta) = 0$, when $-\pi \leqslant \theta < \pi$, $\theta \neq 0$. This issue will be discussed further in Section 2.14.

T H E M O R A L : The total mass of the Poisson kernel is one implies that the solution of the Dirichlet problem for the Laplace equation in the unit disc is an average of the boundary function g with the weight P_r.

We summarize the findings in the following theorem.

Theorem 2.43 (Solution of the Dirichlet problem on the disc). The solution of the Dirichlet problem (2.13) in polar coordinates is

$$u(r,\theta) = \sum_{j=-\infty}^{\infty} a_j r^{|j|} e^{ij\theta}, \quad 0 \leqslant r < 1, \quad -\pi \leqslant \theta < \pi,$$

where

$$a_j = \widehat{g}(j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-ij\theta}\,d\theta, \quad j \in \mathbb{Z}.$$

Moreover, the solution can be written as a convolution with the Poisson kernel as

$$u(r,\theta) = (g * P_r)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)P_r(\theta - t)\,dt,$$

where

$$P_r(\theta) = P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}, \quad 0 \leqslant r < 1, \quad -\pi \leqslant \theta < \pi.$$

The boundary values are obtained in the sense

$$\lim_{r\to 1}(g * P_r)(\theta) = g(\theta), \quad -\pi \leqslant \theta < \pi.$$

T H E M O R A L : Note that we do not obtain the boundary function by inserting $r = 1$ in the convolution formula above, since $P_1(\theta) = 0$, $-\pi \leqslant \theta < \pi$. Instead, we shall consider the limit above. This issue will be discussed further in Section 2.14.

We close this section by considering two explicit examples.

Example 2.44. Consider the steady-state temperature distribution in the unit disc, if the upper half of the circle is kept at 100 °C and the lower half is kept at 0 °C. Then

$$u(1,\theta) = f(\theta) = \begin{cases} 0, & -\pi < \theta < 0, \\ 100, & 0 < \theta < \pi. \end{cases}$$

Observe that f is not continuous, but $f \in L^2([-\pi,\pi])$. The Fourier coefficients of f are

$$\begin{aligned}
\widehat{f}(j) &= \frac{1}{2\pi}\left(\int_{-\pi}^{0} 0e^{-ijt}\,dt + \int_{0}^{\pi} 100e^{-ijt}\,dt\right)\\
&= \frac{100}{2\pi}\bigg|_0^{\pi} \frac{e^{-ijt}}{-ij} = -\frac{100}{2\pi ij}(e^{-ij\pi}-e^0)\\
&= -\frac{100}{2\pi ij}(\cos(j\pi)-1) = \frac{100}{2\pi ij}(1-\cos(j\pi)) = \begin{cases} 0, & j \text{ even},\\ \frac{100i}{\pi j}, & j \text{ odd},\end{cases}
\end{aligned}$$

when $j \neq 0$. For $j = 0$ we have

$$\widehat{f}(0) = \frac{1}{2\pi}\left(-\int_{-\pi}^{0} 0\,dt + \int_{0}^{\pi} 100\,dt\right) = 50.$$

Thus after some simplifications (exercise)

$$\begin{aligned}
u(r,\theta) &= \sum_{j=-\infty}^{\infty} \widehat{f}(j) r^{|j|} e^{ij\theta}\\
&= 50 + \frac{100}{\pi}\sum_{j=1}^{\infty} \frac{1}{j}(1-\cos(j\pi)) r^j \sin(j\theta), \quad 0 \leqslant r \leqslant 1, \quad -\pi \leqslant \theta < \pi.
\end{aligned}$$

By setting $r = 0$, we see that the temperature at the center of the disc is $50\,^\circ$C, which is the average temperature on the boundary of the disc. On the boundary of the disc with $r = 1$ we have

$$u(1,\theta) = 50 + \frac{200}{\pi}\sum_{j=0}^{\infty} \frac{1}{2j+1}\sin((2j+1)\theta), \quad -\pi \leqslant \theta < \pi,$$

which is the Fourier series of f.

Example 2.45. Consider $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in the rectangle $\Omega = [0,a] \times [0,b]$ with the boundary conditions

$$\begin{cases}
u(x,0) = 0, & 0 < x < a,\\
u(x,b) = 0, & 0 < x < a,\\
u(0,y) = 0, & 0 \leqslant y \leqslant b,\\
u(a,y) = g(y), & 0 \leqslant y \leqslant b,
\end{cases}$$

where f is a continuously differentiable function on $[0,b]$. We look for solutions of the form $u(x,t) = A(x)B(y)$. Inserting this into $\Delta u = 0$ gives

$$0 = \Delta u(x,y) = A''(x)B(x) + A(x)B''(x) \Longleftrightarrow \frac{A''(x)}{A(x)} + \frac{B''(y)}{B(y)} = 0$$

for every x and y. This implies

$$\frac{A''(x)}{A(x)} = \lambda = -\frac{B''(y)}{B(y)}, \quad \lambda \in \mathbb{R},$$

for every x and y.

$\boxed{\lambda<0}$ $\lambda=-\mu^2$, $\mu>0$, and

$$B''(y)-\mu^2B(y)=0 \Longleftrightarrow B(y)=c_1\sinh(\mu y)+c_2\cosh(\mu y).$$

The boundary condition $B(0)=B(b)=0$ implies $c_1=c_2=0$ and thus $B(y)=0$.

$\boxed{\lambda=0}$

$$B''(y)=0 \Longleftrightarrow B(y)=c_1y+c_2.$$

Again $B(0)=B(b)=0$ implies $c_1=c_2=0$ and thus $B(y)=0$.

$\boxed{\lambda>0}$ $\lambda=\mu^2$, $\mu>0$, and we have

$$\begin{cases} A''(x)-\mu^2A(x)=0 \\ B''(y)+\mu^2B(y)=0 \end{cases} \Longleftrightarrow \begin{cases} A(x)=c_1\sinh(\mu x)+c_2\cosh(\mu x) \\ B(y)=d_1\sin(\mu y)+d_2\cos(\mu y) \end{cases}$$

Now $A(0)=0 \Longrightarrow c_2=0$ and $B(0)=0 \Longrightarrow d_2=0$. Moreover, $B(b)=0$ implies $d_1=0$ or $\sin(\mu b)=0$. Clearly

$$\sin(\mu b)=0 \Longrightarrow \mu=\frac{j\pi}{b}, \quad j=1,2,\dots,$$

and $d_1=0 \Longrightarrow B(y)=0$. Thus

$$\begin{cases} A(x)=c_1\sinh\frac{j\pi x}{b} \\ B(y)=d_1\sin\frac{j\pi y}{b} \end{cases}$$

and

$$u(x,y)=A(x)B(y)=a_j\sinh\frac{j\pi x}{b}\sin\frac{j\pi y}{b}, \quad j=1,2,\dots,$$

are nontrivial special solutions. We look for the solution of the general problem in the form

$$u(x,y)=\sum_{j=1}^{\infty}a_j\sinh\frac{j\pi x}{b}\sin\frac{j\pi y}{b}.$$

The boundary condition

$$g(y)=u(a,y)=\sum_{j=1}^{\infty}a_j\sinh\frac{j\pi a}{b}\sin\frac{j\pi y}{b}$$

by the Fourier sine series representation gives

$$a_j\sinh\frac{j\pi a}{b}=\frac{2}{b}\int_0^b g(y)\sin\frac{j\pi y}{b}\,dy, \quad j=1,2,\dots,$$

and consequently

$$a_j=\frac{2}{b\sinh\frac{j\pi a}{b}}\int_0^b g(y)\sin\frac{j\pi y}{b}\,dy, \quad j=1,2,\dots.$$

Observe, that the boundary function $g(y)$ is extended as an odd $2b$-periodic function to $\mathbb{R}$ by setting $g(y)=-g(-y), -b<y<0$ and $g(y)=g(y+2b)$, see Remark 2.30. Thus the Fourier cosine coefficients are zero. This gives a representation formula for solution of the problem.

2.12 The heat equation in one-dimension

Let us now consider the time dependent heat diffusion in the one-dimensional case. Suppose we have a ring of radius one centered at the origin. Suppose further that the ring is perfectly insulated so that no heat leaves it. At time $t = 0$ the initial temperature distribution on the ring is given by the function $g : [-\pi, \pi] \to \mathbb{R}$. We can assume the function g to be 2π-periodic since it is defined on the unit circle.

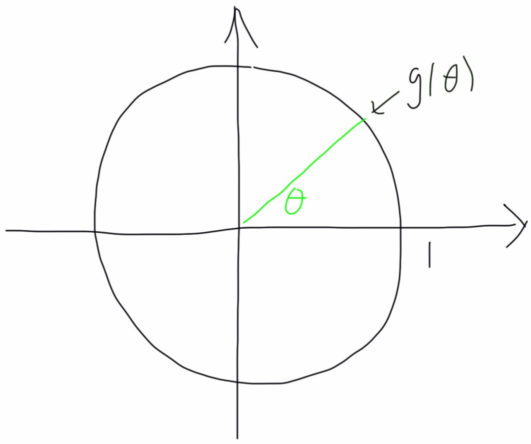


Figure 2.9: Heat distribution in a ring.

The diffusion of heat on the circle is modeled by the heat equation

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial \theta^2} = 0,$$

where a^2 is a positive constant known as the thermal diffusivity, which depends only on the material from which the ring is made. Again we set $a^2 = 1$. We are looking for a solution $u = u(\theta, t)$ where θ describes the position on the ring and $t > 0$ is time. The initial condition is $u(\theta, 0) = g(\theta)$. Thus we have the periodic initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(\theta, t) - \dfrac{\partial^2 u}{\partial \theta^2}(\theta, t) = 0, \quad -\pi \leqslant \theta < \pi, \quad t > 0, \\ u(\theta, 0) = g(\theta), \quad -\pi \leqslant \theta < \pi. \end{cases} \tag{2.18}$$

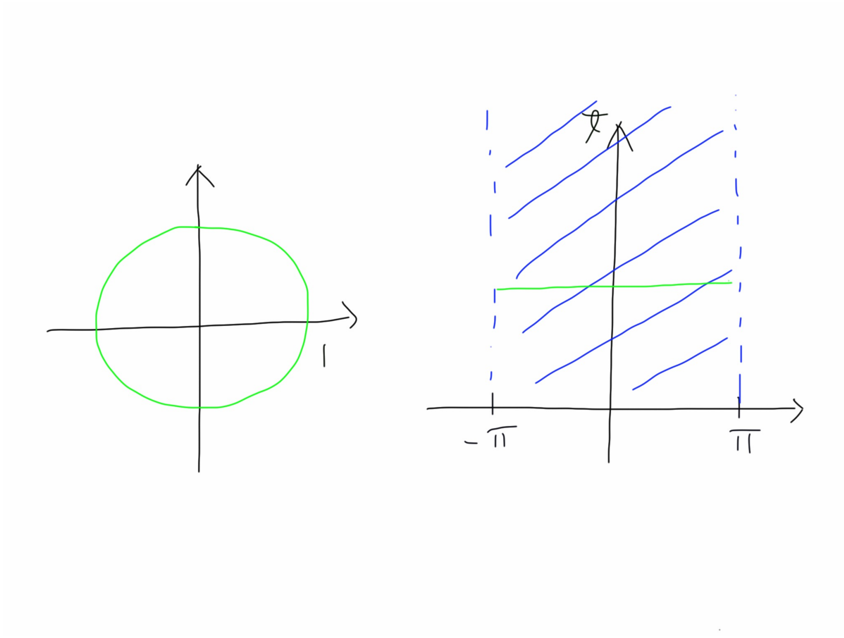


Figure 2.10: The space-time domain for the heat distribution in a ring.

Step 1 (Separation of variables): We use separation of variables again and look for special solutions of the form

$$u(\theta,t) = A(\theta)B(t).$$

By inserting this into the heat equation we get

$$A(\theta)B'(t) - A''(\theta)B(t) = 0 \Longleftrightarrow \frac{B'(t)}{B(t)} = \frac{A''(\theta)}{A(\theta)}.$$

The variables are separated in the sense that the two sides of the equation above depend on different variables and, as in Section 2.11, both have to be constant, which is denoted by λ. Thus

$$\begin{cases} A''(\theta) = \lambda A(\theta), \\ B'(t) = \lambda B(t). \end{cases}$$

Step 2 (Solution to the separated equations): The initial condition

$$u(\theta,0) = A(\theta)B(0) = g(\theta)$$

is a 2π-periodic function. We do a similar case study on the possible values of λ, as we did in the case of the Laplace equation.

$\boxed{\lambda > 0}$ Then $\lambda = \mu^2$, $\mu > 0$, and thus we have $A''(\theta) - \mu^2 A(\theta) = 0$. This ODE has the general solution

$$A(\theta) = c_1 e^{\mu\theta} + c_2 e^{-\mu\theta},$$

which is periodic only if $c_1 = c_2 = 0$. This gives the trivial solution $u = 0$ and thus we exclude this alternative.

$\boxed{\lambda = 0}$ Then $A''(\theta) = 0$, which implies $A(\theta) = c_1\theta + c_2$. Here the only possibility that is compatible with the periodicity of A is the solution $A(\theta) = c_2$ is constant. In this case we also get that $B(t) = c_3$ so that this gives only the constant solution.

$\boxed{\lambda < 0}$ Then $\lambda = -\mu^2$, $\mu > 0$ and the ODE for $A(\theta)$ is $A''(\theta) + \mu^2 A(\theta) = 0$. The general complex solution of this ODE is

$$A(\theta) = c_1 e^{i\mu\theta} + c_2 e^{-i\mu\theta}$$

for some constants c_1 and c_2. Again for A to be periodic we need μ to be an integer which gives thus, considering all cases above, that the only admissible values for λ are of the form $\lambda = -j^2$, $j \in \mathbb{Z}$. For these values we have $B'(t) + j^2 B(t) = 0$, which has the general solution

$$B(t) = ce^{-j^2 t}.$$

Thus we have special solutions, called the normal modes, of the form

$$u(\theta, t) = e^{-j^2 t} e^{ij\theta}, \quad j \in \mathbb{Z}.$$

By the construction, these functions are solutions of the one-dimensional heat equation for every $j \in \mathbb{Z}$.

Step 3 (Fourier series solution of the entire problem): Since the heat equation is linear, any linear combination of the special solutions above will give again a solution. Motivated by the superposition principle, we define

$$u(\theta, t) = \sum_{j=-\infty}^{\infty} a_j e^{-j^2 t} e^{ij\theta}, \quad -\pi \leqslant \theta < \pi, \quad t > 0.$$

The next step is to determine coefficients a_j. By the initial condition $u(\theta, 0) = g(\theta)$, we have

$$\sum_{j=-\infty}^{\infty} a_j e^{ij\theta} = g(\theta)$$

which identifies the coefficients a_j as the Fourier coefficients $\widehat{g}(j)$ of the initial data g. Thus

$$u(\theta, t) = \sum_{j=-\infty}^{\infty} \widehat{g}(j) e^{-j^2 t} e^{ij\theta}.$$

This infinite series converges absolutely, since the Fourier coefficients $\widehat{g}(j)$ are bounded if $g \in L^1([-\pi, \pi])$ (see Lemma 2.20 (2)) and the term $e^{-j^2 t}$ decays extremely fast as $t > 0$ and j is large. Thus

$$\sum_{j=-\infty}^{\infty} \left| \widehat{g}(j) e^{-j^2 t} e^{ij\theta} \right| \leqslant \sum_{j=-\infty}^{\infty} |\widehat{g}(j)| e^{-j^2 t} \leqslant \|g\|_{L^1([-\pi,\pi])} \underbrace{\sum_{j=-\infty}^{\infty} e^{-j^2 t}}_{<\infty}.$$

The last series converges, which shows that the series defining the solution to the heat equation converges absolutely.

Step 4 (Explicit representation formula): Next goal is to derive an integral representation for the solution similar to the convolution with the Poisson kernel for the Dirichlet problem for the Laplace equation in the unit disc. Using the definition of the Fourier coefficients we obtain

$$\begin{aligned}
u(\theta,t) &= \sum_{j=-\infty}^{\infty} \widehat{g}(j)e^{-j^2t}e^{ij\theta} \\
&= \lim_{n\to\infty}\sum_{j=-n}^{n}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(s)e^{-ijs}\,ds\right)e^{-j^2t}e^{ij\theta} \\
&= \lim_{n\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi} g(s)\left(\sum_{j=-n}^{n} e^{-j^2t}e^{ij(\theta-s)}\right)ds.
\end{aligned}$$

We have the uniform estimate

$$\left|\sum_{j=-n}^{n} e^{-j^2t}e^{ij(\theta-s)}\right| \leqslant \sum_{j=-\infty}^{\infty} e^{-j^2t} < \infty.$$

Here it is crucial that $t > 0$, so these estimates do not generalize to negative times. By switching the order of the limit and the integral, we obtain

$$\begin{aligned}
u(\theta,t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(s)\lim_{n\to\infty}\left(\sum_{j=-n}^{n} e^{-j^2t}e^{ij(\theta-s)}\right)ds \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(s)\left(\sum_{j=-\infty}^{\infty} e^{-j^2t}e^{ij(\theta-s)}\right)ds.
\end{aligned}$$

By defining the heat kernel for the circle as

$$H_t(\theta) = \sum_{j=-\infty}^{\infty} e^{-j^2t}e^{ij\theta}, \quad -\pi \leqslant \theta < \pi, \quad t > 0,$$

we see that the solution of the heat equation on the circle is given by the convolution

$$u(\theta,t) = (g * H_t)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(s)H_t(\theta-s)\,ds, \quad -\pi \leqslant \theta < \pi, \quad t > 0.$$

THE MORAL: The solution of the initial value problem for the heat equation on the circle is a convolution of the initial temperature distribution with the heat kernel.

Theorem 2.46 (Solution to the heat equation on the unit circle). The solution of the (periodic) initial value problem (2.18) is

$$u(\theta,t) = (g * H_t)(\theta,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(s)H_t(\theta-s)\,ds, \quad -\pi \leqslant \theta < \pi, \quad t > 0,$$

where $H_t(\theta) = \sum_{j=-\infty}^{\infty} e^{-j^2t}e^{ij\theta}$. The initial values are obtained in the sense that

$$\lim_{t\to 0} u(\theta,t) = g(\theta) \quad -\pi \leqslant \theta \leqslant \pi.$$

Remark 2.47. Another way to derive the solution to a PDE problem is start with the Fourier series expansion, insert it in the PDE and try to determine the coefficients. This approach works for the Laplace equation, the heat equation and the wave equation, but we shall discuss only the initial value problem (2.18) in detail. For a fixed $t > 0$, consider the Fourier series of the function $\theta \mapsto u(\theta, t)$. This gives

$$u(\theta,t) = \sum_{j=-\infty}^{\infty} c_j(t) e^{ij\theta},$$

where the Fourier coefficients are

$$c_j(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(s,t)e^{-ijs}\,ds, \quad j \in \mathbb{Z}.$$

Observe that the Fourier coefficients depend on t. We claim that the PDE for u implies that the Fourier coefficients satisfy an ODE as a function of t. To see this, we switch the order of integral and derivative to have

$$c_j'(t) = \frac{\partial u}{\partial t}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} u(s,t)e^{-ijs}\,ds\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial u}{\partial t}(s,t)e^{-ijs}\,ds, \quad j \in \mathbb{Z}.$$

Then we use the PDE and Lemma 2.31 twice to obtain

$$\begin{aligned}
c_j'(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial u}{\partial t}(s,t)e^{-ijs}\,ds \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial^2 u}{\partial s^2}(s,t)e^{-ijs}\,ds \\
&= ij\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial u}{\partial s}(s,t)e^{-ijs}\,ds \\
&= (ij)^2\frac{1}{2\pi}\int_{-\pi}^{\pi} u(s,t)e^{-ijs}\,ds \\
&= -j^2 c_j(t), \quad j \in \mathbb{Z}.
\end{aligned}$$

THE MORAL: The PDE becomes an ODE on the Fourier side.

The general solution of $c_j'(t)+j^2c(t) = 0$ is $c_j(t) = a_j e^{-j^2 t}$, where a_j is a constant. By the Fourier series representation

$$u(\theta,t) = \sum_{j=-\infty}^{\infty} c_j(t)e^{ij\theta} = \sum_{j=-\infty}^{\infty} a_j e^{-j^2 t} e^{ij\theta}, \quad -\pi \leqslant \theta < \pi, \quad t > 0.$$

By the initial condition $u(\theta,0) = g(\theta)$, we have

$$u(\theta,0) = \sum_{j=-\infty}^{\infty} a_j e^{ij\theta} = g(\theta)$$

which identifies the coefficients a_j as the Fourier coefficients $\widehat{g}(j)$ of the initial data g. Thus

$$u(\theta,t) = \sum_{j=-\infty}^{\infty} \widehat{g}(j) e^{-j^2 t} e^{ij\theta}.$$

THE MORAL: In the this approach we inserted the Fourier series representation of the function in the PDE and determined the Fourier coefficients by the initial condition. In the beginning of this section we derived the same formula for the solution by separation of variables and this lead to the Fourier series. Although the result is same, the main difference is that in the original argument we did not have to assume in the beginning that the solution is given by the Fourier series. However, theory of Fourier series is needed in both approaches.

Let us consider another problem for the one-dimensional heat equation. We study the temperature distribution in a thin uniform bar of given length $L > 0$ with insulated lateral surface and no internal sources of heat, subject to certain boundary and initial conditions. To describe the problem, let $u(x,t)$, $0 < x < L$, $t > 0$, be the temperature at the point x at time t. Assume that the initial temperature distribution of the bar is $u(x,0) = g(x)$ and assume, that the ends of the bar are held at constant temperature $0\,°$C, for example. This is related to the previously considered problem of the heat distribution on a circle, since we can cut the circle and unfold it so that it becomes a bar.

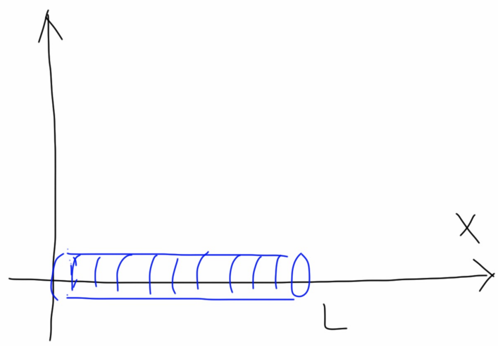


Figure 2.11: The heat distribution in a bar.

The following theorem gives a solution of the heat distribution in the bar of length $L > 0$. We shall prove this in the exercises using the method of separation of variables.

Theorem 2.48 (Solution of the heat equation on a bar). The solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} - a^2 \dfrac{\partial^2 u}{\partial x^2} = 0, & 0 < x < L, \quad t > 0, \\ u(0,t) = u(L,t) = 0, & t \geqslant 0, \\ u(x,0) = g(x), & 0 \leqslant x \leqslant L. \end{cases}$$

is

$$u(x,t) = \sum_{j=1}^{\infty} a_j e^{-\lambda_j^2 t} \sin \frac{j\pi x}{L},$$

where

$$a_j = \frac{2}{L} \int_0^L g(x) \sin \frac{j\pi x}{L} \, dx \quad \text{and} \quad \lambda_j = a \frac{j\pi}{L}, \quad j = 1, 2, \ldots.$$

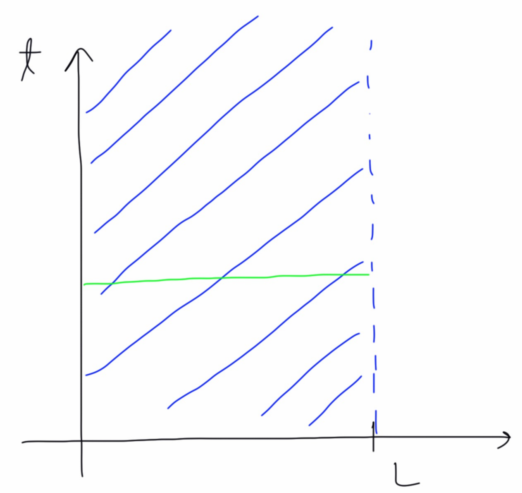


Figure 2.12: The space-time domain for the heat distribution in a bar.

The method of separation of variables can also be used to solve heat conduction problems with other boundary conditions than those given above.

Example 2.49. Consider a bar of lenght π is in boiling water. After reaching the temperature $100\,^\circ$C throughout, the bar is taken out and immersed in a medium with constant freezing temperature $0\,^\circ$C. The the bar are kept insulated and we assume that $a^2 = 1$. Then the temperature distribution is

$$u(x,t) = \sum_{j=1}^{\infty} a_j e^{-j^2 t} \sin(jx),$$

where

$$a_j = \frac{2}{\pi}\int_0^{\pi} 100\sin(jx)\,dx = \frac{200}{j\pi}(1-\cos(j\pi)).$$

After some simplifications (exercise)

$$u(x,t) = \frac{400}{\pi}\sum_{j=0}^{\infty}\frac{e^{-(2j+1)^2 t}}{2j+1}\sin((2j+1)x).$$

Here we used the fact that $1-\cos(j\pi)=0$ if j is even and 2 if j is odd.

2.13 The wave equation in one-dimension

In this section we study the motion of an elastic string of length $L>0$ fixed at its end points and allowed to vibrate freely. The elastic string may be thought of as a violin string, for example. Suppose that the string is set in motion so that it vibrates in a vertical plane and let $u(x,t)$ denote the vertical displacement of the string at the point $x\in[0,L]$ at time $t\geqslant 0$. This is modeled by the wave equation

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0.$$

We are looking for a solution $u=u(x,t)$, where $x\in[0,L]$ describes the position on the string and $t>0$ is time.

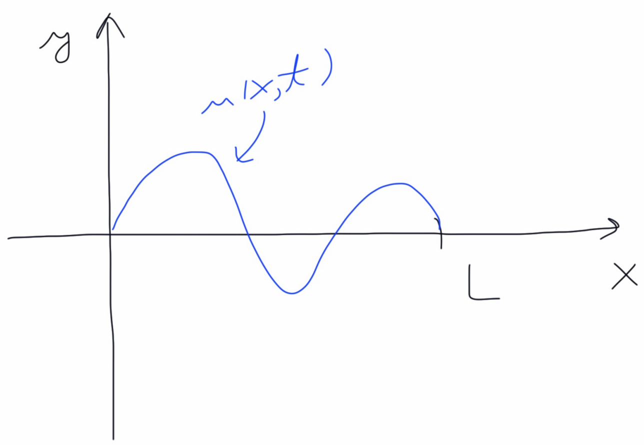


Figure 2.13: A vibrating string.

The initial displacement profile and the velocity of the string at time $t = 0$ are given as the initial conditions

$$u(x,0) = g(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = h(x), \quad 0 \leqslant x \leqslant L.$$

Since the end points of the string are fixed, we have the boundary conditions

$$u(0,t) = u(L,t) = 0 \quad \text{for every} \quad t \geqslant 0.$$

The problem is to determine the solution of the wave equation that also satisfies the the initial conditions and boundary conditions above. This can be considered as a boundary value problem in the strip $0 < x < L$, $t > 0$, in the xt-plane. With this interpretation, the boundary condition is imposed on the sides of the strip and the two initial conditions are imposed on the base of the strip. Thus we have the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < L, \quad t > 0, \\ u(0,t) = u(L,t) = 0, \quad t \geqslant 0, \\ u(x,0) = g(x), \quad 0 \leqslant x \leqslant L \\ \dfrac{\partial u}{\partial t}(x,0) = h(x), \quad 0 \leqslant x \leqslant L. \end{cases} \tag{2.19}$$

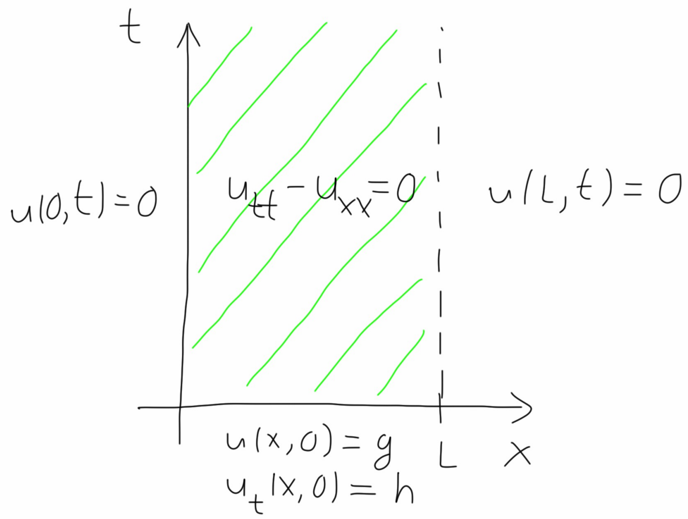


Figure 2.14: The one-dimensional space-time domain.

Step 1 (Separation of variables): We look for special solutions of the form

$$u(x,t) = A(x)B(t),$$

where $A(x)$ is a function of x alone and $B(t)$ is a function of t alone. Plugging this into the wave equation, we get

$$A(x)B''(t)-a^2A''(x)B(t)=0 \Longleftrightarrow \frac{B''(t)}{a^2B(t)}=\frac{A''(x)}{A(x)}.$$

Again we note that the two sides of the identity above depend on different variables and thus both have to be constant and equal to λ. This leads to the ODEs

$$\begin{cases} A''(x)=\lambda A(x), \\ B''(t)=a^2\lambda B(t). \end{cases}$$

We could consider the complex solution of these ODE, as we did in the cases of the Laplace and the heat equations. However, let us show here how to do the analysis using the real Fourier series.

Step 2 (Solution to the separated equations): We do a similar case study on the possible values of λ, as we did in the case of the Laplace and the heat equations.

$\boxed{\lambda>0}$ Then $\lambda=\mu^2$, $\mu>0$, and thus we have $A''-\mu^2A=0$. This ODE has the general real solution

$$A(x)=c_1\cosh(\mu x)+c_2\sinh(\mu x).$$

We show that the only way to satisfy the boundary conditions is to take$c_1=c_2=0$. Indeed, $A(0)=0$ implies $0=c_1\cosh(0)+c_2\sinh(0)=c_1$ so that $A(x)=c_2\sinh(\mu x)$. The condition $A(L)=0$ implies $c_2\sinh(\mu L)=0$. However, $\mu L\neq 0$ and so $\sinh(\mu L)\neq 0$. Thus $c_2=0$ and this gives the trivial solution $u=0$. We exclude this alternative.

$\boxed{\lambda=0}$ Then $A''(x)=0$, which implies $A(x)=c_1x+c_2$. Here the only way to satisfy the boundary condition is to take $c_1=c_2=0$, which again leads to the trivial solution $u=0$.

$\boxed{\lambda<0}$ Then $\lambda=-\mu^2$ for some $\mu>0$ and the ODE for A is $A''+\mu^2A=0$. The general real solution of this ODE is

$$A(x)=c_1\cos(\mu x)+c_2\sin(\mu x)$$

for some constants c_1 and c_2. Since $A(0)=0$, we have $c_1=0$ and $A(x)=c_2\sin(\mu x)$. Since $A(L)=0$, we have $c_2\sin(\mu L)=0$, from which we conclude that $\mu L=j\pi$, $j=0,\pm1,\pm2,\ldots$ (or $c_2=0$, which again gives the trivial solution $u(x,t)=0$). This gives

$$\mu=\frac{j\pi}{L}, \quad j=1,2,\ldots$$

and

$$A(x)=c\sin\left(\frac{j\pi x}{L}\right), \quad j=1,2,\ldots$$

for any constant c that may depend on j. Note that for negative values of j we obtain the same solutions except for a change of sign. Since the general solution will be represented as a linear combination of the special solutions, the solutions with negative j can be discarded without loss.

We have

$$\lambda = -\mu^2 = -\left(\frac{j\pi}{L}\right)^2, \quad j = 1,2,\dots.$$

With this, the ODE for B becomes

$$B''(t) + \left(\frac{aj\pi}{L}\right)^2 B(t) = 0.$$

The general solution of this equation is

$$B(t) = b_1 \cos\left(\frac{aj\pi t}{L}\right) + b_2 \sin\left(\frac{aj\pi t}{L}\right), \quad j = 1,2,\dots$$

for any constants b_1 and b_2 that may depend on j. Thus the product solution is

$$u(x,t) = A(x)B(t) = \sin\left(\frac{j\pi x}{L}\right)\left(a_j \cos\left(\frac{aj\pi t}{L}\right) + b_j \sin\left(\frac{aj\pi t}{L}\right)\right), \quad j = 1,2,\dots$$

Observe, that we have absorbed the constant in front of sine to the constants a_j and b_j.

Step 3 (Fourier series solution of the entire problem): Since the wave equation is linear, any linear combination of these solutions will give again a solution. We thus define

$$u(x,t) = \sum_{j=1}^{\infty} \sin\left(\frac{j\pi x}{L}\right)\left(a_j \cos\left(\frac{aj\pi t}{L}\right) + b_j \sin\left(\frac{aj\pi t}{L}\right)\right)$$

By the initial condition $u(x,0) = g(x)$, we have

$$u(x,0) = \sum_{j=1}^{\infty} a_j \sin\left(\frac{j\pi x}{L}\right) = g(x).$$

which identifies the coefficient a_j as the Fourier sine coefficient of the initial data f on $[0,L]$. Observe, that the boundary function $g(x)$ is extended as an odd $2L$-periodic function to $\mathbb{R}$ by setting $g(x) = -g(-x), -L < x < 0$ and $g(x) = g(x+2L)$, see 2.30. Thus the Fourier cosine coefficients are zero. This gives

$$a_j = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{j\pi x}{L}\right) dx, \quad j = 1,2,\dots$$

On the other hand, by differentiating the series termwise we have

$$\frac{\partial u}{\partial t}(x,t) = \frac{\pi a}{L}\sum_{j=1}^{\infty} \sin\left(\frac{j\pi x}{L}\right)\left(-ja_j \sin\left(\frac{aj\pi t}{L}\right) + jb_j \cos\left(\frac{aj\pi t}{L}\right)\right),$$

from which we obtain

$$\frac{\partial u}{\partial t}(x,0) = \frac{\pi a}{L}\sum_{j=1}^{\infty} jb_j \sin\left(\frac{j\pi x}{L}\right) = h(x).$$

Consequently, $\frac{\pi a}{L} j b_j$ must be the Fourier sine coefficient of h on $[0,L]$. The boundary function $h(x)$ is extended as an odd $2L$-periodic function to $\mathbb{R}$ and thus the Fourier cosine coefficients are zero. This gives

$$\frac{\pi a}{L} j b_j = \frac{2}{L}\int_0^L h(x)\sin\left(\frac{j\pi x}{L}\right)dx, \quad j = 1,2,\dots$$

or equivalently

$$b_j = \frac{2}{\pi a j}\int_0^L h(x)\sin\left(\frac{j\pi x}{L}\right)dx, \quad j = 1,2,\dots$$

We have now determined all the unknown coefficients in the series representation of the solution u. We summarize our findings in here.

Theorem 2.50 (Solution of the one-dimensional wave equation). The solution of the problem (2.19) is

$$u(x,t) = \sum_{j=1}^{\infty} \sin\left(\frac{j\pi x}{L}\right)\left(a_j \cos\left(\frac{aj\pi t}{L}\right) + b_j \sin\left(\frac{aj\pi t}{L}\right)\right),$$

where

$$a_j = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{j\pi x}{L}\right)dx, \quad j = 1,2,\dots,$$

and

$$b_j = \frac{2}{aj\pi}\int_0^L h(x)\sin\left(\frac{j\pi x}{L}\right)dx, \quad j = 1,2,\dots.$$

As in the case of the Laplace equation and the heat equation considered earlier, this is only a formal solution of the problem. To show that the obtained formula actually represents the solution of the problem requires further investigations (exercise).

Remark 2.51. The solution of the vibrating string problem is an infinite sum of the normal modes

$$u_j(x,t) = \sin\left(\frac{j\pi x}{L}\right)\left(a_j \cos\left(\frac{aj\pi t}{L}\right) + b_j \sin\left(\frac{aj\pi t}{L}\right)\right), \quad j = 1,2,\dots.$$

When the string vibrates according to u_j, we say that it is in the jth normal mode of vibration. The first normal mode is called the fundamental mode and the other modes are overtones. The quantities $aj\pi/L$, $j = 1,2,\dots$, are the natural frequencies of the normal mode, which gives the number of oscillations in 2π units of time. The factor $\sin(j\pi x/L)$ is the displacement pattern of the string when it vibrates at the given frequency. When the string vibrates in its normal mode, some points of the string are fixed at all times. These are the solutions of the equation $\sin(j\pi x/L) = 0$. If we do not count the ends of the string, there are $n-1$ equidistant points that do not vibrate in the nth normal mode.

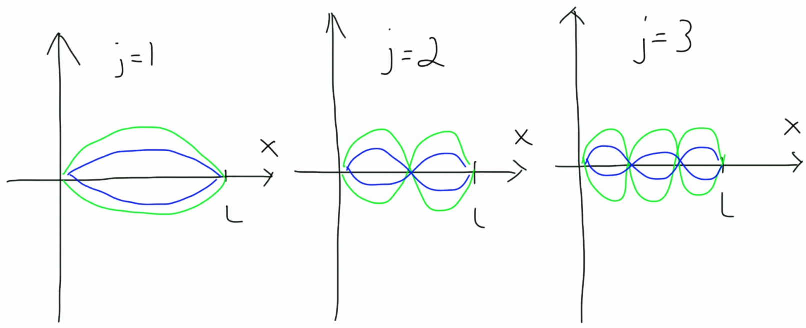


Figure 2.15: Normal modes of a string.

Example 2.52. The ends of a string of length $L = 1$ are fixed at $x = 0$ and $x = 1$. Assume that $a = 1$. The string is set to vibrate from rest with a initial triangular profile

$$g(x) = \begin{cases} \frac{3x}{10}, & 0 \leqslant x \leqslant \frac{1}{3}, \\ \frac{3(1-x)}{20}x, & \frac{1}{3} \leqslant x \leqslant 1. \end{cases}$$

Since $h(x) = 0$, we have $b_j = 0$. Using the formula for the solution and integrating by parts, we have

$$\begin{aligned} a_j &= 2\int_0^L g(x)\sin(j\pi x)\,dx \\ &= \frac{3}{5}\int_0^{1/3} x\sin(j\pi x)\,dx + \frac{3}{10}\int_{1/3}^1 (1-x)\sin(j\pi x)\,dx \\ &= -\frac{\cos\left(\frac{j\pi}{3}\right)}{5j\pi} + \frac{3}{5}\frac{\sin\left(\frac{j\pi}{3}\right)}{j^2\pi^2} + \frac{\cos\left(\frac{j\pi}{3}\right)}{5j\pi} + \frac{3}{10}\frac{\sin\left(\frac{j\pi}{3}\right)}{j^2\pi^2} = \frac{9}{10\pi^2}\frac{\sin\left(\frac{j\pi}{3}\right)}{j^2}. \end{aligned}$$

Thus

$$u(x,t) = \frac{9}{10\pi^2}\sum_{j=1}^{\infty}\frac{\sin\left(\frac{j\pi}{3}\right)}{j^2}\sin(j\pi x)\cos(j\pi t).$$

2.14 Approximations of the identity in $[-\pi,\pi]^*$

There is a common theme in the expressions for the partial sums of the Fourier series of a function, the solution to the Dirichlet problem for the Laplace equation in the disc and the solution to the 2π-periodic initial value problem for the heat equation. Indeed, by (2.12), Theorem 2.43 and Theorem 2.46, we have

$$S_n f(\theta) = (f * D_n)(\theta) = \sum_{j=-n}^{n} \widehat{f}(j)e^{ij\theta}, \quad f = \sum_{j=-\infty}^{\infty} \widehat{f}(j)e^{ij\theta},$$
$$P_r f(\theta) = (f * P_r)(\theta) = \sum_{j=-\infty}^{\infty} \widehat{f}(j)r^{|j|}e^{ij\theta}, \quad 0 < r < 1,$$
$$H_t f(\theta) = (f * H_t)(\theta) = \sum_{j=-\infty}^{\infty} \widehat{f}(j)e^{-j^2 t}e^{ij\theta}, \quad t > 0.$$

The first formula encodes the most basic and natural question in Fourier series: Can we recover a function f from the partial sums of its Fourier series? We have seen that this is not always possible. The pointwise limit

$$\lim_{n\to\infty} S_n f = \lim_{n\to\infty} (D_n * f)$$

may fail to exist even if f is continuous. However, by Theorem 2.37 this limit exists, if f is continuously differentiable. Moreover, for a continuous function f, we are interested in existence of the pointwise limits

$$\lim_{r\to 1}(P_r * f) = f \quad \text{and} \quad \lim_{t\to 0}(H_t * f) = f.$$

These are related to the question in which sense the boundary or initial values are obtained in the corresponding PDE problems. It turns out that the Poisson kernel and the heat kernel are just special cases of a general theory of good kernels.

Definition 2.53. A family of kernels $\{K_\varepsilon\}_{\varepsilon>0}$ which are 2π-periodic functions is said to be a family of good kernels, if it satisfies the following properties.

(1) For every $\varepsilon > 0$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_\varepsilon(x)\,dx = 1.$$

(2) There exists a positive constant M such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |K_\varepsilon(x)|\,dx \leqslant M$$

for every $\varepsilon > 0$.

(3) For every $\delta > 0$ we have

$$\lim_{\varepsilon\to 0}\int_{\delta<|x|\leqslant\pi} |K_\varepsilon(x)|\,dx = 0.$$

T H E M O R A L : Parameter $\varepsilon > 0$ gives the scale at which we take samples of functions. Condition (1) means that the total mass of the kernel is one at all scales and condition (3) means that the mass concentrates near the origin at small scales.

Remarks 2.54:

(1) For nonnegative kernels $K_\varepsilon \geqslant 0$, property (2) is a consequence of (1) with $M = 1$.

(2) Property (3) is the most crucial and describes the fact that, as $\varepsilon \to 0$, the mass of the kernel K_ε concentrates more and more near the origin.

(3) It can be shown that the Poisson kernel $P_{1-\varepsilon}$ and the heat kernel H_t are both good kernels (exercise). It is now natural to ask whether D_n is a good kernel, since if this were true, we would be able to conclude that the Fourier series of a continuous function converges pointwise. Unfortunately this is not the case. Indeed, the Dirichlet kernel is a sign changing function and it does not satisfy condition (2) above.

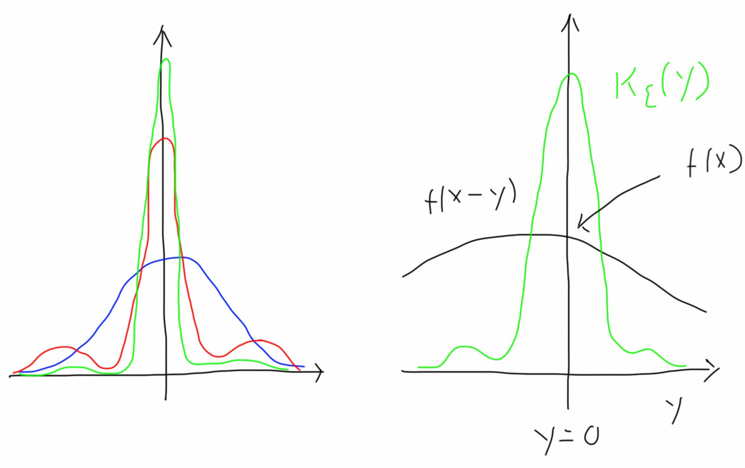


Figure 2.16: A family of good kernels.

The following theorem explains why good kernels are very useful. Because of this result, the family $\{K_\varepsilon\}_{\varepsilon>0}$ is sometimes referred to as an approximation of the identity. Note that the theorem below immediately provides the proof for Theorem 2.43 and Theorem 2.46 for the Poisson and the heat kernel, respectively.

Theorem 2.55. Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of good kernels and $f : [-\pi,\pi] \to \mathbb{C}$ be a bounded 2π-periodic function. Then

$$\lim_{\varepsilon\to 0}(f * K_\varepsilon)(x) = f(x)$$

whenever f is continuous at x. If f is continuous on the whole interval $[-\pi,\pi]$, then the above limit is uniform.

T H E   M O R A L : The fact that the boundary and initial values are obtained is based on a general properties of the convolution with approximations of identity. On the other hand, this can be used to approximate a given function with smoother functions. Several fundamental solutions of PDEs give rise to an approximation of identity.

Proof. Let $\eta > 0$. Since f continuous at x there exists $\delta > 0$ such that

$$|f(x-y) - f(x)| < \eta$$

whenever $|y| < \delta$. Then by property (1) of a good kernel we can write

$$(f * K_\varepsilon)(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x-y) - f(x))K_\varepsilon(y)\,dy.$$

Thus

$$\begin{aligned}
|(K_\varepsilon * f)(x) - f(x)| &\leqslant \frac{1}{2\pi}\int_{|y|<\delta}|K_\varepsilon(y)||f(x-y)-f(x)|\,dy \\
&\quad + \frac{1}{2\pi}\int_{\delta<|y|\leqslant\pi}|K_\varepsilon(y)||f(x-y)-f(x)|\,dy. \\
&\leqslant \frac{\eta}{2\pi}\int_{-\pi}^{\pi}|K_\varepsilon(y)|\,dy + \frac{1}{\pi}\sup_{t\in[-\pi,\pi]}|f(t)|\int_{\delta<|y|\leqslant\pi}|K_\varepsilon(y)|\,dy.
\end{aligned}$$

Now letting $\varepsilon \to 0$ the second term tends to zero by property (3) of the good kernels. Since $\eta > 0$ was arbitrary, this completes the proof of the first part of the theorem. For the second part note that if f is continuous on $[-\pi,\pi]$ and 2π-periodic, then it is uniformly continuous. Thus the $\delta > 0$ in the argument above can be chosen independently of x, which shows that the convergence is uniform in this case. □

Example 2.56. Let

$$K_\varepsilon(x) = \frac{1}{\varepsilon}\mathbf{1}_{[-\pi,\pi]}\left(\frac{x}{\varepsilon}\right), \quad 0 < \varepsilon < 1,$$

where

$$\mathbf{1}_{[a,b]}(x) = \begin{cases} 1, & \text{if } \; x \in [a,b], \\ 0, & \text{otherwise,} \end{cases}$$

is the indicator function (or the characteristic function) of the set $[a,b]$.

Observe that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}K_\varepsilon(x)\,dx = \frac{1}{2\pi}\int_{-\varepsilon\pi}^{\varepsilon\pi}\frac{1}{\varepsilon}\,dx = 1, \quad \varepsilon > 0.$$

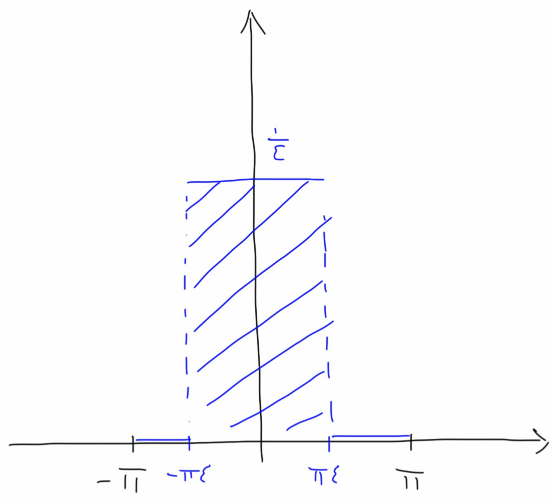


Figure 2.17: A good kernel related to the integral average.

Thus for all $\varepsilon > 0$ the function K_ε has the same total mass one, but as $\varepsilon \to 0$ this mass is concentrated more and more around the origin. Now for any $f \in L^1([-\pi,\pi])$ we have

$$\begin{aligned}(f * K_\varepsilon)(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_\varepsilon(t)\,dt \\ &= \frac{1}{2\pi\varepsilon}\int_{-\pi\varepsilon}^{\pi\varepsilon} f(x-t)\,dt \\ &= \frac{1}{2\pi\varepsilon}\int_{x-\pi\varepsilon}^{x+\pi\varepsilon} f(t)\,dt.\end{aligned}$$

Thus the value $(f * K_\varepsilon)(x)$ is equal to the average of f on a symmetric interval of length $2\pi\varepsilon$ around the point x. In this sense we can think of the convolution as an averaging process. Observe that, if f is continuous at x, then

$$\lim_{\varepsilon\to 0}(f * K_\varepsilon)(x) = \lim_{\varepsilon\to 0}\frac{1}{2\pi\varepsilon}\int_{x-\pi\varepsilon}^{x+\pi\varepsilon} f(t)\,dt = f(x).$$

This means that the pointwise value of a continuous function is the limit of the integral averages.

2.15 Summary

The main steps in the application of the Fourier series to PDE problems are the following.

(1) The task is to find the solution u of an initial or boundary value problem in a rectangular domain.
(2) Separate variables and insert the product solution to the PDE.
(3) This reduces the problem to two ODE.
(4) The two ODE are solved explicitly to find nontrivial special solutions.
(5) The general solution of a problem is a linear combination of nontrivial special solutions.
(6) The initial or boundary conditions are used to represent the coefficients using Fourier series.
(7) It is a matter of taste whether one wants to work with the real or the complex form of the Fourier series.
(8) The solution of the original problem is represented as a convolution of the data with a kernel function, which is a fundamental solution of the corresponding problem.
(9) This gives an integral representation of the solution to the original problem and the initial or boundary values are attained by using approximations of the unity.

Fourier transform gives a method to solve PDEs in the higher dimensional case. It has many properties analogous to the Fourier series, but the functions do not need to be periodic. Convolution plays a central role in the theory.

3 Fourier transform and PDEs

The theory of the Fourier series applies to periodic functions in the one-dimensional case. In this chapter, we develop an analogous theory for functions defined on the whole n-dimensional Euclidean space $\mathbb{R}^n$, which are not assumed to be periodic. The Fourier series of a periodic function associates the Fourier coefficients to that function and, under certain assumptions, a function can be represented as a Fourier series. In this section we shall see that, under certain assumptions, a function can be represented by its Fourier transform, which is a function associated to the original function.

3.1 The L^p-space on $\mathbb{R}^n$*

We consider functions defined in the n-dimensional Euclidean space $\mathbb{R}^n$, $n = 1,2,\ldots$. We discuss briefly some aspects of the integration of such functions. A function $f : \mathbb{R}^n \to \mathbb{C}$ is integrable, if

$$\int_{\mathbb{R}^n} |f(x)|\,dx < \infty.$$

Here the n-dimensional integral is computed as an iterated integral

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1,\ldots,x_n)\,dx_1 \cdots dx_n.$$

We begin with listing some basic properties of the integral.

(1) For $f : \mathbb{R}^n \to \mathbb{C}$ and $A \subset \mathbb{R}^n$, we have

$$\left|\int_A f(x)\,dx\right| \leqslant \int_A |f(x)|\,dx.$$

(2) If $A \subset B \subset \mathbb{R}^n$, then

$$\int_A |f(x)|\,dx \leqslant \int_B |f(x)|\,dx.$$

(3) If $|f| \leqslant |g|$ and $A \subset \mathbb{R}^n$, then

$$\int_A |f(x)|\,dx \leqslant \int_A |g(x)|\,dx.$$

WARNING: It is essential in (2) and (3) that we consider the absolute values of the functions. The corresponding claims do not hold for real valued functions that are allowed to change signs.

We recall the the definition of the $L^p(\mathbb{R}^n)$ spaces. For $1 \leqslant p < \infty$, the space $L^p(\mathbb{R}^n)$ consists of functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$\|f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p\,dx\right)^{\frac{1}{p}} < \infty.$$

For $p = \infty$ we set

$$\|f\|_{L^\infty(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} |f(x)|.$$

It can be shown that $L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, is a complete normed space (Banach space) with the norm defined above, but this is out of the scope of this course.

Examples 3.1:

(1) Let $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = 1$. Then $\|f\|_{L^\infty(\mathbb{R}^n)} = 1 < \infty$, so that $f \in L^\infty(\mathbb{R}^n)$. However, $\|f\|_{L^p(\mathbb{R}^n)} = \infty$, so that $f \notin L^p(\mathbb{R}^n)$ whenever $1 \leqslant p < \infty$.

(2) Let $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 1, & |x| \leqslant 1, \\ \frac{1}{|x|^2}, & |x| > 1. \end{cases}$$

Then

$$\int_{\mathbb{R}} |f(x)|\,dx = \int_{|x|\leqslant 1} 1\,dx + 2\int_1^\infty \frac{1}{x^2}\,dx = 2 + 2\Big|_1^{+\infty} \frac{-1}{x} = 4 < \infty.$$

Thus $f \in L^1(\mathbb{R})$.

(3) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{1+|x|}$.

Claim: $f \in L^\infty(\mathbb{R})$.

Reason. For any $x \in \mathbb{R}$ we have $1 + |x| \geqslant 1$ and

$$|f(x)| = \frac{1}{1+|x|} \leqslant 1,$$

so that $\|f\|_{L^\infty(\mathbb{R})} = \sup_{x\in\mathbb{R}} |f(x)| \leqslant 1$. ∎

Claim: $f \notin L^1(\mathbb{R})$.

Reason. For $x \geqslant 1$ we have

$$\frac{1}{1+|x|} \geqslant \frac{1}{2x}.$$

Thus

$$\int_{\mathbb{R}} |f(x)|\,dx \geqslant \int_1^\infty \frac{1}{1+|x|}\,dx \geqslant \int_1^\infty \frac{1}{2x}\,dx = \infty.$$ ∎

Claim: $f \in L^2(\mathbb{R})$.

Reason.

$$\begin{aligned}\int_{\mathbb{R}} |f(x)|^2\,dx &= \int_{\mathbb{R}} \frac{1}{(1+|x|)^2}\,dx \\ &= \int_{|x|\leqslant 1} \frac{1}{(1+|x|)^2}\,dx + \int_{|x|>1} \frac{1}{(1+|x|)^2}\,dx \\ &\leqslant \int_{|x|\leqslant 1} 1\,dx + \int_{|x|>1} \frac{1}{x^2}\,dx \\ &= 2 + 2\left.\frac{-1}{x}\right|_1^{\infty} = 2+2 = 4 < \infty.\end{aligned}$$ ■

T H E   M O R A L : $L^2(\mathbb{R})$ is not contained in $L^1(\mathbb{R})$. Note carefully, that this is different from the case with an interval of finite length when we have $L^2([-\pi,\pi]) \subset L^1([-\pi,\pi])$.

(4) Let $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & \text{if } \quad 0 < x \leqslant 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $f \notin L^\infty(\mathbb{R})$, because f is unbounded near $x = 0$.
Claim: $f \in L^1(\mathbb{R})$.
Reason.

$$\int_{\mathbb{R}} |f(x)|\,dx = \int_0^1 \frac{1}{\sqrt{x}}\,dx < \infty.$$ ■

Claim: $f \notin L^2(\mathbb{R})$.
Reason.

$$\int_{\mathbb{R}} |f(x)|^2\,dx = \int_0^1 \frac{1}{x}\,dx = \infty$$ ■

T H E   M O R A L : $L^1(\mathbb{R})$ is not contained in $L^2(\mathbb{R})$.

Remark 3.2. There are essentially two reasons why a function may fail to belong to $L^1(\mathbb{R})$. The first reason is the decay of the function near infinity. For example, the function $1/(1+|x|)^2$ decays fast enough at infinity so that it belongs to $L^1(\mathbb{R})$. On the other hand, the function $1/(1+|x|)$ does not decay fast enough so that it would belong to $L^1(\mathbb{R})$. A second reason is that the function may blow up at a given point. Typical example is the function that equals $1/|x|$ when $|x| \leqslant 1$ and 0 otherwise. It blows up near $x = 0$ too fast in order to belong to $L^1(\mathbb{R})$. On the other hand, the function which agrees with $1/\sqrt{|x|}$ when $|x| \leqslant 1$ and is 0 otherwise, also bows up near $x = 0$. However, $1/\sqrt{|x|}$ is integrable on $|x| \leqslant 1$ and thus it belongs to $L^1(\mathbb{R})$. The borderline case for $L^1(\mathbb{R})$, both at infinity and close to a point is the function $1/|x|$. This function does not decay fast enough at infinity in order to belong to $L^1(\mathbb{R})$. At the same time it blows up too fast near $x = 0$ in order to belong to $L^1(\mathbb{R})$.

3.2 The Fourier transform*

Let $f:\mathbb{R}^n\to\mathbb{C}$ be a complex valued integrable function, that is $f\in L^1(\mathbb{R}^n)$. The Fourier transform $\widehat{f}:\mathbb{R}^n\to\mathbb{C}$ of f is defined as

$$\widehat{f}(\xi)=\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx,$$

where $x\cdot\xi=x_1\xi_1+\dots+x_n\xi_n$ is the standard inner product in $\mathbb{R}^n$.

T H E M O R A L : The Fourier transform is a replacement of the Fourier coefficients for a function defined on the whole space.

Remark 3.3. There are several alternative definitions for the Fourier transform in the literature, for example,

$$\int_{\mathbb{R}^n} f(x)e^{ix\cdot\xi}\,dx,\quad \int_{\mathbb{R}^n} f(x)e^{-2\pi ix\cdot\xi}\,dx\quad\text{and}\quad(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx.$$

There is an analogous theory for these definitions and, as we shall see, the factor 2π appears somewhere in each of these choices.

We gather many useful properties of the Fourier transform in the following proposition. Note that there are many similar properties for the Fourier coefficients of a function in $L^1([-\pi,\pi])$, compare to Lemma 2.20.

Lemma 3.4. Let $f,g\in L^1(\mathbb{R}^n)$.

(1) (Linearity) $\widehat{af+bg}(\xi)=a\widehat{f}(\xi)+b\widehat{g}(\xi)$, $a,b\in\mathbb{C}$.

(2) (Boundedness) For every $\xi\in\mathbb{R}^n$ we have

$$|\widehat{f}(\xi)|\leqslant\int_{\mathbb{R}^n}|f(x)|\,dx=\|f\|_{L^1(\mathbb{R}^n)}.$$

This implies $\|\widehat{f}\|_{L^\infty(\mathbb{R}^n)}\leqslant\|f\|_{L^1(\mathbb{R}^n)}$.

(3) $\widehat{f}(0)=\displaystyle\int_{\mathbb{R}^n} f(x)\,dx$.

(4) (Continuity) If $f\in L^1(\mathbb{R}^n)$, then $\widehat{f}$ is continuous.

(5) (Dilation) Let $f_a(x)=\frac{1}{a^n}f\left(\frac{x}{a}\right)$, $a>0$. Then $\widehat{f_a}(\xi)=\widehat{f}(a\xi)$.

(6) (Translation) For $y\in\mathbb{R}^n$ we have $\widehat{f(x+y)}(\xi)=e^{iy\cdot\xi}\widehat{f}(\xi)$.

(7) (Modulation) For $\eta\in\mathbb{R}^n$ we have $\widehat{e^{ix\cdot\eta}f(x)}(\xi)=\widehat{f}(\xi-\eta)$.

Proof. $\boxed{(1)}$ Follows from linearity of integral.

$\boxed{(2)}$ $|\widehat{f}(\xi)|=\left|\displaystyle\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx\right|\leqslant\int_{\mathbb{R}^n}\left|f(x)e^{-ix\cdot\xi}\right|dx=\int_{\mathbb{R}^n}|f(x)|\,dx.$

$\boxed{(3)}$ Follows immediately by inserting $\xi=0$ in the definition of the Fourier transform.

$\boxed{(4)}$ **Claim:** $\xi_k\to\xi$ implies $\widehat{f}(\xi_k)\to\widehat{f}(\xi)$ as $k\to\infty$.

Reason. Since the complex exponential function is continuous, we have

$$\lim_{k\to\infty} f(x)e^{-ix\cdot\xi_k} = f(x)e^{-ix\cdot\xi}.$$

This implies that

$$\begin{aligned}\lim_{k\to\infty}\widehat{f}(\xi_k) &= \lim_{k\to\infty}\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi_k}\,dx\\ &= \int_{\mathbb{R}^n}\lim_{k\to\infty} f(x)e^{-ix\cdot\xi_k}\,dx\\ &= \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx = \widehat{f}(\xi).\end{aligned}$$

The fact that we can switch the order of the integral and the limit follows from the Lebesgue dominated convergence theorem, since $|f(x)e^{-ix\cdot\xi_k}| = |f(x)| \in L^1(\mathbb{R}^n)$ for every $k = 1,2,\dots$. ■

$\boxed{(5)}$

$$\widehat{f_a}(\xi) = \int_{\mathbb{R}^n}\frac{1}{a^n}f\left(\frac{x}{a}\right)e^{-ix\cdot\xi}\,dx = \int_{\mathbb{R}^n}\frac{1}{a^n}f(y)e^{-iay\cdot\xi}a^n\,dy = \widehat{f}(a\xi).$$

Here we made a change of variables $y = \frac{x}{a}$, which implies that $dx = a^n\,dy$.

$\boxed{(6)}$ By a change of variable $z = x+y$, we have

$$\begin{aligned}\widehat{g}(\xi) = \widehat{f(x+y)}(\xi) &= \int_{\mathbb{R}^n} f(x+y)e^{-ix\cdot\xi}\,dx\\ &= \int_{\mathbb{R}^n} f(z)e^{-i(z-y)\cdot\xi}\,dz = \int_{\mathbb{R}^n} f(z)e^{-iz\cdot\xi+iy\cdot\xi}\,dz\\ &= e^{iy\cdot\xi}\int_{\mathbb{R}^n} f(z)e^{-iz\cdot\xi}\,dz = e^{iy\cdot\xi}\widehat{f}(\xi).\end{aligned}$$

$\boxed{(7)}$

$$\begin{aligned}\widehat{e^{ix\cdot\eta}f(x)}(\xi) &= \int_{\mathbb{R}^n} e^{ix\cdot\eta}f(x)e^{-ix\cdot\xi}\,dx = \int_{\mathbb{R}^n} f(x)e^{ix\cdot\eta-ix\cdot\xi}\,dx\\ &= \int_{\mathbb{R}^n} f(x)e^{-ix\cdot(\xi-\eta)}\,dx = \widehat{f}(\xi-\eta).\end{aligned}$$ □

3.3 The Fourier transform and differentiation*

The following proposition shows how the Fourier transform interacts with derivatives. This turns out to be extremely useful as in many cases the Fourier transform transforms a PDE to an ODE.

Theorem 3.5. Let $f \in C^1(\mathbb{R}^n)$ and assume that $\frac{\partial f}{\partial x_j} \in L^1(\mathbb{R}^n)$, $j = 1, 2, \dots, n$. We also assume that $\lim_{|x|\to\infty} f(x) = 0$. Then

$$\widehat{\frac{\partial f}{\partial x_j}}(\xi) = i\xi_j \widehat{f}(\xi), \quad j = 1, 2, \dots, n.$$

THE MORAL: Differentiation is multiplication on the Fourier side. This is the main reason why the Fourier transform is useful in the PDE theory. The smoothness of f is reflected to the decay of $\widehat{f}(\xi)$ as $|\xi| \to \infty$. Indeed, if f is very smooth and several integrations by parts are allowed, then the theorem above shows that powers of $|\xi|$ multiplied by $\widehat{f}$ are bounded functions, because the Fourier transform on the left-hand side is bounded.

Proof. Since $\lim_{|x|\to\infty} f(x) = 0$, an integration by parts gives

$$\begin{aligned}
\widehat{\frac{\partial f}{\partial x_j}}(\xi) &= \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x) e^{-ix\cdot\xi}\, dx \\
&= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x_j}(x) e^{-ix\cdot\xi}\, dx_j \right) dx_1 \dots dx_{j-1}\, dx_{j+1} \dots dx_n \\
&= \int_{\mathbb{R}^{n-1}} \left(\lim_{a\to\infty} \Big|_{x_j=-a}^{a} f(x) e^{-ix\cdot\xi} \right) dx_1 \dots dx_{j-1}\, dx_{j+1} \dots dx_n \\
&\qquad - \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} f(x) \frac{\partial}{\partial x_j}(e^{-ix\cdot\xi})\, dx_j \right) dx_1 \dots dx_{j-1}\, dx_{j+1} \dots dx_n \\
&= -\int_{\mathbb{R}^n} f(x)(-i\xi_j) e^{-ix\cdot\xi}\, dx = i\xi_j \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx = i\xi_j \widehat{f}(\xi).
\end{aligned}$$

□

Remarks 3.6:

(1) In particular, the previous theorem applies to a compactly supported smooth function $f \in C_0^\infty(\mathbb{R}^n)$. Recall, that a function is compactly supported, if it is zero outside a compact (closed and bounded) set.

(2) A vector of the form $\alpha = (\alpha_1, \dots, \alpha_n)$, where each component α_j is a non-negative integer, is called a multi-index of order $|\alpha| = \alpha_1 + \dots + \alpha_n$. For a multi-index α, we define

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}(x)$$

Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ and denote $\lambda^\alpha = \lambda_1^{\alpha_1} \dots \lambda_n^{\alpha_n}$. The function $e_\lambda : \mathbb{R}^n \to \mathbb{C}$,

$$e_\lambda(x) = e^{x\cdot\lambda} = e^{x_1\lambda_1 + \dots + x_n\lambda_n},$$

belongs to $C^\infty(\mathbb{R}^n)$ and

$$D^\alpha e_\lambda = \lambda^\alpha e_\lambda.$$

Consequently, every linear partial differential operator with constant coefficients

$$P = P(D) = \sum_{|\alpha|\le k} a_\alpha D^\alpha : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$$

we have

$$P(D)e_\lambda = \sum_{|\alpha|\leqslant k} a_\alpha \lambda^\alpha e_\lambda.$$

In other words, e_λ is an eigenvector corresponding the eigenvalue λ^α for every operator $P(D)$. Note that the PDE related to the operator P is

$$P(D)u = \sum_{|\alpha|\leqslant k} a_\alpha D^\alpha u = 0.$$

The idea behind the Fourier transform is that the partial differential operator P can be better understood if the functions on which they act are represented as linear combinations of the eigenvectors. Observe that the set of eigenvalues is the whole $\mathbb{C}^n$ and for this reason it is natural to replace the linear combinations by integrals over λ. Indeed, P acts as a scalar multiplication on each eigenvector and the scalar is the eigenvalue corresponding the eigenvector. In practice this means that it is better to consider the operator P on the Fourier side.

The following result deals with the case that the derivatives are on the Fourier side.

Theorem 3.7. Suppose that $f \in L^1(\mathbb{R}^n)$ is such that the function $-ix_j f(x) \in L^1(\mathbb{R}^n)$. Then $\widehat{f}$ is differentiable and

$$\frac{\partial \widehat{f}}{\partial \xi_j}(\xi) = -\widehat{ix_j f(x)}(\xi), \quad j = 1,2,\dots,n.$$

Proof. By the chain rule

$$\frac{\partial}{\partial \xi_j}(e^{-ix\cdot\xi}) = e^{-ix\cdot\xi}\frac{\partial}{\partial \xi_j}(-ix\cdot\xi) = -ix_j e^{-ix\cdot\xi}.$$

Since $-ix_j f(x) \in L^1(\mathbb{R}^n)$ we can calculate the Fourier transform as

$$\begin{aligned}
\widehat{-ix_j f(x)}(\xi) &= \int_{\mathbb{R}^n} -ix_j f(x) e^{-ix\cdot\xi}\,dx \\
&= \int_{\mathbb{R}^n} f(x)\frac{\partial}{\partial \xi_j}(e^{-ix\cdot\xi})\,dx \\
&= \frac{\partial}{\partial \xi_j}\left(\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx\right) = \frac{\partial \widehat{f}}{\partial \xi_j}(\xi).
\end{aligned}$$

The order of the limit and the integral can be changed by the Lebesgue dominated convergence theorem. ☐

THE MORAL: The decay of $f(x)$ as $|x| \to \infty$ is reflected to the smoothness of $\widehat{f}$. Indeed, if f decays rapidly as $|x| \to \infty$, then large powers of $|x|$ multiplied by f are integrable and an iteration of the theorem above shows that $\widehat{f}$ can be differentiated several times.

Example 3.8. Let $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \mathbf{1}_{(-a,a)}(x) = \begin{cases} 1, & |x| < a, \\ 0, & |x| \geqslant a, \end{cases}$$

that is, f is the indicator function of the interval $(-a,a)$. Then the Fourier transform of f is

$$\widehat{f}(\xi) = \int_{-a}^{a} f(x)e^{-ix\xi}\,dx = \bigg|_{-a}^{a} \frac{e^{-ix\xi}}{-i\xi} = \frac{e^{i\xi a} - e^{-i\xi a}}{i\xi} = \frac{2\sin(a\xi)}{\xi}, \quad \xi \neq 0.$$

Moreover

$$\widehat{f}(0) = \int_{\mathbb{R}} \mathbf{1}_{(-a,a)}(x)\,dx = 2a.$$

Observe that $\lim_{|\xi|\to\infty} \widehat{f}(\xi) = 0$ and $\widehat{f} \in L^2(\mathbb{R})$, but $\widehat{f} \notin L^1(\mathbb{R})$. The function f has a compact support and thus $\widehat{f} \in C^\infty(\mathbb{R})$, that is, it has derivatives of every order. This is in accordance with the previous theorem.

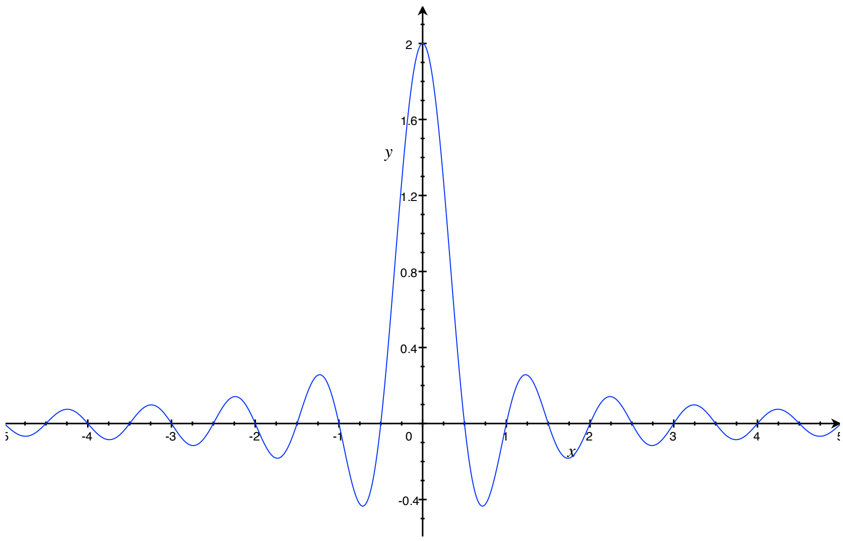


Figure 3.1: The graph of the Fourier transform $\widehat{\mathbf{1}_{(-1,1)}}$.

WARNING: This example shows that $f \in L^1(\mathbb{R})$ does not imply that $\widehat{f} \in L^1(\mathbb{R})$.

Example 3.9. Let $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & x \leqslant 0, \end{cases}$$

and $g : \mathbb{R} \to \mathbb{R}$,

$$g(x) = \begin{cases} xe^{-x}, & x > 0, \\ 0, & x \leqslant 0. \end{cases}$$

Then

$$\begin{aligned}\widehat{f}(\xi) &= \int_0^\infty e^{-x(1+i\xi)}\,dx = -\lim_{a\to\infty}\left|_0^a \frac{e^{-x(1+i\xi)}}{1+i\xi}\right. \\ &= -\lim_{a\to\infty}\frac{e^{-a(1+i\xi)}-1}{1+i\xi} = \frac{1}{1+i\xi}.\end{aligned}$$

THE MORAL: The Fourier transform of a real valued function may be a complex valued function.

On the other hand,

$$\begin{aligned}\widehat{g}(\xi) &= \widehat{xf(x)}(\xi) = \frac{1}{-i}\widehat{-ixf(x)}(\xi) = \frac{1}{-i}\frac{\partial\widehat{f}}{\partial\xi}(\xi) \\ &= \frac{1}{-i}\frac{\partial}{\partial\xi}\left(\frac{1}{1+i\xi}\right) = \frac{1}{-i}\frac{-i}{(1+i\xi)^2} = \frac{1}{(1+i\xi)^2}.\end{aligned}$$

Here we used Theorem 3.7, but a direct computation works as well.

Example 3.10. Let $f:\mathbb{R}\to\mathbb{R}$, $f(x)=e^{-|x|}$. Then

$$\widehat{f}(\xi) = \int_{-\infty}^0 e^x e^{-ix\xi}\,dx + \int_0^\infty e^{-x}e^{-ix\xi}\,dx = \frac{1}{1-i\xi}+\frac{1}{1+i\xi} = \frac{2}{1+\xi^2}.$$

3.4 The Fourier transform of the Gaussian*

Next we calculate the Fourier transform of the Gaussian function. This will be very useful for us later.

Example 3.11. Let $f:\mathbb{R}\to\mathbb{R}$, $f(x)=e^{-x^2}$. Then

$$f'(x) = -2xe^{-x^2} = -2xf(x)\in L^1(\mathbb{R})$$

and

$$\widehat{f'}(\xi) = -\widehat{2xf(x)}(\xi) = 2i\,\widehat{ixf(x)}(\xi).$$

Theorem 3.7 and Theorem 3.5 imply

$$\frac{\partial\widehat{f}}{\partial\xi}(\xi) = -\widehat{ixf(x)}(\xi) = -\frac{1}{2i}\widehat{f'}(\xi) = -\frac{1}{2i}i\xi\widehat{f}(\xi) = -\frac{\xi}{2}\widehat{f}(\xi).$$

Thus $\widehat{f}$ satisfies the ODE

$$\frac{\partial\widehat{f}}{\partial\xi}(\xi)+\frac{\xi}{2}\widehat{f}(\xi)=0 \iff \frac{\partial}{\partial\xi}\left(\widehat{f}(\xi)e^{\xi^2/4}\right)=0 \iff \widehat{f}(\xi)=ce^{-\xi^2/4}$$

for some constant c. In order to specify the constant, we observe that

$$c = \widehat{f}(0) = \int_{\mathbb{R}} e^{-x^2}\,dx.$$

Claim: $\int_{\mathbb{R}} e^{-x^2}\,dx = \sqrt{\pi}$.

Reason. An integration in the polar coordinates gives

$$\begin{aligned}\left(\int_{\mathbb{R}} e^{-x^2}\,dx\right)^2 &= \left(\int_{\mathbb{R}} e^{-x^2}\,dx\right)\left(\int_{\mathbb{R}} e^{-y^2}\,dy\right) = \int_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy \\ &= \int_0^\infty \int_{\partial B(0,r)} e^{-r^2}\,dS\,dr = \int_0^\infty e^{-r^2} \underbrace{\int_{\partial B(0,r)} 1\,dS}_{=2\pi r}\,dr \\ &= 2\pi \int_0^\infty r e^{-r^2}\,dr = 2\pi \lim_{a\to\infty} -\frac{1}{2}\Big|_0^a e^{-r^2} = \pi.\end{aligned}$$

■

Thus $c = \widehat{f}(0) = \sqrt{\pi}$ and

$$\widehat{f}(\xi) = \sqrt{\pi} e^{-\xi^2/4}.$$

Remark 3.12. By Lemma 3.4 (5), we have

$$\widehat{f\left(\frac{x}{\sqrt{2}}\right)}(\xi) = \sqrt{2}\widehat{f}(\sqrt{2}\xi),$$

which implies that

$$\widehat{e^{-x^2/2}}(\xi) = \sqrt{2\pi}\, e^{-\xi^2/2}.$$

This shows that $g(x) = e^{-x^2/2}$ is an eigenfunction for the Fourier transform corresponding to the eigenvalue $\sqrt{2\pi}$, that is,

$$\widehat{g} = \sqrt{2\pi} g.$$

Then we consider a higher dimensional version of the previous example.

Example 3.13. Let $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = e^{-|x|^2}$. Then

$$\begin{aligned}\widehat{f}(\xi) &= \int_{\mathbb{R}^n} e^{-|x|^2} e^{-ix\cdot\xi}\,dx = \int_{\mathbb{R}^n} e^{-x\cdot x - ix\cdot\xi}\,dx = \int_{\mathbb{R}^n} e^{\sum_{j=1}^n(-x_j^2 - ix_j\xi_j)}\,dx \\ &= \int_{\mathbb{R}^n} \prod_{j=1}^n e^{-x_j^2 - ix_j\xi_j}\,dx = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{j=1}^n e^{-x_j^2 - ix_j\xi_j}\,dx_1 \cdots dx_n \\ &= \prod_{j=1}^n \int_{\mathbb{R}} e^{-x_j^2 - ix_j\xi_j}\,dx_j = \prod_{j=1}^n \int_{\mathbb{R}} e^{-x_j^2} e^{-ix_j\xi_j}\,dx_j \\ &= \prod_{j=1}^n \widehat{e^{-x_j^2}}(\xi_j) = \prod_{j=1}^n \sqrt{\pi} e^{-\xi_j^2/4} = (\sqrt{\pi})^n e^{-\sum_{j=1}^n \xi_j^2/4} = \pi^{\frac{n}{2}} e^{-|\xi|^2/4}\end{aligned}$$

by the one-dimensional result in the previous example.

3.5 The Fourier inversion formula*

The problem we consider in this section is the following: If we are given $\widehat{f}$ can we determine f? The Fourier inversion theorem will state that, under certain assumptions, we have

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{ix\cdot\xi}\,d\xi.$$

Note a beautiful analogy to the definition of the Fourier transform. This is a deep result and it is instructive to see what happens if we try to prove directly by substituting the formula for $\widehat{f}(\xi)$ into the integral above. If we do this, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f(y)e^{-iy\cdot\xi}\,dy\right)e^{ix\cdot\xi}\,d\xi\\ &= \int_{\mathbb{R}^n} f(y)\underbrace{\left(\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\,d\xi\right)}_{=?}\,dy.\end{aligned}$$

This does not work out, because the inner integral does not exist, that is, the function is not integrable. Thus we have to choose another approach.

Theorem 3.14 (Fourier inversion theorem). Let $f\in L^1(\mathbb{R}^n)$ be a bounded and continuous function and assume that $\widehat{f}\in L^1(\mathbb{R}^n)$. Then

$$f(x) = (2\pi)^{-n}\int_{\mathbb{R}^n}\widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi$$

for all $x\in\mathbb{R}^n$.

THE MORAL: A function can be recovered from its Fourier transform. This corresponds to the Fourier series representation of a periodic function.

Proof. **Step 1:** Let $K:\mathbb{R}^n\to\mathbb{R}$, $K(x)=\pi^{-\frac{n}{2}}e^{-|x|^2}$ and

$$K_a(x) = \frac{1}{a^n}K\left(\frac{x}{a}\right) = \pi^{-\frac{n}{2}}a^{-n}e^{-\frac{|x|^2}{a^2}},\quad a>0.$$

Then

$$\int_{\mathbb{R}^n} K(x)\,dx = \widehat{K}(0) = \pi^{-\frac{n}{2}}\pi^{\frac{n}{2}} = 1,$$

since $\widehat{K}(\xi)=e^{-\frac{|\xi|^2}{4}}$ by Example 3.13.

Claim: $\int_{\mathbb{R}^n} K_a(x)\,dx = 1$ for all $a>0$.

Reason. By the change of variables $y=\frac{x}{a}$, $dx=a^n\,dy$, we have

$$\int_{\mathbb{R}^n} K_a(x)\,dx = \int_{\mathbb{R}^n}\frac{1}{a^n}K\left(\frac{x}{a}\right)dx = \int_{\mathbb{R}^n}\frac{1}{a^n}K(y)a^n\,dy = 1.$$

■

Claim: For every $r>0$, we have $\int_{\{|x|\geqslant r\}} K_a(x)\,dx\to 0$ as $a\to 0$.

Reason.

$$\int_{\{|x|\geqslant r\}} K_a(x)\,dx = \int_{\{|x|\geqslant r\}}\frac{1}{a^n}K\left(\frac{x}{a}\right)dx = \int_{\{|y|\geqslant\frac{r}{a}\}}\frac{1}{a^n}K(y)a^n\,dy\to 0,$$

as $a\to 0$. Observe that

$$\begin{aligned}\lim_{a\to 0}\int_{\{|y|\geqslant\frac{r}{a}\}}\frac{1}{a^n}K(y)a^n\,dy &= \lim_{a\to 0}\int_{\mathbb{R}^n}K(y)\mathbf{1}_{\{|y|\geqslant\frac{r}{a}\}}(y)\,dy\\ &= \int_{\mathbb{R}^n}K(y)\lim_{a\to 0}\mathbf{1}_{\{|y|\geqslant\frac{r}{a}\}}(y)\,dy = 0.\end{aligned}$$

Here the order of the limes and the integral can be switched by the Lebesgue dominated convergence theorem. ■

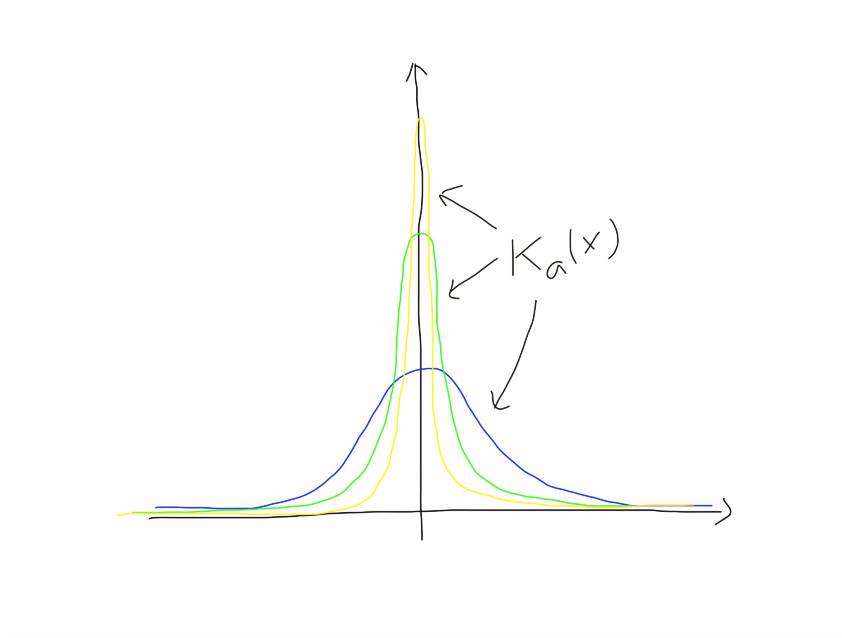


Figure 3.2: Kernel $K_a(x)$ with different values of parameter a.

Claim: $\int_{\mathbb{R}^n} f(x)K_a(x)\,dx \to f(0)$ as $a \to 0$.

Reason. Since $f \in C(\mathbb{R}^n)$, for every $\varepsilon > 0$ there exists $r > 0$ such that

$$|f(x) - f(0)| < \frac{\varepsilon}{2}$$

whenever $|x - 0| < r$. This implies

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} f(x)K_a(x)\,dx - f(0)\right| = \left|\int_{\mathbb{R}^n} f(x)K_a(x)\,dx - f(0)\underbrace{\int_{\mathbb{R}^n} K_a(x)\,dx}_{=1}\right| \\
&= \left|\int_{\mathbb{R}^n} K_a(x)(f(x) - f(0))\,dx\right| \\
&\leqslant \int_{|x|<r} K_a(x)|f(x) - f(0)|\,dx + \int_{\{|x|\geqslant r\}} K_a(x)|f(x) - f(0)|\,dx \\
&\leqslant \frac{\varepsilon}{2}\underbrace{\int_{\mathbb{R}^n} K_a(x)\,dx}_{=1} + \underbrace{2\sup_{x\in\mathbb{R}^n}|f(x)|\int_{\{|x|\geqslant r\}} K_a(x)\,dx}_{<\frac{\varepsilon}{2}} < \varepsilon
\end{aligned}$$

by choosing $a > 0$ small enough. Here we used the previous claim. ∎

Step 2: By changing the order of integration

$$\begin{aligned}\int_{\mathbb{R}^n} \widehat{f}(\xi)g(\xi)\,d\xi &= \int_{\mathbb{R}^n} g(\xi)\left(\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx\right)d\xi\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(\xi)e^{-ix\cdot\xi}\,d\xi\,dx\\ &= \int_{\mathbb{R}^n} f(x)\left(\int_{\mathbb{R}^n} g(\xi)e^{-i\xi\cdot x}\,d\xi\right)dx\\ &= \int_{\mathbb{R}^n} f(x)\widehat{g}(x)\,dx.\end{aligned}$$

Let $g_a : \mathbb{R}^n \to \mathbb{R}^n$,

$$g_a(x) = (2\pi)^{-n} e^{-\frac{|ax|^2}{4}}, \quad a > 0.$$

Then

$$\begin{aligned}\widehat{g_a}(\xi) &= (2\pi)^{-n}\widehat{e^{-\left|\frac{ax}{2}\right|^2}}(\xi) = (2\pi)^{-n}\left(\frac{2}{a}\right)^n \widehat{e^{-|x|^2}}\left(\frac{2}{a}\xi\right)\\ &= (2\pi)^{-n}\left(\frac{2}{a}\right)^n \pi^{\frac{n}{2}} e^{-\frac{1}{4}\left|\frac{2\xi}{a}\right|^2} = \pi^{-\frac{n}{2}} a^{-n} e^{-\left|\frac{\xi}{a}\right|^2} = K_a(\xi).\end{aligned}$$

In the second equality we used Lemma 3.4 (5) and in the third equality Example 3.13.

Step 3:

$$\begin{aligned} f(0) &= \lim_{a\to 0}\int_{\mathbb{R}^n} f(x)K_a(x)\,dx \quad \text{(Step 1)}\\ &= \lim_{a\to 0}\int_{\mathbb{R}^n} f(x)\widehat{g}_a(x)\,dx \quad \text{(Step 2)}\\ &= \lim_{a\to 0}\int_{\mathbb{R}^n} \widehat{f}(\xi)g_a(\xi)\,d\xi \quad \text{(Step 2)}\\ &= \int_{\mathbb{R}^n} \widehat{f}(\xi)\underbrace{\lim_{a\to 0} g_a(\xi)}_{=(2\pi)^{-n}}\,d\xi\\ &= (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{f}(\xi)\,d\xi.\end{aligned}$$

This proves the claim for $x = 0$. The general case follows by denoting $F(y) = f(x+y)$. Then

$$\begin{aligned} f(x) = F(0) &= (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{F}(\xi)\,d\xi\\ &= (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{f(x+y)}(\xi)\,d\xi\\ &= (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi.\end{aligned}$$

The last equality follows from Lemma 3.4 (6). □

Remark 3.15. The previous theorem holds true under the substantially weaker hypothesis that $f \in L^1(\mathbb{R}^n)$ and $\widehat{f} \in L^1(\mathbb{R}^n)$. This is not so surprising since the

Fourier inversion formula can be written as

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{-i(-x)\cdot\xi}\, d\xi = (2\pi)^{-n} \widehat{\widehat{f}}(-x).$$

Thus the function f is given by a Fourier transform of the function $\widehat{f} \in L^1(\mathbb{R}^n)$ and it is automatically continuous and bounded.

As a corollary we get a uniqueness result for Fourier transforms. Namely, if two functions $f, g \in L^1(\mathbb{R}^n)$ have the same Fourier transform, then they must be the same function.

Theorem 3.16. Let $f, g \in L^1(\mathbb{R}^n)$ be continuous functions. If $\widehat{f}(\xi) = \widehat{g}(\xi)$ for every $\xi \in \mathbb{R}^n$ then $f(x) = g(x)$ for every $x \in \mathbb{R}^n$.

THE MORAL: A function is uniquely defined by its Fourier transform.

Proof. We note that $\widehat{f-g} = \widehat{f} - \widehat{g} = 0$, which implies $\widehat{f-g} \in L^1(\mathbb{R}^n)$. By the Fourier inversion theorem, see Theorem 3.14, we have

$$\begin{aligned} f(x) - g(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{(f-g)}(\xi) e^{ix\cdot\xi}\, d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \underbrace{(\widehat{f}(\xi) - \widehat{g}(\xi))}_{=0} e^{ix\cdot\xi}\, d\xi = 0 \end{aligned}$$

for every $x \in \mathbb{R}^n$ and we are done. □

3.6 The Fourier transformation and convolution

Let $f, g : \mathbb{R}^n \to \mathbb{C}$. The convolution $f * g : \mathbb{R}^n \to \mathbb{C}$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y)\, dy,$$

whenever this integral exists.

THE MORAL: The convolution on $\mathbb{R}^n$ has a similar role in representation formulas for solutions of PDEs as in the one-dimensional case.

This operation is commutative $f * g = g * f$ and associative $(f * g) * h = f * (g * h)$. To prove commutativity, fix $x \in \mathbb{R}^n$ and make the change of variables $z = x - y$, which implies $y = x - z$ and $dy = dz$. This implies

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y)\, dy = \int_{\mathbb{R}^n} f(z) g(x-z)\, dz = (g * f)(x).$$

We leave it as an exercise to show that $(f * g) * h = f * (g * h)$.

Then we consider the question, that for which functions the definition makes sense. Assume that $f, g \in L^1(\mathbb{R}^n)$. Then

$$\begin{aligned}\int_{\mathbb{R}^n} |(f * g)(x)|\,dx &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y)g(y)\,dy \right| dx \\ &\leqslant \int_{\mathbb{R}^n} |g(y)| \left(\int_{\mathbb{R}^n} |f(x-y)|\,dx \right) dy \\ &= \int_{\mathbb{R}^n} |g(y)| \left(\int_{\mathbb{R}^n} |f(z)|\,dz \right) dy \\ &= \int_{\mathbb{R}^n} |f(z)|\,dz \int_{\mathbb{R}^n} |g(y)|\,dy,\end{aligned}$$

from which it follows that

$$\|f * g\|_{L^1(\mathbb{R}^n)} \leqslant \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)} < \infty.$$

In particular, this implies that $f * g \in L^1(\mathbb{R}^n)$.

WARNING: Note that this is not obvious. In general, a product of two integrable function is not necessarily integrable.

Theorem 3.17. Assume that $f, g \in L^1(\mathbb{R}^n)$. Then

$$\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$$

for every $\xi \in \mathbb{R}^n$.

THE MORAL: Convolution becomes the standard multiplication on the Fourier side. This will be very useful when we derive representation formulas for solutions to PDE and this is one motivation for the definition of a convolution.

Proof. Since

$$\begin{aligned}(f * g)(x)e^{-ix\cdot\xi} &= \int_{\mathbb{R}^n} f(y)g(x-y)e^{-ix\cdot\xi}\,dy \\ &= \int_{\mathbb{R}^n} f(y)e^{-iy\cdot\xi} g(x-y)e^{-i(x-y)\cdot\xi}\,dy,\end{aligned}$$

we have

$$\begin{aligned}\widehat{f * g}(\xi) &= \int_{\mathbb{R}^n} (f * g)(x)e^{-ix\cdot\xi}\,dx \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y)e^{-iy\cdot\xi} g(x-y)e^{-i(x-y)\cdot\xi}\,dy\,dx \\ &= \int_{\mathbb{R}^n} f(y)e^{-iy\cdot\xi} \left(\int_{\mathbb{R}^n} g(x-y)e^{-i(x-y)\cdot\xi}\,dx \right) dy \\ &= \int_{\mathbb{R}^n} f(y)e^{-iy\cdot\xi} \underbrace{\left(\int_{\mathbb{R}^n} g(z)e^{-iz\cdot\xi}\,dz \right)}_{=\widehat{g}(\xi)} dy = \widehat{f}(\xi)\widehat{g}(\xi).\end{aligned}$$

□

Next we present a corresponding result for the Fourier transform of the product. This can be seen as a dual result of Theorem 3.17.

Theorem 3.18. Assume that $f, g, \widehat{f}, \widehat{g} \in L^1(\mathbb{R}^n)$. Then

$$\widehat{fg}(\xi) = (2\pi)^{-n}(\widehat{f} * \widehat{g})(\xi)$$

for every $\xi \in \mathbb{R}^n$.

THE MORAL: Product becomes convolution on the Fourier side.

Proof. By the Fourier inversion theorem

$$\begin{aligned}
\widehat{fg}(\xi) &= \int_{\mathbb{R}^n} f(x)g(x)e^{-ix\cdot\xi}\,dx \\
&= \int_{\mathbb{R}^n} \left((2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{f}(\eta)e^{ix\cdot\eta}\,d\eta\right) g(x)e^{-ix\cdot\xi}\,dx \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{f}(\eta)g(x)e^{-ix\cdot(\xi-\eta)}\,dx\,d\eta \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{f}(\eta)\underbrace{\left(\int_{\mathbb{R}^n} g(x)e^{-ix\cdot(\xi-\eta)}\,dx\right)}_{=\widehat{g}(\xi-\eta)} d\eta \\
&= (2\pi)^{-n}(\widehat{f} * \widehat{g})(\xi).
\end{aligned}$$

□

3.7 Plancherel's formula*

Assume that $f \in L^1(\mathbb{R}^n)$ is such a function that $\widehat{f} \in L^1(\mathbb{R}^n)$ and denote $f^\#(x) = \overline{f(-x)}$. Then

$$\widehat{f^\#}(\xi) = \overline{\widehat{f}(\xi)}.$$

Define $g(x) = (f * f^\#)(x)$. By Theorem 3.17 we have

$$\widehat{g}(\xi) = \widehat{f * f^\#}(\xi) = \widehat{f}(\xi)\widehat{f^\#}(\xi) = \widehat{f}(\xi)\overline{\widehat{f}(\xi)} = |\widehat{f}(\xi)|^2.$$

On the other hand

$$g(0) = \int_{\mathbb{R}^n} f(y)f^\#(0-y)\,dy = \int_{\mathbb{R}^n} f(y)\overline{f(y)}\,dy = \int_{\mathbb{R}^n} |f(y)|^2\,dy$$

and the Fourier inversion formula implies

$$\int_{\mathbb{R}^n} |f(y)|^2\,dy = g(0) = (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{g}(\xi)\,d\xi = (2\pi)^{-n}\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2\,d\xi$$

THE MORAL: The $L^2(\mathbb{R}^n)$ norm of the Fourier transform is the same as the $L^2(\mathbb{R}^n)$ norm of the function up to a multiplicative constant. This can be used to define the Fourier transform of a function an $L^2(\mathbb{R}^n)$ function, but this is out of the scope of this course. The factor $(2\pi)^{-n}$ appears with our definition for the Fourier transform, but there are different scalings in the literature.

3.8 Approximations of the identity in $\mathbb{R}^n$*

In this section we study approximations of the identity in the higher dimensional case.

Definition 3.19. Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of functions $K_\varepsilon : \mathbb{R}^n \to \mathbb{C}$, with the following properties.

(1) For every $\varepsilon > 0$ we have $\int_{\mathbb{R}^n} K_\varepsilon(x)\,dx = 1$.

(2) There exists some constant $M > 0$ such that, for every $\varepsilon > 0$ we have

$$\int_{\mathbb{R}^n} |K_\varepsilon(x)|\,dx \leqslant M.$$

(3) For every $\delta > 0$, we have

$$\lim_{\varepsilon\to 0}\int_{\{|x|>\delta\}} |K_\varepsilon(x)|\,dx = 0.$$

Then the family $\{K_\varepsilon\}_{\varepsilon>0}$ will be called a family of good kernels.

These families of good kernels play the same role as the good kernels in the 2π-periodic case, see Section 2.14. In fact, we shall consider the analogous theory in the higher dimensional case. The importance of good kernels is contained in the following result.

Theorem 3.20. Let $f \in L^1(\mathbb{R}^n)$ be bounded and continuous at $x \in \mathbb{R}^n$ and let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of good kernels. Then

$$\lim_{\varepsilon\to 0}(K_\varepsilon * f)(x) = f(x).$$

If $f \in C_0(\mathbb{R}^n)$ (a compactly supported continuous function) then $K_\varepsilon * f \to f$ uniformly as $\varepsilon \to 0$.

THE MORAL: Approximations of the identity give an interpretation that the convolution $K_\varepsilon * f$ can be seen as a weighted integral average. The pointwise value of a function is replaced with an integral average, which converges to the value of the function as $\varepsilon \to 0$.

Proof. The proof is similar to the one in the 2π-periodic case. Let $\eta > 0$. By the continuity of f at x there exists $\delta > 0$ such that

$$|f(x-y) - f(x)| < \frac{\eta}{2M} \quad \text{when} \quad |y| < \delta, \tag{3.1}$$

where M is from property (2) of a good kernel.

For $x \in \mathbb{R}^n$ we use property (1) of the family of good kernels to have

$$\begin{aligned}|(K_\varepsilon * f)(x) - f(x)| &= \left| \int_{\mathbb{R}^n} K_\varepsilon(y) f(x-y)dy - f(x)\left(\int_{\mathbb{R}^n} K_\varepsilon(y)\,dy\right)\right| \\ &\leqslant \int_{\{|y|<\delta\}} |f(x-y) - f(x)||K_\varepsilon(y)|\,dy \\ &\quad + \int_{\{|y|>\delta\}} |f(x-y) - f(x)||K_\varepsilon(y)|\,dy \\ &= I_1 + I_2.\end{aligned}$$

For I_1 we use (3.1) to obtain

$$|I_1| \leqslant \frac{\eta}{2M} \int_{\mathbb{R}^n} |K_\varepsilon(y)|\,dy \leqslant \frac{\eta}{2}.$$

For I_2 we choose ε small enough so that

$$\int_{\{|y|>\delta\}} |K_\varepsilon(y)|\,dy < \frac{\eta}{4\|f\|_{L^\infty(\mathbb{R}^n)}},$$

which is possible by property (3) of the good kernel. Observe that since f is bounded we have $\|f\|_{L^\infty(\mathbb{R}^n)} < \infty$. On the other hand, we may assume that $\|f\|_{L^\infty(\mathbb{R}^n)} > 0$, otherwise $f = 0$ and the claim is clear. Thus

$$|I_2| \leqslant 2\|f\|_{L^\infty(\mathbb{R}^n)} \int_{\{|y|>\delta\}} |K_\varepsilon(y)|\,dy < \frac{\eta}{2}.$$

Summing up the estimates for I_1 and I_2 we obtain

$$|(K_\varepsilon * f)(x) - f(x)| < \eta$$

when $\varepsilon > 0$ is small enough. This proves the desired convergence at x.

If f is continuous with compact support it is bounded. Furthermore, it is vanishing outside a compact set so it is uniformly continuous. Thus the δ in (3.1) can be chosen to be the same for all $x \in \mathbb{R}^n$ which shows that the convergence is uniform in this case. □

Example 3.21. Let $K : \mathbb{R}^n \to \mathbb{R}$, $K(x) = e^{-\pi|x|^2}$ be the Gaussian function and

$$K_a(x) = \frac{1}{a^n} K\left(\frac{x}{a}\right) = \pi^{-\frac{n}{2}} a^{-n} e^{-\frac{|x|^2}{a^2}}, \quad a > 0.$$

Then the proof of the Fourier inversion theorem 3.14 shows that $\{K_a\}_{a>0}$ is a family of good kernels.

Example 3.22. For $t > 0$, let $H_t : \mathbb{R}^n \to \mathbb{R}$,

$$H_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}.$$

Then H_t is called the heat kernel for the upper half plane and, as we shall see, it is related to the time dependent heat equation in the upper half plane. Furthermore, H_t can be given as

$$H_t(x) = \frac{1}{(\sqrt{4\pi t})^n} K\left(\frac{x}{\sqrt{4\pi t}}\right) = \frac{1}{(\sqrt{4\pi t})^n} e^{-\frac{\pi|x|^2}{4\pi t}} = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}.$$

where $K(x) = e^{-\pi|x|^2}$. Then $\{H_t\}_{t>0}$ is a family of good kernels as in Example 3.21.

Example 3.23. For $r > 0$, let $K_r : \mathbb{R}^n \to \mathbb{R}$,

$$K_r(x) = \frac{\mathbf{1}_{B(0,r)}(x)}{|B(0,r)|},$$

where $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$ is the open ball with the center $x \in \mathbb{R}^n$ and radius $r > 0$. $|B(x,r)|$ denotes the n-dimensional volume of the ball. Then $\{K_r\}_{r>0}$ is a family of good kernels and

$$\begin{aligned}(K_r * f)(x) &= \frac{1}{|B(0,r)|}\int_{\mathbb{R}^n} f(y)\mathbf{1}_{B(0,r)}(x-y)\,dy \\ &= \frac{1}{|B(x,r)|}\int_{\mathbb{R}^n} f(y)\mathbf{1}_{B(x,r)}(y)\,dy \\ &= \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy\end{aligned}$$

is the integral average of f over the ball $B(x,r)$. Here we also used the fact that the volume of a ball is independent of the location. The previous theorem tells that

$$\lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy = f(x)$$

if f is continuous. This means that the pointwise value $f(x)$ is a limit of the integral averages over balls centered at x as the radius tends to zero. It is also possible to show this directly form the definition of continuity.

THE MORAL: Approximations of the identity can be constructed by using one function by rescaling.

3.9 The Laplace equation in the upper half-space

In this section we consider the Laplace equation in the upper half-space

$$\mathbb{R}^{n+1}_+ = \{(x,y) : x \in \mathbb{R}^n,\ y > 0\}.$$

We assume that the boundary function g is a continuous and bounded function defined on $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \{0\}$ and consider the following Dirichlet problem

$$\begin{cases} \Delta u(x,y) = \sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_j^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0, & (x,y) \in \mathbb{R}^{n+1}_+, \\ u(x,0) = g(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.2}$$

Observe that here the Laplacian is taken with respect to all $n+1$ variables. Physically this models the case when the temperature does not change in time and that the temperature $u(x,0)$ at the boundary is given by the function $g(x)$. We look for solutions $u \in C^2(\mathbb{R}^{n+1}_+) \cap C(\overline{\mathbb{R}^{n+1}_+})$ of this problem. This means that u is twice continuously differentiable (all partial derivatives of u of second order exist and are continuous) and that u is continuous up to the boundary. The differentiability condition guarantees that the partial derivatives in the Laplace operator make sense and the continuity up to the boundary is needed for the boundary condition.

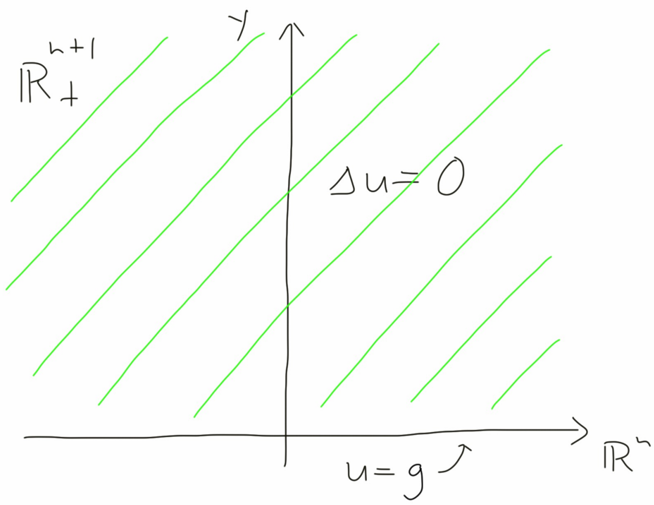


Figure 3.3: The Laplace equation in the upper half-space.

Step 1 (PDE on the Fourier side): To solve this problem let $y>0$ be fixed and denote

$$\widehat{u}(\xi,y) = \int_{\mathbb{R}^n} u(x,y)e^{-ix\cdot\xi}\,dx, \quad \xi \in \mathbb{R}^n, \quad y>0.$$

Observe that we take the Fourier transform of u in the x variable only. By Theorem 3.5, we obtain

$$\widehat{\frac{\partial^2 u}{\partial x_j^2}}(\xi,y) = i\xi_j \widehat{\frac{\partial u}{\partial x_j}}(\xi,y) = (i\xi_j)^2\widehat{u}(\xi,y) = -\xi_j^2\widehat{u}(\xi,y), \quad j=1,\dots,n,$$

and, by switching the order of differentiation and integration, we have

$$\begin{aligned}\widehat{\frac{\partial^2 u}{\partial y^2}}(\xi,y) &= \int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial y^2}(x,y)e^{-ix\cdot\xi}\,dx \\ &= \frac{\partial^2}{\partial y^2}\left(\int_{\mathbb{R}^n} u(x,y)e^{-ix\cdot\xi}\,dx\right) = \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi,y).\end{aligned}$$

It follows that

$$\begin{aligned}
0 &= \widehat{\Delta u}(\xi, y) \\
&= -\xi_1^2 \widehat{u}(\xi, y) - \dots - \xi_n^2 \widehat{u}(\xi, y) + \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y) \\
&= -|\xi|^2 \widehat{u}(\xi, y) + \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y).
\end{aligned}$$

This is the Laplace equation on the Fourier side. Compare to Remark 2.47 for the Fourier series. Note that, under appropriate assumptions, the Fourier inversion theorem 3.14 implies $\Delta u = 0 \iff \widehat{\Delta u} = 0$.

THE MORAL: The Laplace equation becomes an ODE on the Fourier side.

Step 2 (Solution on the Fourier side): For a fixed ξ the solutions of the ODE for $\widehat{u}$ are of the form

$$\widehat{u}(\xi, y) = c_1(\xi) e^{|\xi| y} + c_2(\xi) e^{-|\xi| y}.$$

Note here that, as ξ is fixed and we solve the ODE in y, the constants c_1 and c_2 may depend on $\xi \in \mathbb{R}^n$. We disregard the first term on the right hand side, since $e^{|\xi| y} \to \infty$ as $|\xi| \to \infty$. This corresponds to the physically irrelevant unbounded solution and we are left with

$$\widehat{u}(\xi, y) = c_2(\xi) e^{-|\xi| y}.$$

The boundary condition on the Fourier side implies

$$c_2(\xi) = \widehat{u}(\xi, 0) = \widehat{g}(\xi)$$

and thus

$$\widehat{u}(\xi, y) = \widehat{g}(\xi) e^{-|\xi| y}, \quad \xi \in \mathbb{R}^n, \quad y > 0.$$

This is the candidate for a solution of the Dirichlet problem on the Fourier side.

Step 3 (Solution to the original problem): The Fourier inversion theorem 3.14 gives

$$\begin{aligned}
u(x, y) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{u}(\xi, y) e^{i x \cdot \xi}\, d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-|\xi| y} \widehat{g}(\xi) e^{i x \cdot \xi}\, d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{P_y}(\xi) \widehat{g}(\xi) e^{i x \cdot \xi}\, d\xi,
\end{aligned}$$

where $\widehat{P_y}(\xi) = e^{-|\xi| y}$. The function $P_y(x)$ is called the Poisson kernel for the upper half-space. Observe that at this point we only know the Fourier transform of the

Poisson kernel. By Theorem 3.17, we obtain

$$
\begin{aligned}
u(x,y) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{P_y}(\xi)\widehat{g}(\xi)e^{ix\cdot\xi}\,d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{P_y * g}(\xi)e^{ix\cdot\xi}\,d\xi \\
&= (P_y * g)(x) = \int_{\mathbb{R}^n} P_y(x-z)g(z)\,dz.
\end{aligned}
$$

Here we used the Fourier inversion formula again.

THE MORAL: The solution of the Dirichlet problem in the upper half-space is a convolution of the boundary function with a Poisson kernel.

Step 4 (Explicit representation formula): We thus need to find, for every $y > 0$, a function $P_y(x)$, $x \in \mathbb{R}^n$, such that $\widehat{P_y}(\xi) = e^{-y|\xi|}$. Indeed, it is possible to find an explicit formula for the function P_y will be the Poisson kernel for the upper half-space.

Lemma 3.24. For $x \in \mathbb{R}^n$ and $y > 0$ we define

$$
P_y(x) = P(x,y) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{y}{(|x|^2+y^2)^{\frac{n+1}{2}}}, \quad x \in \mathbb{R}^n, \quad y > 0.
$$

Here $\Gamma(\frac{n+1}{2})$ is a dimensional constant given by the Γ-function

$$
\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\,dx.
$$

The following evaluation of an integral plays a role in the proof of Lemma 3.24.

Lemma 3.25.

$$
e^{-\beta} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}} e^{-\frac{\beta^2}{4s}}\,ds, \qquad \beta > 0.
$$

Proof. First we observe that, for $\xi \in \mathbb{R}$, we have

$$
\int_0^\infty e^{-s}e^{-s\xi^2}\,ds = \bigg|_0^\infty \frac{e^{-s(1+\xi^2)}}{-(1+\xi^2)} = \frac{1}{1+\xi^2}.
$$

By Example 3.10

$$
\widehat{e^{-|x|}}(\xi) = \frac{2}{1+\xi^2}.
$$

The Fourier inversion theorem gives

$$
\begin{aligned}
e^{-|x|} &= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{e^{-|x|}}(\xi)e^{ix\xi}\,d\xi = \frac{1}{\pi} \int_{\mathbb{R}} \frac{e^{ix\xi}}{1+\xi^2}\,d\xi \\
&= \frac{1}{\pi} \int_{\mathbb{R}} \left(\int_0^\infty e^{-s}e^{-s\xi^2}\,ds \right) e^{ix\xi}\,d\xi \\
&= \frac{1}{\pi} \int_0^\infty e^{-s} \left(\int_{\mathbb{R}} e^{-s\xi^2}e^{ix\xi}\,d\xi \right) ds.
\end{aligned}
$$

By Example 3.11 and Theorem 3.4 (5), we have

$$\widehat{e^{-\frac{|x|^2}{4s}}}(\xi) = \sqrt{4s\pi}e^{-s\xi^2}.$$

By the Fourier inversion theorem

$$e^{-\frac{|x|^2}{4s}} = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{e^{-\frac{|x|^2}{4s}}}(\xi)e^{ix\xi}\,d\xi = \sqrt{\frac{s}{\pi}}\int_{\mathbb{R}} e^{-s\xi^2}e^{ix\xi}\,d\xi.$$

This implies

$$e^{-|x|} = \int_{\mathbb{R}} \frac{1}{\sqrt{\pi s}}e^{-s}e^{-\frac{x^2}{4s}}\,ds.$$

The claim follows by choosing $x = \beta$. $\square$

Proof (of Lemma 3.24). First we prove the claim for $y = 1$. Since $\widehat{P}(\xi,1) = \widehat{P_1}(\xi) = e^{-|\xi|}$, the Fourier inversion theorem and Lemma 3.25 give

$$\begin{aligned}
P(x,1) &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-|\xi|}e^{ix\cdot\xi}\,d\xi \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\left(\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-s}}{\sqrt{s}}e^{-\frac{|\xi|^2}{4s}}\,ds\right)d\xi \\
&= (2\pi)^{-n}\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-s}}{\sqrt{s}}\left(\int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{4s}}e^{ix\cdot\xi}\,d\xi\right)ds.
\end{aligned}$$

We shall evaluate the inner integral in the formula above. By Example 3.13 and Theorem 3.4 (5)

$$\widehat{e^{-s|x|^2}}(\xi) = \widehat{e^{-|\sqrt{s}x|^2}}(\xi) = (\sqrt{s})^{-n}\widehat{e^{-|x|^2}}\left(\frac{\xi}{\sqrt{s}}\right) = (\sqrt{s})^{-n}\pi^{\frac{n}{2}}e^{-\frac{|\xi|^2}{4s}}.$$

By the Fourier inversion theorem

$$\begin{aligned}
e^{-s|x|^2} &= (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{e^{-s|x|^2}}(\xi)e^{ix\cdot\xi}\,d\xi \\
&= (2\pi)^{-n}\pi^{\frac{n}{2}}s^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{4s}}e^{ix\cdot\xi}\,d\xi.
\end{aligned}$$

Thus

$$\begin{aligned}
P(x,1) &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-|\xi|}e^{ix\cdot\xi}\,d\xi \\
&= (2\pi)^{-n}\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-s}}{\sqrt{s}}\frac{1}{(2\pi)^{-n}\pi^{\frac{n}{2}}s^{-\frac{n}{2}}}e^{-s|x|^2}\,ds \\
&= \frac{1}{\pi^{\frac{n+1}{2}}}\frac{y}{(|x|^2+1)^{1+\frac{n-1}{2}}}\int_0^\infty e^{-s}s^{\frac{n-1}{2}}\,ds \\
&= \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\frac{1}{(|x|^2+1)^{\frac{n+1}{2}}}.
\end{aligned}$$

This proves the claim for $y = 1$.

The case $y > 0$ follows by a change of variables and the previous formula

$$\begin{aligned}P(x,y) &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-y|\xi|}e^{ix\cdot\xi}\,d\xi\\ &= (2\pi)^{-n}y^{-n}\int_{\mathbb{R}^n} e^{-|\xi|}e^{ix\cdot\frac{\xi}{y}}\,d\xi\\ &= \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}y^{-n}\frac{1}{\left(\left|\frac{x}{y}\right|^2+1\right)^{\frac{n+1}{2}}}\\ &= \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\frac{y}{(|x|^2+y^2)^{\frac{n+1}{2}}}.\end{aligned}$$

This completes the proof. □

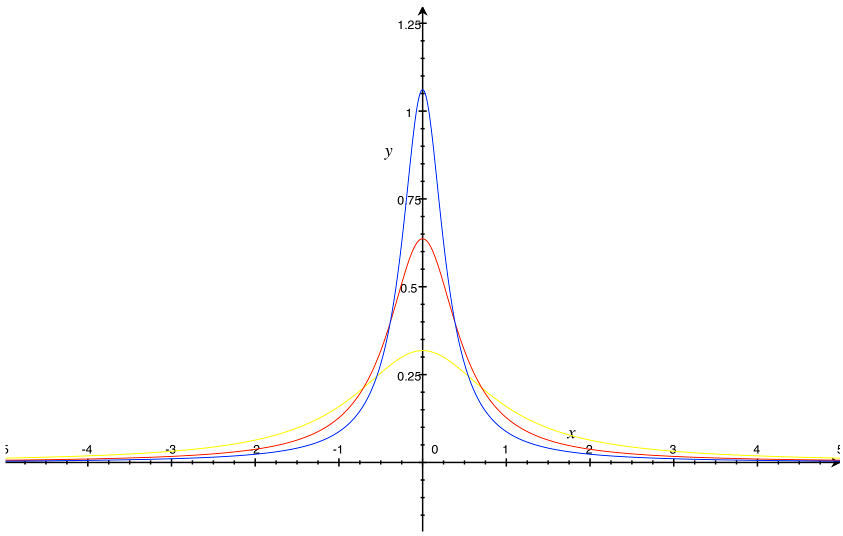


Figure 3.4: The graph of the Poisson kernel in dimension $n = 1$ for $y = 1$ (yellow), $y = 0.5$ (red) and $y = 0.3$ (blue).

The Poisson kernel has the following properties.

(1) $P_y(x) > 0$ for every $x \in \mathbb{R}^n$ and $y > 0$.

(2) For every $y > 0$ we have

$$\int_{\mathbb{R}^n} P_y(x)\,dx = \widehat{P_y}(0) = 1.$$

(3) The Poisson kernel $P(x,y) = P_y(x)$ is a solution of the Laplace equation in the upper half-space $\mathbb{R}^{n+1}_+$. It is called the fundamental solution in the upper half-space $\mathbb{R}^{n+1}_+$, since all other solutions can be represented as a convolution with it.

Remark 3.26. Denote $P(x) = P_1(x)$. Then

$$P_y(x) = \frac{1}{y^n} P\left(\frac{x}{y}\right), \quad x \in \mathbb{R}^n, \quad y > 0.$$

This means that the Poisson kernel $P_y(x)$ with $y > 0$ can be obtained from $P_1(y)$ by rescaling, compare to Example 3.21. This formula has the following consequences. First, by the change of variables $z = \frac{x}{y}$, $dz = \frac{1}{y^n} dx$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} P_y(x)\,dx &= \frac{1}{y^n} \int_{\mathbb{R}^n} P\left(\frac{x}{y}\right) dx \\
&= \frac{1}{y^n} \int_{\mathbb{R}^n} P(z) y^n \,dz \\
&= \int_{\mathbb{R}^n} P(x)\,dx = 1, \quad y > 0.
\end{aligned}$$

Thus the dimensional constant $c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}$ appearing in front of the Poisson kernel gives

$$c_n \int_{\mathbb{R}^n} P_y(x)\,dx = c_n \int_{\mathbb{R}^n} P(x)\,dx = c_n \int_{\mathbb{R}^n} \frac{1}{(|x|^2+1)^{\frac{n+1}{2}}}\,dx = 1, \quad y > 0.$$

Second, from this we can see that $\{P_y\}_{y>0}$ is a family of good kernels. Indeed, all the properties of a family of good kernels in Definition 3.19 are satisfied except maybe the third. To see the third property in Definition 3.19, we show that

$$\lim_{y\to 0} \int_{|x|>\delta} |P_y(x)|\,dx = 0.$$

for every $\delta > 0$. By the change of variables $z = \frac{x}{y}$, $dz = \frac{1}{y^n} dx$, we have

$$\begin{aligned}
\int_{|x|>\delta} |P_y(x)|\,dx &= \frac{1}{y^n} \int_{\{|x|>\delta\}} P\left(\frac{x}{y}\right) dx \\
&= \int_{\left\{|z|>\frac{\delta}{y}\right\}} P(z)\,dz \\
&= \int_{\mathbb{R}^n} \mathbf{1}_{\left\{|x|>\frac{\delta}{y}\right\}}(x) P(x)\,dx.
\end{aligned}$$

For every $x \in \mathbb{R}^n$ we have

$$\lim_{y\to 0} \mathbf{1}_{\left\{|x|>\frac{\delta}{y}\right\}}(x) P(x) = 0.$$

Furthermore, for every $y > 0$ and $x \in \mathbb{R}^n$ we have

$$|\mathbf{1}_{\left\{|x|>\frac{\delta}{y}\right\}}(x) P(x)| \leqslant |P(x)| \in L^1(\mathbb{R}^n).$$

Thus we may change the order of limit and integral by the Lebesgue dominated convergence theorem and obtain

$$\lim_{y\to 0} \int_{\{\}} |P_y(x)|\,dx = \int_{\mathbb{R}^n} \lim_{y\to 0} \mathbf{1}_{\left\{|x|>\frac{\delta}{y}\right\}}(x) P(x)\,dx = 0,$$

which shows the third property of good kernels.

We collect our findings in the following result.

Theorem 3.27. Let $g \in C_0^\infty(\mathbb{R}^n)$. The solution to the Dirichlet problem (3.2) is

$$u(x,y) = (g * P_y)(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} \frac{g(z)y}{(|x-z|^2+y^2)^{\frac{n+1}{2}}}\,dz.$$

The boundary condition is taken in the sense that

$$\lim_{y\to 0} u(x,y) = g(x) \quad \text{for every} \quad x \in \mathbb{R}^n.$$

W A R N I N G : The boundary condition cannot be verified by inserting directly $y = 0$ to the formula above. Thus the limit interpretation with the approximation of the identity is needed for the boundary values.

3.10 The heat equation in the upper half-space

The general form of the heat equation is

$$u_t(x,t) - \Delta u(x,t) = 0,$$

where the Laplace operator is only in the x-variable

$$\Delta u(x,t) = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2}(x,t).$$

We consider the initial value problem

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = 0, & x \in \mathbb{R}^n, \quad t > 0, \\ u(x,0) = g(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.3}$$

This is called the Cauchy problem for the heat equation. Physically this models the case when the initial temperature $u(x,0)$ at the moment $t = 0$ is given by the function $g(x)$ and we would like to know the temperature $u(x,t)$ for $t > 0$. Here we look for solutions $u \in C^2(\mathbb{R}^{n+1}_+) \cap C(\overline{\mathbb{R}^{n+1}_+})$. This means that u has continuous second order partial derivatives in the upper half-space $\mathbb{R}^{n+1}$ and that u is continuous up to the initial boundary $\mathbb{R}^n \times \{y = 0\}$.

Step 1 (PDE on the Fourier side): Let $t > 0$ be fixed and denote by $\widehat{u}(\xi,t)$ the Fourier transform of $u(x,t)$ in the x-variable

$$\widehat{u}(\xi,t) = \int_{\mathbb{R}^n} u(x,t)e^{-ix\cdot\xi}\,dx.$$

On the Fourier side, the heat equation becomes

$$0 = \widehat{u_t - \Delta u}(\xi,t) = \widehat{u_t}(\xi,t) - \widehat{\Delta u}(\xi,t) = \frac{\partial \widehat{u}}{\partial t}(\xi,t) + |\xi|^2 \widehat{u}(\xi,t).$$

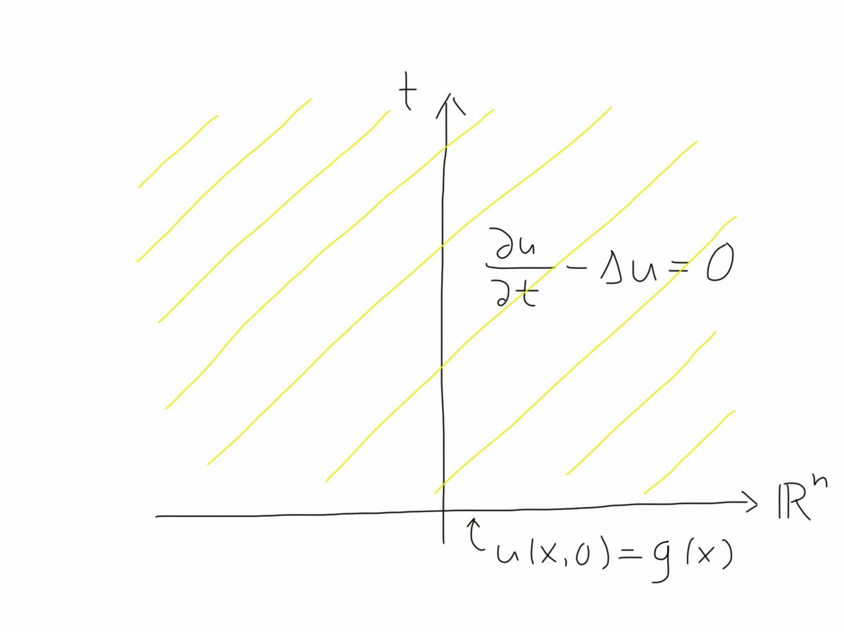


Figure 3.5: The heat equation in the upper half-space.

This can be seen in the same way as for the Laplace equation. Compare this to Remark 2.47 for the Fourier series.

Step 2 (Solution on the Fourier side): The solution of this ODE for a fixed $\xi \in \mathbb{R}^n$ is

$$\widehat{u}(\xi, t) = c(\xi)e^{-|\xi|^2 t}.$$

The initial condition $u(x,0) = g(x)$ gives

$$c(\xi) = \widehat{u}(\xi, 0) = \widehat{g}(\xi)$$

and thus

$$\widehat{u}(\xi, t) = \widehat{g}(\xi)e^{-|\xi|^2 t}.$$

This is the solution on the Fourier side.

Step 3 (Solution to the original problem): By the Fourier inversion theorem

$$\begin{aligned}
u(x,t) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{u}(\xi, t) e^{ix\cdot\xi}\, d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-|\xi|^2 t} \widehat{g}(\xi) e^{ix\cdot\xi}\, d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{H_t}(\xi) \widehat{g}(\xi) e^{ix\cdot\xi}\, d\xi,
\end{aligned}$$

where $\widehat{H_t}(\xi) = e^{-|\xi|^2 t}$. Observe that $\widehat{H_t}$ is a Gaussian function. By Theorem 3.17, we have

$$\begin{aligned} u(x,t) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{H_t}(\xi)\widehat{g}(\xi) e^{ix\cdot\xi}\,d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{H_t * g}(\xi) e^{ix\cdot\xi}\,d\xi \\ &= (H_t * g)(x,t) = \int_{\mathbb{R}^n} H_t(x-y)g(y)\,dy. \end{aligned}$$

Here we used the Fourier inversion formula again.

Step 4 (Explicit representation formula): We claim that if

$$H_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^n, \quad t > 0,$$

then $\widehat{H_t}(\xi) = e^{-|\xi|^2 t}$. The function H_t is called the heat kernel in the upper half-space. Observe that it is a Gaussian function.

Reason.

$$\begin{aligned} \widehat{H_t}(\xi) &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \widehat{e^{-\frac{|x|^2}{4t}}}(\xi) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \widehat{e^{-(|x|/(2\sqrt{t}))^2}}(\xi) \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}} (2\sqrt{t})^n \widehat{e^{-|x|^2}}(2\sqrt{t}\xi) \quad \text{(Lemma 3.4 (5))} \\ &= \pi^{-\frac{n}{2}} \pi^{\frac{n}{2}} e^{-|2\sqrt{t}\xi|^2/4} = e^{-|\xi|^2 t}. \quad \text{(Example 3.13)} \end{aligned}$$

■

T H E M O R A L : The solution of the initial value problem for the heat equation in the upper half-space is a convolution of the initial value function with the heat kernel.

The family $\{H_t\}_{t>0}$ is a family of good kernels related to approximations of the identity, see Example 3.22. Moreover, the heat kernel $H(x,t) = H_t(x)$ is a solution of the heat equation in the upper half-space $\mathbb{R}^{n+1}_+$. It is called the fundamental solution in the upper half-space $\mathbb{R}^{n+1}_+$, since all other solutions can be represented as a convolution with it.

Theorem 3.28. Assume that $g \in C_0^\infty(\mathbb{R}^n)$. The solution to the Cauchy problem (3.3) is

$$u(x,t) = (H_t * g)(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy, \quad x \in \mathbb{R}^n, \quad t > 0.$$

The initial values are attained in the sense

$$\lim_{t\to 0} u(x,t) = g(x) \quad \text{for every} \quad x \in \mathbb{R}^n.$$

W A R N I N G : The initial condition cannot be verified by inserting directly $t = 0$ to the formula above. Thus the limit interpretation with the approximation of the identity is needed for the initial values.

Remark 3.29. We consider the nonhomogeneous initial value problem

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = f(x,t), & x \in \mathbb{R}^n, \quad t > 0, \\ u(x,0) = g(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.4}$$

On the Fourier side, the equation becomes

$$\frac{\partial \widehat{u}}{\partial t}(\xi,t) + |\xi|^2 \widehat{u}(\xi,t) = \widehat{f}(\xi)$$

and after a multiplication through by $e^{|\xi|^2 t}$ we have

$$e^{|\xi|^2 t} \frac{\partial \widehat{u}}{\partial t}(\xi,t) + e^{|\xi|^2 t}|\xi|^2 \widehat{u}(\xi,t) = e^{|\xi|^2 t}\widehat{f}(\xi).$$

For the left-hand side, we observe, that

$$e^{|\xi|^2 t} \frac{\partial \widehat{u}}{\partial t}(\xi,t) + e^{|\xi|^2 t}|\xi|^2 \widehat{u}(\xi,t) = \frac{\partial}{\partial t}\left(e^{|\xi|^2 t}\widehat{u}(\xi,t)\right).$$

Thus by the fundamental theorem of calculus, we have

$$e^{|\xi|^2 t}\widehat{u}(\xi,t) + \widehat{u}(\xi,0) = \int_0^t \frac{\partial}{\partial s}\left(e^{|\xi|^2 s}\widehat{u}(\xi,s)\right) ds = \int_0^t e^{|\xi|^2 s}\widehat{f}(\xi,s)\,ds$$

and, using the fact that $\widehat{u}(\xi,0) = \widehat{g}(\xi)$, we obtain

$$\widehat{u}(\xi,t) = e^{-|\xi|^2 t}\widehat{g}(\xi) + \int_0^t e^{-|\xi|^2(t-s)}\widehat{f}(\xi,s)\,ds.$$

By the Fourier inversion theorem

$$\begin{aligned}
u(x,t) &= (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{u}(\xi,t)e^{ix\cdot\xi}\,d\xi \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-|\xi|^2 t}\widehat{g}(\xi)e^{ix\cdot\xi}\,d\xi + (2\pi)^{-n}\int_{\mathbb{R}^n}\left(\int_0^t e^{-|\xi|^2(t-s)}\widehat{f}(\xi,s)\,ds\right)e^{ix\cdot\xi}\,d\xi \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{H_t}(\xi)\widehat{g}(\xi)e^{ix\cdot\xi}\,d\xi + \int_0^t (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{H_{t-s}}(\xi)\widehat{f}(\xi,s)e^{ix\cdot\xi}\,d\xi\,ds \\
&= (H_t * g)(x) + \int_0^t (H_{t-s} * f(\cdot,s))(x)\,ds \\
&= \frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy \\
&\qquad + \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-|\frac{x-y|^2}{4(t-s)}} f(y,s)\,dy\,ds.
\end{aligned}$$

Thus the solution of the inhomogeneous problem can be represented using the solutions of the homogeneous problem given by Theorem 3.28. This is so-called Duhamel's principle. We shall return to this later in Section 5.3.

Example 3.30. Let us consider the initial value problem for the Schrödinger equation

$$\begin{cases} iu_t(x,t) + \Delta u(x,t) = 0, & x \in \mathbb{R}^n, \quad t > 0, \\ u(x,0) = g(x), & x \in \mathbb{R}^n. \end{cases}$$

Here u and g are complex valued functions. If $v(x,t) = u(x,it)$, where u is as in the representation formula above, we have

$$v(x,t) = \frac{1}{(4\pi i t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} g(y)\,dy, \quad x \in \mathbb{R}^n, \quad t > 0.$$

Here

$$i^{\frac{1}{2}} = e^{\frac{1}{2}\log i} = e^{\frac{1}{2}(\log|i| + i \arg i)} = e^{\frac{i}{2}\arg i} = e^{\frac{i\pi}{4}}.$$

We can check by a direct calculation, that u is a solution of the Schrödinger equation above. Indeed, if $v(x,t) = u(x,it)$, then

$$\frac{\partial v}{\partial t}(x,t) = \frac{\partial}{\partial t}(u(x,it)) = i u_t(x,it) = i\Delta u(x,it) = i\Delta v(x,t).$$

The function

$$\Psi(x,t) = \frac{1}{(4\pi i t)^{\frac{n}{2}}} e^{i\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^n, \quad t \neq 0,$$

is called the fundamental solution of the Schrödinger equation. Note that the formula $u = f * \Psi$ makes sense for all $t \neq 0$. Thus this gives a solution to the problem

$$\begin{cases} i u_t(x,t) + \Delta u(x,t) = 0, & x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \\ u(x,0) = g(x), & x \in \mathbb{R}^n. \end{cases}$$

In particular, the Schrödinger equation is reversible in time, whereas the heat equation is not. We shall return to this point later.

3.11 The wave equation in the upper half-space

The general form of the wave equation is

$$u_{tt}(x,t) - \Delta u(x,t) = 0,$$

where the Laplace operator is only in the x-variable

$$\Delta u(x,t) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x,t).$$

One can think of the wave equation as describing the displacement of a vibrating string (in dimension $n = 1$), a vibrating membrane (in dimension $n = 2$) or an elastic solid (dimension $n = 3$). In dimension $n = 3$ this equation also determines the behavior of electromagnetic waves in vacuum and the propagation of sound waves.

The general goal is to find a solution of the Cauchy problem for the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \mathbb{R}^n, \quad t > 0, \\ u(x,0) = g(x), & x \in \mathbb{R}^n, \\ u_t(x,0) = h(x), & x \in \mathbb{R}^n. \end{cases}$$

Step 1 (PDE on the Fourier side): Let $t > 0$ be fixed and denote by $\widehat{u}(\xi, t)$ the Fourier transform of $u(x,t)$ in the x-variable

$$\widehat{u}(\xi,t) = \int_{\mathbb{R}^n} u(x,t) e^{-ix\cdot\xi}\, dx.$$

Step 2 (Solution on the Fourier side): On the Fourier side, the wave equation becomes

$$0 = \widehat{(u_{tt} - \Delta u)}(\xi,t) = \widehat{u_{tt}}(\xi,t) - \widehat{\Delta u}(\xi,t) = \frac{\partial^2 \widehat{u}}{\partial t^2}(\xi,t) + |\xi|^2 \widehat{u}(\xi,t).$$

The solution of this ODE for a fixed $\xi \in \mathbb{R}^n$ is

$$\widehat{u}(\xi,t) = c_1(\xi)\cos(|\xi|t) + c_2(\xi)\sin(|\xi|t)$$

for some functions $c_1(\xi)$ and $c_2(\xi)$. Taking the Fourier transforms of the initial conditions we obtain

$$\widehat{g}(\xi) = \widehat{u}(\xi,0) = c_1(\xi) \quad \text{and} \quad \widehat{h}(\xi) = \widehat{u_t}(\xi,0) = |\xi| c_2(\xi).$$

Thus the solution on the Fourier side can be written in the form

$$\widehat{u}(\xi,t) = \widehat{g}(\xi)\cos(|\xi|t) + \widehat{h}(\xi)\frac{\sin(|\xi|t)}{|\xi|}.$$

Step 3 (Solution to the original problem): By the Fourier inversion formula

$$u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi)\cos(|\xi|t) + \widehat{h}(\xi)\frac{\sin(|\xi|t)}{|\xi|} \right) e^{ix\cdot\xi}\, d\xi. \tag{3.5}$$

Denote

$$\widehat{\Phi_t}(\xi) = \frac{\sin(|\xi|t)}{|\xi|} \quad \text{and} \quad \widehat{\Psi_t}(\xi) = \cos(|\xi|t) = \frac{\partial \widehat{\Phi_t}}{\partial t}(\xi).$$

Then

$$\begin{aligned} u(x,t) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi)\widehat{\Psi_t}(\xi) + \widehat{h}(\xi)\widehat{\Phi_t}(\xi) \right) e^{ix\cdot\xi}\, d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g * \Psi_t}(\xi) + \widehat{h * \Phi_t}(\xi) \right) e^{ix\cdot\xi}\, d\xi \quad \text{(Theorem 3.17)} \\ &= (g * \Psi_t)(x) + (h * \Phi_t)(x). \quad \text{(Fourier inversion formula)} \end{aligned}$$

Now the problem is how to determine functions Ψ_t and Φ_t and what is the interpretation of the representation formula above. This is a hard problem and we shall return to this later. A direct calculation shows that the u given by the formula above does indeed solve the Cauchy problem for the wave equation. Furthermore, the solution to the Cauchy problem is unique, but we will not prove this here.

3.12 Summary

The main steps in the application of the Fourier transform to PDE problems are the following.

(1) The task is to find a solution of an initial or boundary value problem in the half-space.

(2) Take the Fourier transform of the PDE and of the initial conditions, with respect to the space variables.

(3) This reduces the problem to an ODE.

(4) The ordinary ODE is solved on the Fourier side.

(5) The initial or boundary conditions are used to determine the free parameters.

(6) The Fourier inversion formula gives the solution of the original problem.

(7) The solution of the original problem is represented as a convolution of the data with the fundamental solution.

(8) This gives a solution to the original problem and the initial or boundary values are attained by using approximations of the unity.

Boundary value problems for the Laplace equation in subdomains of the higher dimensional Euclidean space appear frequently in natural sciences and engineering. We derive representation formulas and study general properties of solutions to the Laplace equation. In this process we shall encounter fundamental solutions, Green's functions, mean value property, Harnack's inequality and maximum principles.

4 Laplace equation

The n-dimensional Laplace equation

$$\Delta u = 0$$

and the Poisson equation

$$-\Delta u = f$$

appear frequently in natural sciences and engineering. Recall that

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

Let Ω be an open subset of $\mathbb{R}^n$. The problem is to find a function $u \in C^2(\Omega)$ such that it is a solution to the Laplace or Poisson equation in Ω. Physically, solutions of the Poisson equation correspond to steady states for evolutions in time such as heat flow or wave motion, with f corresponding to external driving forces such as heat sources or wave generators.

Definition 4.1. A function $u \in C^2(\Omega)$, which satisfies $\Delta u = 0$ in Ω, is called a harmonic function in Ω.

We have already seen examples of harmonic functions, see Theorem 3.27. In this section we take a more systematic approach to the Laplacian.

4.1 Gauss-Green theorem

We shall need certain integral formulas to be able to study the Laplacian. Here we assume that Ω is a bounded and open subset of $\mathbb{R}^n$ and that the boundary $\partial\Omega$ is smooth. This means that the boundary can be locally represented as a graph of a smooth function. The closure of the domain is a union of the domain and its boundary, that is, $\overline{\Omega} = \Omega \cup \partial\Omega$. We say that $u \in C^1(\overline{\Omega})$, if $u \in C^1(\Omega)$ is such that u and all partial derivatives $\frac{\partial u}{\partial x_j}$, $j = 1,\dots,n$, can be extended continuously up to the boundary $\partial\Omega$.

Theorem 4.2 (Gauss-Green theorem). Assume that $u \in C^1(\overline{\Omega})$. Then

$$\int_\Omega \frac{\partial u}{\partial x_j}(x)\,dx = \int_{\partial\Omega} u(x)\nu_j(x)\,dS(x), \quad j = 1,\dots,n,$$

where dS denotes the surface measure on $\partial\Omega$. Here $\nu(x) = (\nu_1(x),\dots,\nu_n(x))$ is the outward pointing unit normal vector on $\partial\Omega$,

$$\frac{\partial u}{\partial \nu}(x) = \nabla u(x)\cdot\nu(x), \quad x \in \partial\Omega,$$

is the outward normal derivative of u and

$$\nabla u(x) = \left(\frac{\partial u}{\partial x_1}(x),\dots,\frac{\partial u}{\partial x_n}(x)\right), \quad x \in \Omega,$$

is the gradient of u.

Remark 4.3. Another way to write the Gauss-Green theorem is

$$\int_\Omega \operatorname{div} F(x)\,dx = \int_{\partial\Omega} F(x)\cdot\nu(x)\,dS(x),$$

where $F = (F_1,\dots,F_n)$ is a vector field, whose component functions satisfy the assumption in the Gauss-Green theorem. This is called the divergence theorem. Recall, that

$$\operatorname{div} F(x) = \sum_{j=1}^n \frac{\partial F_j}{\partial x_j}(x)$$

is the divergence of F.

Reason.

$$\begin{aligned}\int_\Omega \operatorname{div} F(x)\,dx &= \int_\Omega \sum_{j=1}^n \frac{\partial F_j}{\partial x_j}(x)\,dx = \sum_{j=1}^n \int_\Omega \frac{\partial F_j}{\partial x_j}(x)\,dx \\ &= \sum_{j=1}^n \int_{\partial\Omega} F_j(x)\nu_j(x)\,dS(x) = \int_{\partial\Omega} \sum_{j=1}^n F_j(x)\nu_j(x)\,dS(x) \\ &= \int_{\partial\Omega} F(x)\cdot\nu(x)\,dS(x).\end{aligned}$$

■

THE MORAL: The Gauss-Green theorem gives information about the divergence of a vector field inside the domain by its values on the boundary of the domain. More precisely, the integral of the divergence of a vector field over a domain is equal to the total flow through the boundary. This is useful in boundary value problems for PDEs.

Theorem 4.4 (Integration by parts). Assume that $u, v \in C^1(\overline{\Omega})$. Then

$$\int_\Omega \frac{\partial u}{\partial x_j}(x)v(x)\,dx = -\int_\Omega \frac{\partial v}{\partial x_j}(x)u(x)\,dx + \int_{\partial\Omega} u(x)v(x)\nu_j(x)\,dS(x), \quad j = 1,\dots,n.$$

Proof. Apply the Gauss-Green theorem for uv (exercise). □

THE MORAL: The Gauss-Green theorem can be seen as an integration by parts formula in $\mathbb{R}^n$.

Theorem 4.5 (Green's identities). Assume that $u, v \in C^2(\overline{\Omega})$. Then

(1) $\displaystyle\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx = -\int_\Omega u(x)\Delta v(x)\,dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu}(x)u(x)\,dS(x),$

(2) $\displaystyle\int_\Omega (u(x)\Delta v(x) - v(x)\Delta u(x))\,dx = \int_{\partial\Omega}\left(u(x)\frac{\partial v}{\partial\nu}(x) - v(x)\frac{\partial u}{\partial\nu}(x)\right)dS(x),$

(3) $\displaystyle\int_\Omega \Delta u(x)\,dx = \int_{\partial\Omega}\frac{\partial u}{\partial\nu}(x)\,dS(x).$

These are called Green's first, second and third identities, respectively.

Proof. $\boxed{(1)}$ By replacing v with $\frac{\partial v}{\partial x_j}$ in Theorem 4.4, we have

$$\int_\Omega \frac{\partial v}{\partial x_j}(x)\frac{\partial u}{\partial x_j}(x)\,dx = -\int_\Omega u(x)\frac{\partial^2 v}{\partial x_j^2}(x)\,dx + \int_{\partial\Omega} u(x)\frac{\partial v}{\partial x_j}(x)\nu_j(x)\,dS(x),$$

$j = 1,\dots,n$. The claim follows by summing over $j = 1,\dots,n$.

$\boxed{(2)}$ Switch u and v in (1) and subtract (exercise).

$\boxed{(3)}$ By replacing u with $\frac{\partial u}{\partial x_j}$ and $v = 1$ in Theorem 4.4, we have

$$\int_\Omega \frac{\partial^2 u}{\partial x_j^2}(x)\,dx = \int_{\partial\Omega}\frac{\partial u}{\partial x_j}(x)\nu_j(x)\,dS(x), \quad j = 1,\dots,n.$$

The claim follows by summing over $j = 1,\dots,n$, since

$$\begin{aligned}\int_\Omega \Delta u(x)\,dx &= \sum_{j=1}^n \int_\Omega \frac{\partial^2 u}{\partial x_j^2}(x)\,dx = \sum_{j=1}^n \int_{\partial\Omega}\frac{\partial u}{\partial x_j}(x)\nu_j(x)\,dS(x)\\ &= \int_{\partial\Omega}\nabla u(x)\cdot\nu(x)\,dS(x) = \int_{\partial\Omega}\frac{\partial u}{\partial\nu}(x)\,dS(x).\end{aligned}$$

□

Remarks 4.6:

(1) If u is harmonic in Ω, then the Green's first identity gives

$$\int_\Omega |\nabla u(x)|^2\,dx = \int_\Omega \nabla u(x)\cdot\nabla u(x)\,dx = \int_{\partial\Omega}\frac{\partial u}{\partial\nu}(x)u(x)\,dS(x).$$

Thus for every harmonic function we have

$$\frac{1}{2}\int_{\partial\Omega}\frac{\partial u^2}{\partial\nu}(x)\,dS(x) = \int_{\partial\Omega}\frac{\partial u}{\partial\nu}(x)u(x)\,dS(x) \geqslant 0.$$

(2) Green's third identity tells that the integral of the Laplacian of a function over a domain is equal to the total flow through the boundary. In particular, if u is harmonic in Ω, then

$$\int_{\partial V} \frac{\partial u}{\partial \nu}(x)\, dS(x) = 0$$

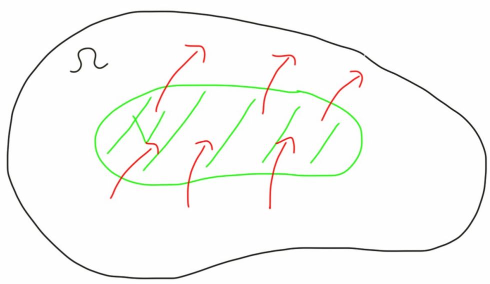
for every subdomain V for which $\overline{V} \subset \Omega$. This means that the total flow is zero through the boundary of any subdomain V. Physically this means that there are no heat sources or electric charges in the domain.

4.2 PDEs and physics

In a typical case u is a function that denotes the density of some quantity in steady state. For example u may denote temperature, chemical concentration or electrostatic potential. If V is any smooth subdomain of Ω, the total flow through the boundary ∂V is zero

$$\int_{\partial V} F(x) \cdot \nu(x)\, dS(x) = 0,$$

where $F = (F_1, \ldots, F)$ is the flux density and ν is the unit outer normal of ∂V.



By the Gauss-Green theorem we have

$$\int_V \operatorname{div} F(x)\, dx = \int_{\partial V} F(x) \cdot \nu(x)\, dS(x) = 0.$$

Since this holds for every subdomain V of Ω, we have

$$\operatorname{div} F(x) = 0 \quad \text{for every} \quad x \in \Omega.$$

It is physically reasonable to assume that the flux F is proportional to the gradient ∇u but in the opposite direction, since the flow is from regions of high temperature to regions of low temperature or high concentration to low concentration. Thus

$$F(x) = -a\nabla u(x), \quad a > 0.$$

This gives

$$\operatorname{div} F(x) = -a \operatorname{div} \nabla u(x) = -a\Delta u(x) = 0 \quad \text{for every} \quad x \in \Omega,$$

which implies $\Delta u = 0$ in Ω. A similar argument can be done for the Poisson equation. In this case, the function f describes the heat source or an electric charge distribution.

Examples 4.7:

(1) If u is chemical concentration, then $\Delta u = 0$ is Fick's law of diffusion.

(2) If u is temperature, then $\Delta u = 0$ is Fourier's law of heat conduction (steady state).

(3) If u is electrostatic potential, then $\Delta u = 0$ is Ohm's law of electrical conduction.

(4) Maxwell's equations for the electric field E with a source f are

$$\begin{cases} \operatorname{div} E = f, \\ \operatorname{curl} E = 0. \end{cases}$$

Now $\operatorname{curl} E = 0$ implies that $E = -\nabla u + c$. Thus $\operatorname{div} E = f$ implies that

$$\operatorname{div} \nabla u = \Delta u = -f,$$

which is the Poisson equation.

4.3 Boundary values and physics

The Dirichlet problem

Assume that Ω is a bounded open set in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. We shall consider two different boundary value problems. The Dirichlet problem for the Laplace equation is

$$\begin{cases} \Delta u = 0 \quad \text{in} \quad \Omega, \\ u = g \quad \text{on} \quad \partial\Omega. \end{cases}$$

Here $u \in C^2(\overline{\Omega})$ and g is referred to as boundary data. It is useful to keep in mind the physical significance of the Dirichlet problem. Let the function u describe the steady state temperature distribution in a homogeneous isotropic body in the

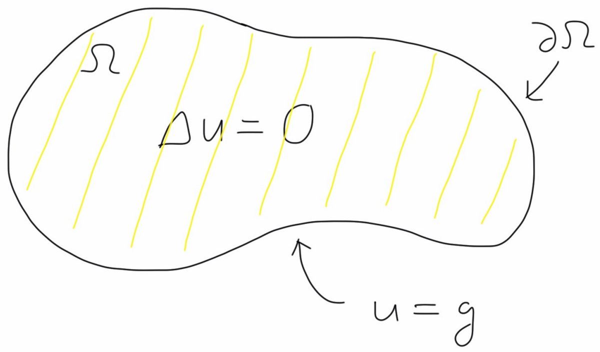


Figure 4.1: The Dirichlet problem.

interior, which is the domain Ω, and suppose that the temperature distribution g is given on the boundary $\partial\Omega$. In electrostatics the Dirichlet boundary condition specifies the values of the potential u on the boundary $\partial\Omega$, which induces the electric field $E = -\nabla u$ in Ω.

Remarks 4.8:

(1) If u is harmonic in Ω and $u = 0$ on $\partial\Omega$, then $u = 0$ in Ω.

Reason. By Green's first identity

$$\int_\Omega |\nabla u(x)|^2\,dx = \int_\Omega \nabla u(x)\cdot\nabla u(x)\,dx$$
$$= -\int_\Omega u(x)\underbrace{\Delta u(x)}_{=0}\,dx + \int_{\partial\Omega}\frac{\partial u}{\partial \nu}(x)\underbrace{u(x)}_{=0}\,dS(x) = 0.$$

This implies that $|\nabla u(x)| = 0$ and thus $u(x) = c$ for every $x \in \Omega$. Since $u(x) = 0$ for $x \in \partial\Omega$, the constant c has to be zero and we have $u(x) = 0$ for every $x \in \Omega$. ■

(2) If u and v are harmonic functions in Ω and $u(x) = v(x)$ for every $x \in \partial\Omega$, then $u(x) = v(x)$ for every $x \in \Omega$.

Reason. Let $w = u - v$. Then

$$\Delta w = \Delta(u - v) = \Delta u - \Delta v = 0 \quad \text{in} \quad \Omega$$

and $w = u - v = 0$ on $\partial\Omega$. (1) implies $w = 0$ and thus $u = v$ in Ω. ■

Thus the solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega, \end{cases}$$

for the Laplace equation is unique.

(3) The solution of the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega, \end{cases}$$

for the Poisson equation is unique.

Reason. Assume that u and v are solutions to the problem. Let $w = u - v$. Then

$$\Delta w = \Delta(u - v) = \Delta u - \Delta v = 0 \quad \text{in} \quad \Omega$$

and $w = u - v = g - g = 0$ on $\partial\Omega$. (1) implies $w = 0$ and thus $u = v$ in Ω. ■

THE MORAL: Green's formulas imply that the solution $u \in C^2(\overline{\Omega})$ of the Dirichlet problem for the Laplace and Poisson equations in a bounded smooth domain is unique. Observe that this qualitative result is not based on representation formulas for solutions and holds in all domains and boundary values. However, we would prefer not to assume $u \in C^2(\overline{\Omega})$ for uniqueness of the Dirichlet problem. Later we shall prove a stronger uniqueness results as a consequence of the maximum principle.

Example 4.9. Let Ω be the unit disc in $\mathbb{R}^2$ and consider the Dirichlet problem (2.13) with smooth enough boundary values. The solution in polar coordinates, as a convolution with the Poisson kernel, given by Theorem 2.43 is unique.

Example 4.10. Consider the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = c & \text{on} \quad \partial\Omega, \end{cases}$$

where $c \in \mathbb{R}$ is a given constant. It is obvious that $u(x) = c$ is a solution to this problem. Since the solution is unique, it is the only solution to this problem.

The Neumann problem

The Neumann problem for the Laplace equation is

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = h & \text{on} \quad \partial\Omega. \end{cases}$$

Physically the Neumann problem describes the steady state temperature distribution in Ω when the heat flow through $\partial\Omega$ is given by the normal derivative $\frac{\partial u}{\partial \nu} = h$. For example, if the surface of the body $\partial\Omega$ is insulated, the function h in the Neumann boundary condition is zero.

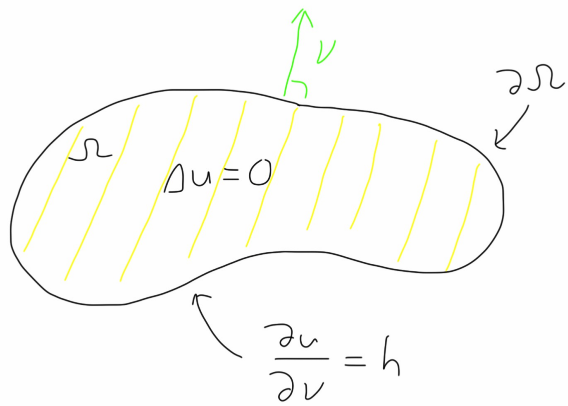


Figure 4.2: The Neumann problem.

Example 4.11. If u is a solution to the Neumann problem above, then $u(x)+c$, where $c \in \mathbb{R}$, is a solution to the same problem.

Reason.

$$\Delta(u+c) = \Delta u \quad \text{in} \quad \Omega$$

and

$$\frac{\partial}{\partial \nu}(u+c) = \frac{\partial u}{\partial \nu} = h \quad \text{on} \quad \partial\Omega.$$

■

Thus the Neumann problem has infinitely many solutions.

THE MORAL: The solution of the Neumann problem for the Laplace equation is not unique.

We need one more definition before we proceed. An open set Ω is connected, if every pair of points in Ω can be connected by a piecewise linear path in Ω. Thus a connected open set consists of one component.

Remarks 4.12:

(1) If u is a harmonic function in a connected domain Ω and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, then $u = c$ in Ω.

Reason. By Green's first identity

$$\int_\Omega |\nabla u(x)|^2\,dx = \int_\Omega \nabla u(x)\cdot\nabla u(x)\,dx$$
$$= -\int_\Omega u(x)\underbrace{\Delta u(x)}_{=0}\,dx + \int_{\partial\Omega} \underbrace{\frac{\partial u}{\partial \nu}(x)}_{=0} u(x)\,dS(x) = 0.$$

This implies that $|\nabla u(x)| = 0$ and thus $u(x) = c$ for every $x \in \Omega$. The assumption that Ω is connected is used here. If Ω has several components, then u may be a different constant in every component. ■

(2) If u and v are harmonic functions in a connected domain Ω and $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu}$ on $\partial\Omega$, then $u = v + c$ in Ω.

Reason. Let $w = u - v$. Then

$$\Delta w = \Delta(u-v) = \Delta u - \Delta v = 0 \quad \text{in} \quad \Omega$$

and

$$\frac{\partial w}{\partial \nu} = \frac{\partial}{\partial \nu}(u-v) = 0 \quad \text{on} \quad \partial\Omega.$$

(1) implies $w = c$ and thus $u = v + c$ in Ω. ■

THE MORAL: The Green's formulas imply that the solution $u \in C^2(\overline{\Omega})$ of the Neumann problem is unique up to an additive constant in a connected smooth domain. If the domain has several components, the solution may be different constant in every component of the domain. Observe that this qualitative result is not based on representation formulas for solutions and holds in all domains and boundary values.

Example 4.13. Consider the Neumann problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

It is obvious that $u(x) = c$, where $c \in \mathbb{R}$ is any constant, is a solution to this problem. Thus this problem has infinitely many solutions.

Remarks 4.14:

(1) Green's third identity gives the following compatibility condition of the Neumann problem

$$0 = \int_\Omega \underbrace{\Delta u}_{=0}\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\,dS = \int_{\partial\Omega} h\,dS.$$

Note that the solution does not exist without the condition that the total heat flow through the boundary is zero.

(2) It is also possible to consider a combination of the Dirichlet and Neumann boundary conditions

$$a(x)\frac{\partial u}{\partial \nu}(x)+b(x)u(x)=g(x), \quad x\in\partial\Omega,$$

but we shall not discuss this issue here.

Remark 4.15. We discuss the Dirichlet and Neumann problems for the Poisson equation $-\Delta u = f$. However, it is enough to consider boundary value problems in which either the equation is homogeneous ($\Delta u = 0$) or the boundary condition is homogeneous ($g = 0$ or $h = 0$). For example, to solve

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega, \end{cases}$$

we may write $u = u_1 + u_2$ with

$$\begin{cases} -\Delta u_1 = f & \text{in} \quad \Omega, \\ u_1 = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta u_2 = 0 & \text{in} \quad \Omega, \\ u_2 = g & \text{on} \quad \partial\Omega. \end{cases}$$

If we are able to solve these problems separately, the solution of the original problem is a sum of the solutions.

Reason.

$$-\Delta u = -\Delta(u_1+u_2) = -\Delta u_1 = f \quad \text{in} \quad \Omega$$

and

$$u = u_1+u_2 = 0+g = g \quad \text{on} \quad \partial\Omega.$$

■

Recall that the solution is unique by Remark 4.8 (3). A similar argument applies to the Neumann problem as well (exercise).

THE MORAL: It is enough to consider the nonhomogeneous PDE with zero boundary values and the homogeneous PDE with nonzero boundary values.

4.4 Fundamental solution of the Laplace equation

In Section 3.9 we derived the fundamental solution of the Laplace equation in the upper half-space using the Fourier transform. In this section, we derive a formula for the fundamental solution of the Laplace equation in $\mathbb{R}^n$. Since the equation is linear, any linear combination, or integral, of fundamental solutions will be a solution to the Laplace equation as well. This will give us a method to represent all other solutions as integrals, or convolutions, with the fundamental solution.

We assume that $\Omega = \mathbb{R}^n$ and we are looking for special solutions of the form

$$u(x) = v(|x|) = v(r(x)),$$

where $r(x) = |x| = (x_1^2 + \cdots + x_n^2)^{1/2}$. This means that we are looking for radial solutions, that is, solutions that only depend on the distance from the origin.

Let us see what the Laplace equation is for the radial solutions. By the chain rule

$$\frac{\partial r}{\partial x_j}(x) = \frac{\partial}{\partial x_j}(|x|) = \frac{1}{2}(x_1^2 + \cdots + x_n^2)^{-\frac{1}{2}} 2x_j = \frac{x_j}{|x|} = \frac{x_j}{r(x)}, \quad j = 1, \dots, n, \quad x \neq 0.$$

Again, by the chain rule

$$\frac{\partial u}{\partial x_j}(x) = \frac{\partial}{\partial x_j}(v(r(x))) = v'(r(x))\frac{\partial r}{\partial x_j}(x) = v'(r(x))\frac{x_j}{r(x)}$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial x_j^2}(x) &= v''(r(x))\frac{x_j^2}{r(x)^2} + \frac{\partial}{\partial x_j}\left(\frac{x_j}{r(x)}\right)v'(r(x)) \\ &= v''(r(x))\frac{x_j^2}{r(x)^2} + v'(r(x))\left(\frac{1}{r(x)} - \frac{x_j^2}{r(x)^3}\right), \quad j = 1, \dots, n, \quad x \neq 0.\end{aligned}$$

Thus

$$\begin{aligned}\Delta u(x) &= \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x) \\ &= v''(r(x)) \underbrace{\sum_{j=1}^{n} \frac{x_j^2}{r(x)^2}}_{=1} + n\frac{v'(r(x))}{r(x)} - v'(r(x)) \underbrace{\sum_{j=1}^{n} \frac{x_j^2}{r(x)^3}}_{=1/r(x)} \\ &= v''(r(x)) + \frac{n-1}{r(x)} v'(r(x)) = 0, \quad x \neq 0.\end{aligned}$$

Hence for a radial function

$$\Delta u(x) = 0, \quad x \neq 0 \Longleftrightarrow v''(r) + \frac{n-1}{r}v'(r) = 0, \quad r > 0.$$

THE MORAL: This is the radial version of the Laplace equation. Note that the Laplace equation for a radial function becomes an ODE.

If $v' \neq 0$, we have

$$(\ln v')' = \frac{v''}{v'} = -\frac{n-1}{r} = -(n-1)(\ln r)' = \left(\ln \frac{1}{r^{n-1}}\right)',$$

which gives

$$v'(r) = \frac{c}{r^{n-1}}$$

for some constant c. Note also that we can assume that $n \geqslant 2$ otherwise the ODE is completely trivial $v'' = 0$. An integration gives

$$v(r) = \begin{cases} a \ln r + b, & n = 2, \\ \frac{c}{r^{n-2}} + d, & n \geqslant 3, \end{cases}$$

where a, b, c, d are constants. These functions give radial solutions of the Laplace equation in $\mathbb{R}^n \setminus \{0\}$. In the following definition, we set $b = 0$ and $d = 0$, and choose a and c in an appropriate manner.

Definition 4.16. The function $\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$,

$$\Phi(x) = \begin{cases} -\dfrac{1}{2\pi} \ln |x|, & n = 2, \\ \dfrac{1}{n(n-2)\alpha(n)} \dfrac{1}{|x|^{n-2}}, & n \geqslant 3, \end{cases}$$

is called the fundamental solution of the Laplace equation. Here we denote the volume of the unit ball in $\mathbb{R}^n$ by $\alpha(n) = |B(0,1)|$.

T H E M O R A L : The goal is to represent all solutions to boundary value problems for the Laplace and Poisson equations using the fundamental solution. Physically the fundamental solution is the potential induced by a unit point mass at the origin.

Remarks 4.17:

(1) Note that Φ is harmonic in $\mathbb{R}^n \setminus \{0\}$, that is, $\Delta\Phi(x) = 0$ for every $x \in \mathbb{R}^n \setminus \{0\}$ (exercise). Moreover, Φ has a singularity at the origin in the sense that Φ is unbounded in every neighbourhood of the origin. Note that Φ and its first partial derivatives are integrable in any neighbourhood of the origin, even though Φ has a singularity there. However, the second partial derivatives of Φ are not integrable near the origin (exercise).

(2) The choice of the constants for the fundamental solution is a normalization that gives

$$-\int_{\partial B(0,r)} \frac{\partial \Phi}{\partial \nu}(x)\, dS(x) = 1 \quad \text{for every} \quad r > 0,$$

where ν is the outward pointing unit normal on $\partial B(0,r)$. The negative sign becomes positive for the inward pointing unit normal on $\partial B(0,r)$, see the proof of Theorem 4.19. Essentially this normalization has the same role as the normalization $\int K_\varepsilon(x)\, dx = 1$ for a family of good kernels so that $f * K_\varepsilon$ will converge to f instead of cf for some constant c. We shall return to this soon.

Example 4.18. Let $n = 3$. Then the volume of the unit ball in $\mathbb{R}^3$ is $\alpha(3) = \frac{4\pi}{3}$ and

$$\Phi(x) = \frac{1}{4\pi}\frac{1}{|x|}, \quad x \in \mathbb{R}^3, \quad x \neq 0,$$

is the Coulomb (or Newton) potential. Thus the induced electric field is

$$E(x) = -\nabla\Phi(x) = -\frac{1}{4\pi}\nabla(|x|^{-1}) = \frac{1}{4\pi}|x|^{-2}\frac{x}{|x|} = \frac{1}{4\pi}\frac{x}{|x|^3}, \quad x \in \mathbb{R}^3, \quad x \neq 0.$$

4.5 The Poisson equation

We derive a representation formula for a solution to the nonhomogeneous Laplace equation, called the Poisson equation,

$$-\Delta u = f \quad \text{in} \quad \mathbb{R}^n,$$

where $f \in C_0^\infty(\mathbb{R}^n)$, that is, f is a compactly supported smooth function. Recall, that the support of a function is compact, if the function is identically zero outside a closed and a bounded set. In practice, this set can be chosen to be a ball. For a fixed $y \in \mathbb{R}^n$, the function $x \mapsto \Phi(x-y)f(y)$ is harmonic in $\mathbb{R}^n \setminus \{y\}$. Here Φ is the fundamental solution to the Laplace equation. Since the Laplace equation is linear, we could think that the convolution

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,dy$$

is also a solution of the Laplace equation.

WARNING: This is wrong. We are not allowed switch the order of differentiation and integration and conclude that

$$\Delta u(x) = \int_{\mathbb{R}^n} \Delta_x\Phi(x-y)f(y)\,dy = 0.$$

The problem is that the second order partial derivatives of Φ behave as $c/|x|^n$, which is not an integrable function.

However, we have the following result, which shows that the function above is a solution to the Poisson equation.

Theorem 4.19 (Solution to the Poisson equation in the whole space). Let $f \in C_0^\infty(\mathbb{R}^n)$ and define

$$u(x) = (f * \Phi)(x) = \int_{\mathbb{R}^n} f(y)\Phi(x-y)\,dy,$$

where Φ is the fundamental solution of the Laplace equation. Then $u \in C^2(\mathbb{R}^n)$ and $-\Delta u = f$ in $\mathbb{R}^n$.

WARNING: The problem above does not have a unique solution, since we can always add a function v with $\Delta v = 0$ to the solution.

THE MORAL: A convolution of the source term with the fundamental solution is a solution to the Poisson equation in the whole space. Physically f describes a charge density, that is, a distribution of electric charges and u is the potential of the electric field induced by f. Observe that the potential is harmonic outside the support of f.

Examples 4.20:

(1) In the plane $\mathbb{R}^2$ we have the logarithmic potential

$$u(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} f(y)\ln|x-y|\,dy.$$

(2) In the space $\mathbb{R}^3$ we have the Newton (or Coulomb) potential

$$u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|}\,dy.$$

Remark 4.21. The theorem gives a solution u in the whole space without a specification of the boundary values. However, if Ω is an open and bounded subset of $\mathbb{R}^n$ and v is a solution of the Dirichlet problem

$$\begin{cases} \Delta v = 0 & \text{in} \quad \Omega, \\ v = -u & \text{on} \quad \partial\Omega. \end{cases}$$

Then $w = u + v$ is a solution to the problem

$$\begin{cases} -\Delta w = f & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Thus we have reduced the Dirichlet problem for the Poisson equation to the Dirichlet problem for the Laplace equation. This observation will be useful later in the construction of Green's function.

Proof. **Step 1:** Let us first calculate the partial derivatives of u. The jth difference quotient is

$$\frac{u(x+he_j)-u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y)\frac{f(x-y+he_j)-f(x-y)}{h}\,dy$$

where $h \neq 0$ and $e_j = (0,\dots,0,1,0,\dots,0)$ with 1 in the jth slot. Since f is twice continuously differentiable this difference quotient converges and

$$\lim_{h\to 0}\frac{f(x-y+he_j)-f(x-y)}{h} = \frac{\partial f}{\partial x_j}(x-y).$$

This partial derivative is bounded and has compact support since $f \in C_0^\infty(\mathbb{R}^n)$. The Lebesgue dominated convergence theorem allows us to interchange the limit and the integral to get

$$\begin{aligned}\frac{\partial u}{\partial x_j}(x) &= \lim_{h\to 0} \frac{u(x+he_j)-u(x)}{h} \\ &= \lim_{h\to 0}\int_{\mathbb{R}^n} \Phi(y)\frac{f(x-y+he_j)-f(x-y)}{h}\,dy \\ &= \int_{\mathbb{R}^n} \Phi(y)\lim_{h\to 0}\frac{f(x-y+he_j)-f(x-y)}{h}\,dy \\ &= \int_{\mathbb{R}^n} \Phi(y)\frac{\partial f}{\partial x_j}(x-y)\,dy = \left(\Phi * \frac{\partial f}{\partial x_j}\right)(x), \quad j=1,\dots,n.\end{aligned}$$

Arguing in exactly the same way, which is possible because f has second order continuous derivatives with compact support, we get

$$\frac{\partial^2 u}{\partial x_j \partial x_k}(x) = \int_{\mathbb{R}^n} \Phi(y)\frac{\partial^2 f}{\partial x_j \partial x_k}(x-y)\,dy = \left(\Phi * \frac{\partial^2 f}{\partial x_j \partial x_k}\right)(x), \quad j,k=1,\dots,n.$$

Thus $u \in C^2(\mathbb{R}^n)$ and we have a formula for calculating the derivatives of u.

THE MORAL: This argument shows that the differentiation can be taken inside the convolution. This is why the convolution inherits smoothness properties of the functions.

Step 2: To show that u is a solution to the Poisson equation, we would like to argue in the same way, pass the derivatives inside the integral, apply them to Φ instead of f and use the fact that Φ is harmonic. The warning before the statement of the theorem shows that we have to be careful here. We thus split the convolution integral into two parts, one close to the origin and the other far away from the origin.

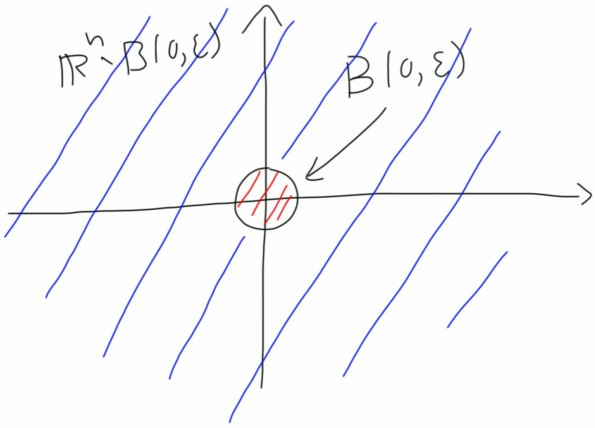
In order to carry out this plan, let $0<\varepsilon<\frac{1}{2}$ and write

$$\begin{aligned}\Delta u(x) &= \Delta_x \int_{\mathbb{R}^n} \Phi(y) f(x-y)\,dy = \int_{\mathbb{R}^n} \Phi(y)\Delta_x f(x-y)\,dy \\ &= \int_{B(0,\varepsilon)} \Phi(y)\Delta_x f(x-y)\,dy + \int_{\mathbb{R}^n\setminus B(0,\varepsilon)} \Phi(y)\Delta_x f(x-y)\,dy \\ &= I_\varepsilon + J_\varepsilon.\end{aligned}$$

Step 3: First we estimate I_ε.

$$\begin{aligned}|I_\varepsilon| &\leqslant \int_{B(0,\varepsilon)} |\Phi(y)||\Delta_x f(x-y)|\,dy \\ &\leqslant \sup_{y\in B(0,\varepsilon)} |\Delta_x f(x-y)| \int_{B(0,\varepsilon)} |\Phi(y)|\,dy \leqslant \sup_{x\in\mathbb{R}^n} |\Delta f(x)| \int_{B(0,\varepsilon)} |\Phi(y)|\,dy,\end{aligned}$$

where $\sup_{x\in\mathbb{R}^n} |\Delta f(x)| < \infty$ since $f \in C_0^\infty(\mathbb{R}^n)$. Next we shall compute the remaining integral.



$\boxed{n=2}$

$$\begin{aligned}
\int_{B(0,\varepsilon)} |\Phi(y)|\,dy &= -\frac{1}{2\pi}\int_{B(0,\varepsilon)} \ln|y|\,dy = -\frac{1}{2\pi}\int_0^\varepsilon \int_{\partial B(0,r)} \ln|y|\,dS(y)\,dr \\
&= -\frac{1}{2\pi}\int_0^\varepsilon \ln r\left(\int_{\partial B(0,r)} 1\,dS(y)\right)dr = -\frac{1}{2\pi}\int_0^\varepsilon 2\pi r\ln r\,dr \\
&= -\Big|_0^\varepsilon \left(\frac{1}{2}r^2\ln r - \frac{1}{4}r^2\right) = -\left(\frac{1}{2}\varepsilon^2\ln\varepsilon - \frac{1}{4}\varepsilon^2\right) \\
&\leqslant \varepsilon^2|\ln\varepsilon| + \frac{1}{4}\varepsilon^2.
\end{aligned}$$

Since $\varepsilon < 1/2 \Longleftrightarrow |\ln\varepsilon| > |\ln 1/2| \Longleftrightarrow 1 < \frac{|\ln\varepsilon|}{|\ln 1/2|}$ the previous estimate implies that

$$\int_{B(0,\varepsilon)} |\Phi(y)|\,dy \leqslant c\varepsilon^2|\ln\varepsilon|$$

for some constant c.

$\boxed{n\geqslant 3}$

$$\begin{aligned}
\int_{B(0,\varepsilon)} |\Phi(y)|\,dy &= c\int_{B(0,\varepsilon)} |y|^{2-n}\,dy = c\int_0^\varepsilon \int_{\partial B(0,r)} |y|^{2-n}\,dS(y)\,dr \\
&= c\int_0^\varepsilon r^{2-n}\left(\int_{\partial B(0,r)} 1\,dS(y)\right)dr = c\int_0^\varepsilon r^{2-n}|\partial B(0,r)|\,dr,
\end{aligned}$$

where c is a constant that depends only on the dimension. Now the $(n-1)$-dimensional volume of the sphere $\partial B(0,r)$ is $|\partial B(0,r)| = cr^{n-1}$ for some constant c that depends only on dimension. Thus

$$\int_{B(0,\varepsilon)} |\Phi(y)|\,dy = c\int_0^\varepsilon r\,dr = c\varepsilon^2$$

for some constant c that depends only on the dimension. In both cases we see that

$$\lim_{\varepsilon\to 0} I_\varepsilon = 0.$$

Step 4: In this step we estimate J_ε. First we note that

$$J_\varepsilon(x) = \int_{\mathbb{R}^n\setminus B(0,\varepsilon)} \Phi(y)\Delta_x(f(x-y))\,dy = \int_{\mathbb{R}^n\setminus B(0,\varepsilon)} \Phi(y)\Delta_y(f(x-y))\,dy.$$

This equality follows from

$$\frac{\partial}{\partial x_j}(f(x-y)) = -\frac{\partial}{\partial y_j}(f(x-y)), \quad \frac{\partial^2}{\partial x_j^2}(f(x-y)) = \frac{\partial^2}{\partial y_j^2}(f(x-y)), \quad j=1,\dots,n.$$

We apply Green's first identity with $\Omega = \mathbb{R}^n \setminus B(0,\varepsilon)$ so that the exterior unit vector points inwards to the ball

$$\nu(y) = -\frac{y}{|y|} = -\frac{y}{\varepsilon}, \quad y \in \partial B(0,\varepsilon).$$

Note carefully, that this is the outward pointing unit normal of $B(0,\varepsilon)$ with a negative sign. We apply Green's first identity to the functions $\Phi(y)$ and $y \mapsto f(x-y)$ with x fixed. This gives

$$\begin{aligned} J_\varepsilon(x) &= \int_{\partial B(0,\varepsilon)} \Phi(y)\frac{\partial}{\partial \nu}(f(x-y))\,dS(y) - \int_{\mathbb{R}^n\setminus B(0,\varepsilon)} \nabla_y\Phi(y)\cdot\nabla_y(f(x-y))\,dy \\ &= K_\varepsilon + L_\varepsilon. \end{aligned}$$

Step 5: In this step we estimate K_ε as

$$\begin{aligned} |K_\varepsilon| &\leqslant \int_{\partial B(0,\varepsilon)} |\Phi(y)| \left|\frac{\partial}{\partial \nu}(f(x-y))\right| dS(y) \\ &= \int_{\partial B(0,\varepsilon)} |\Phi(y)||\nabla_y(f(x-y))\cdot\nu(x-y)|\,dS(y) \\ &\leqslant \int_{\partial B(0,\varepsilon)} |\Phi(y)||\nabla f(x-y)|\underbrace{|\nu(x-y)|}_{=1}\,dS(y) \quad \text{(Cauchy-Schwarz inequality)} \\ &\leqslant \sup_{y\in\mathbb{R}^n}|\nabla f(y)| \int_{\partial B(0,\varepsilon)} |\Phi(y)|\,dS(y), \end{aligned}$$

where $\sup_{y\in\mathbb{R}^n}|\nabla f(y)| < \infty$ because $f \in C_0^\infty(\mathbb{R}^n)$. Now we estimate the remaining integral. Since Φ is a radial function and $|x| = \varepsilon$ on $\partial B(0,\varepsilon)$ we get

$$\int_{\partial B(0,\varepsilon)} |\Phi(y)|\,dS(y) = |\Phi(\varepsilon)||\partial B(0,\varepsilon)| \leqslant \begin{cases} 2\pi\varepsilon|\ln\varepsilon|, & n=2, \\ c\varepsilon^{2-n}\varepsilon^{n-1} = c\varepsilon, & n \geqslant 3. \end{cases}$$

From this we conclude that

$$\lim_{\varepsilon\to 0} K_\varepsilon = 0.$$

Step 6: We compute L_ε. Green's first identity gives

$$\begin{aligned}
L_\varepsilon &= -\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \nabla_y \Phi(y) \cdot \nabla_y (f(x-y))\, dy \\
&= \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \underbrace{\Delta\Phi(y)}_{=0,\, y\neq 0} f(x-y)\, dy - \int_{\partial B(0,\varepsilon)} f(x-y) \frac{\partial \Phi}{\partial \nu}(y)\, dS(y) \\
&= -\int_{\partial B(0,\varepsilon)} f(x-y) \frac{\partial \Phi}{\partial \nu}(y)\, dS(y)
\end{aligned}$$

Let us calculate the normal derivative

$$\frac{\partial \Phi}{\partial \nu}(y) = \nabla \Phi(y) \cdot \nu(y),$$

where $\nu(y) = -\frac{y}{|y|}$, $y \in \partial B(0,\varepsilon)$, is the outward pointing unit normal of the set $\mathbb{R}^n \setminus B(0,\varepsilon)$.

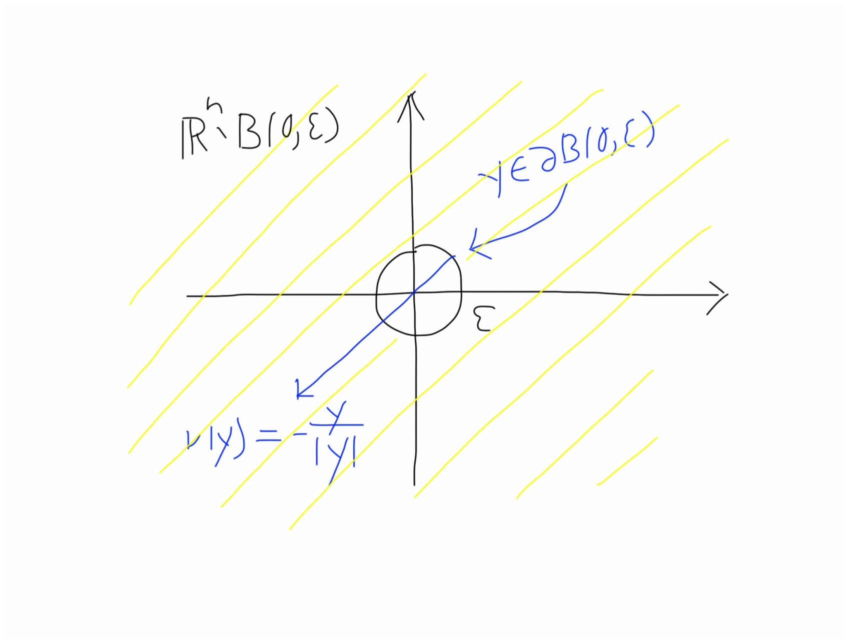


Figure 4.3: The outward normal of $\mathbb{R}^n \setminus B(0,\varepsilon)$.

$\boxed{n \geqslant 3}$

$$\begin{aligned}
\frac{\partial \Phi}{\partial \nu}(y) &= \frac{1}{n(n-2)\alpha(n)} (2-n)|y|^{1-n} \nabla(|y|) \cdot \nu(y) \\
&= -\frac{1}{n\alpha(n)} \frac{y}{|y|^n} \cdot \left(-\frac{y}{|y|}\right) = \frac{1}{n\alpha(n)} \frac{|y|^2}{|y|^{n+1}} \\
&= \frac{1}{n\alpha(n)} |y|^{1-n} = \frac{1}{n\alpha(n)} \varepsilon^{1-n}, \quad y \in \partial B(0,\varepsilon).
\end{aligned} \tag{4.1}$$

This implies

$$
\begin{aligned}
L_\varepsilon &= -\frac{1}{n\alpha(n)\varepsilon^{n-1}}\int_{\partial B(0,\varepsilon)} f(x-y)\,dS(y) \\
&= -\frac{1}{|\partial B(0,\varepsilon)|}\int_{\partial B(0,\varepsilon)} f(x-y)\,dS(y) \\
&= -\frac{1}{|\partial B(x,\varepsilon)|}\int_{\partial B(x,\varepsilon)} f(z)\,dS(z)
\end{aligned}
$$

by the change of variables $z = x - y$. Recall that $|\partial B(x,\varepsilon)|$ denotes the $(n-1)$-dimensional volume of the sphere $\partial B(x,r)$. Thus the last quantity is an integral average of f over a sphere of radius ε centered at x. Let us take a closer look at this average.

Claim: $\lim\limits_{\varepsilon\to 0}\frac{1}{|\partial B(x,\varepsilon)|}\int_{\partial B(x,\varepsilon)} f(z)\,dS(z) = f(x).$

Reason. We have

$$
\frac{1}{|\partial B(x,\varepsilon)|}\int_{\partial B(x,\varepsilon)} f(z)\,dS(z) - f(x) = \frac{1}{|\partial B(x,\varepsilon)|}\int_{\partial B(x,\varepsilon)} (f(z)-f(x))\,dS(z).
$$

Let $\eta > 0$. By continuity there exists $\varepsilon > 0$ such that

$$
|z-x| \leqslant \varepsilon \Rightarrow |f(x)-f(z)| \leqslant \eta.
$$

This implies

$$
\left|\frac{1}{|\partial B(x,\varepsilon)|}\int_{\partial B(x,\varepsilon)} f(z)\,dS(z) - f(x)\right| \leqslant \frac{1}{|\partial B(x,\varepsilon)|}\int_{\partial B(x,\varepsilon)} |f(z)-f(x)|\,dS(z) \leqslant \eta,
$$

which proves the claim. Observe, that this is the same argument as in approximation of the identity. ■

This implies that

$$
\lim_{\varepsilon\to 0} L_\varepsilon = -f(x).
$$

$\boxed{n=2}$ Observe that

$$
\begin{aligned}
\frac{\partial\Phi}{\partial\nu}(y) &= \nabla\Phi(y)\cdot\nu(y) = -\frac{1}{2\pi}\nabla(\ln|y|)\cdot\nu(y) \\
&= -\frac{1}{2\pi}\frac{1}{|y|}\frac{y}{|y|}\cdot\left(-\frac{y}{|y|}\right) = \frac{1}{2\pi|y|} = \frac{1}{2\pi\varepsilon}, \quad y\in\partial B(0,\varepsilon), \quad j = 1,\dots,n.
\end{aligned}
$$

This is precisely the same formula as in the case $n \geqslant 3$ for $n = 2$. Thus for $y \in \partial B(0,\varepsilon)$ we have

$$
\frac{\partial\Phi}{\partial\nu}(y) = \frac{1}{2\pi\varepsilon} = \frac{1}{|\partial B(0,\varepsilon)|}
$$

and the rest is exactly the same as in the higher dimensional case. Thus

$$
\lim_{\varepsilon\to 0} L_\varepsilon = -f(x)
$$

also when $n = 2$.

Step 7: Gathering all estimates together we see that

$$\Delta u(x) = I_\varepsilon + J_\varepsilon = \underbrace{I_\varepsilon}_{\to 0} + \underbrace{K_\varepsilon}_{\to 0} + \underbrace{L_\varepsilon}_{\to -f(x)} \to -f(x) \quad \text{as} \quad \varepsilon \to 0$$

and we find that u solves the Poisson equation $\Delta u = -f$. □

Remark 4.22. We collect here basic results on volumes related to balls and spheres. Since the n-dimensional volume with $n \geqslant 2$ of a ball $B(x,r)$ is translation invariant and scales to the power n, we have

$$|B(x,r)| = |B(0,r)| = r^n |B(0,1)| = \alpha(n) r^n,$$

where $\alpha(n)$ is the n-dimensional volume of the unit ball $B(0,1)$. On the other hand, since the $(n-1)$-dimensional volume of the sphere $\partial B(x,r)$ is translation invariant and scales to the power $n-1$, we have

$$|\partial B(x,r)| = |\partial B(0,r)| = r^{n-1} |\partial B(0,1)| = \beta(n) r^{n-1},$$

where $\beta(n)$ is the $(n-1)$-dimensional volume of the unit sphere $\partial B(0,1)$. Moreover,

$$\alpha(n) = \int_{B(0,1)} 1\,dx = \int_0^1 \int_{\partial B(0,r)} 1\,dS\,dr = \int_0^1 \beta(n) r^{n-1}\,dr = \frac{\beta(n)}{n}$$

Thus

$$|\partial B(0,1)| = \beta(n) = n\alpha(n) = n|B(0,1)|.$$

By Example 3.13

$$\int_{\mathbb{R}^n} e^{-|x|^2}\,dx = \pi^{\frac{n}{2}}.$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-|x|^2}\,dx &= \int_0^\infty \int_{\partial B(0,r)} e^{-|x|^2}\,dS\,dr = \beta(n) \int_0^\infty e^{-r^2} r^{n-1}\,dr \\ &= \frac{\beta(n)}{2} \int_0^\infty e^{-s} s^{\frac{n}{2}-1}\,ds = \frac{\beta(n)}{2} \Gamma\left(\frac{n}{2}\right).\end{aligned}$$

Thus

$$\beta(n) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

and

$$\alpha(n) = \frac{\beta(n)}{n} = \frac{2\pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)} = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

The last equality follows by a change of variables in the definition of the Gamma function. This shows that the volumes of a ball and a sphere can be represented in terms of the gamma function.

4.6 The Green's function

In this section we show how we can use the fundamental solution of the Laplace equation in the whole space to solve a Dirichlet problem in a subdomain. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with a smooth boundary and assume that $f \in C_0^\infty(\Omega)$. We consider the Dirichlet problem for the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega. \end{cases}$$

Our goal is to derive a general representation formula for the solution of this problem using potential functions and so-called Green's function.

Assume that $u \in C^2(\overline{\Omega}))$. Let $\varepsilon > 0$ be small enough so that $\overline{B(x,\varepsilon)} \subset \Omega$. Recall that Φ is the fundamental solution of the Laplace equation in Defintion 4.16. By Green's second identity, we have

$$\begin{aligned} &\int_{\Omega \setminus B(x,\varepsilon)} \big(u(y) \underbrace{\Delta_y(\Phi(y-x))}_{=0} - \Phi(y-x)\Delta_y u(y)\big)\, dy \\ &= \int_{\partial B(x,\varepsilon) \cup \partial\Omega} u(y) \frac{\partial \Phi}{\partial \nu}(y-x)\, dS(y) - \int_{\partial B(x,\varepsilon) \cup \partial\Omega} \Phi(y-x) \frac{\partial u}{\partial \nu}(y)\, dS(y) \qquad (4.2) \\ &= \int_{\partial B(x,\varepsilon)} \cdots dS(y) + \int_{\partial\Omega} \cdots dS(y) - \int_{\partial B(x,\varepsilon)} \cdots dS(y) - \int_{\partial\Omega} \cdots dS(y) \\ &= I_1(\varepsilon) + I_2 - I_3(\varepsilon) - I_4. \end{aligned}$$

Here ν denotes, as usual, the outward pointing unit normal of $\Omega \setminus B(x,\varepsilon)$ on $\partial\Omega \cup \partial B(x,\varepsilon)$. Let us consider the terms above separately.

$\boxed{I_3(\varepsilon)}$

$$\begin{aligned} |I_3(\varepsilon)| &= \left| \int_{\partial B(x,\varepsilon)} \Phi(y-x) \frac{\partial u}{\partial \nu}(y)\, dS(y) \right| \\ &\leqslant \int_{\partial B(x,\varepsilon)} |\Phi(y-x)||\nabla u(y)||\nu(y)|\, dS(y) \\ &\leqslant \sup_{y \in \overline{\Omega}} |\nabla u(y)| \int_{\partial B(x,\varepsilon)} |\Phi(\varepsilon)|\, dy \\ &\leqslant \begin{cases} \sup_{y\in\overline{\Omega}} |\nabla u(y)| \varepsilon |\ln \varepsilon|, & n = 2, \\ c \sup_{y\in\overline{\Omega}} |\nabla u(y)| \varepsilon^{n-1} \varepsilon^{2-n}, & n \geqslant 3. \end{cases} \end{aligned}$$

Thus

$$\lim_{\varepsilon \to 0} I_3(\varepsilon) = 0.$$

$\boxed{I_1(\varepsilon)}$ By the calculation in (4.1), we obtain

$$\frac{\partial \Phi}{\partial \nu}(y-x) = \frac{1}{n\alpha(n)|y-x|^{n-1}} = \frac{1}{n\alpha(n)\varepsilon^{n-1}}, \quad y \in \partial B(x,\varepsilon),$$

where ν is the outward pointing unit normal of $\Omega \setminus B(x,\varepsilon)$ on $\partial B(x,\varepsilon)$. A substitution of this in $I_1(\varepsilon)$ gives

$$\begin{aligned}
I_1(\varepsilon) &= \int_{\partial B(x,\varepsilon)} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y)\\
&= \frac{1}{n\alpha(n)\varepsilon^{n-1}}\int_{\partial B(x,\varepsilon)} u(y)\,dS(y)\\
&= \frac{1}{|\partial B(x,\varepsilon)|}\int_{\partial B(x,\varepsilon)} u(y)\,dS(y) \to u(x) \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}$$

Thus

$$\lim_{\varepsilon\to 0} I_1(\varepsilon) = u(x).$$

Finally we recall that Φ is harmonic away from zero so that $\Delta_y \Phi(y-x) = 0$ for $y \neq x$. Thus by rearranging terms in (4.2) and letting $\varepsilon \to 0$ we conclude that

$$\begin{aligned}
u(x) &= \lim_{\varepsilon\to 0} I_1(\varepsilon)\\
&= \underbrace{\lim_{\varepsilon\to 0} I_3(\varepsilon)}_{=0} + I_4 - I_2 - \lim_{\varepsilon\to 0}\int_{\Omega\setminus B(x,\varepsilon)} \Phi(y-x)\Delta_y u(y)\,dy\\
&= \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) - \int_{\partial\Omega} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y)\\
&\qquad - \int_{\Omega} \Phi(y-x)\Delta u(y)\,dy, \quad x \in \Omega.
\end{aligned}$$

We state this result as a theorem.

Theorem 4.23. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set with a smooth boundary and $u \in C^2(\overline{\Omega})$. Then

$$\begin{aligned}
u(x) &= \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) - \int_{\partial\Omega} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y)\\
&\qquad - \int_{\Omega} \Phi(y-x)\Delta u(y)\,dy,
\end{aligned} \tag{4.3}$$

for every $x \in \Omega$. Here ν denotes, as usual, the outward pointing unit normal vector of Ω.

THE MORAL: This representation formula holds for any function $u \in C^2(\overline{\Omega})$. In particular, the function u does not need to be a solution to a PDE. This gives a representation formula for a function inside the domain by its values on the boundary of the domain. More precisely, this allows us to determine u if we know the value of Δu in Ω as well as the value of u and the normal derivative on the boundary $\partial\Omega$. This is useful in boundary value problems for PDEs. Observe, that for the Dirichlet problem for the Laplace equation we cannot describe both u and $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$, since the solution is uniquely determined already by u on $\partial\Omega$.

Remarks 4.24:

(1) If $u \in C_0^2(\mathbb{R}^n)$, then by applying (4.3) in a ball $B(0,r)$ where $r > 0$ is chosen so large that $\operatorname{supp} u \subset B(0,r)$, the integrals in (4.3) over the boundary $\partial B(0,r)$ are zero and thus we have

$$u(x) = -\int_{\mathbb{R}^m} \Phi(y-x)\Delta_y u(y)\,dy$$

for every $x \in B(0,r)$. Observe that $u(x) = 0$ for every $x \in \mathbb{R}^n \setminus B(0,r)$.

(2) If $\Delta u = 0$ in Ω, then by (4.3), we have

$$u(x) = \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) - \int_{\partial\Omega} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y), \quad x \in \Omega. \tag{4.4}$$

The first integral on the right-hand side is called the single layer potential with charge density $\frac{\partial u}{\partial \nu}(y)$ and the second integral is called the double layer potential with dipole moment density $u(y)$. The latter represents the potential induced by a double layer of charges of opposite sign on $\partial\Omega$ and the former represents the potential induced by a single layer of charges on $\partial\Omega$. Note that these potentials are harmonic in Ω and in $\mathbb{R}^n \setminus \overline{\Omega}$.

(3) Representation formula (4.4) implies that if $u \in C^2(\Omega)$ is a harmonic function in Ω, then $u \in C^\infty(\Omega)$. This means that every harmonic function is smooth. Indeed, since there is no singularity in the integrand, the derivatives can be taken inside the integral.

(4) By choosing $u = 1$ in (4.4), we have

$$-\int_{\partial\Omega} \frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y) = 1$$

for every $x \in \Omega$. This is related to the normalization of the fundamental solution in Defintion 4.16.

Let us look at the representation formula (4.3) in connection with the Dirichlet problem. We require that $\Delta u = -f$ in Ω. The boundary condition specifies the values of $u = g$ on the boundary $\partial\Omega$, but the normal derivative $\frac{\partial u}{\partial \nu}$ is unknown. We solve this problem by adding a harmonic function to the fundamental solution. For a fixed $x \in \Omega$, let $\phi^x = \phi^x(y)$ be a corrector function, which is a solution the Dirichlet problem

$$\begin{cases} \Delta_y \phi^x(y) = 0, & y \in \Omega, \\ \phi^x(y) = \Phi(y-x), & y \in \partial\Omega. \end{cases}$$

T H E M O R A L : We solve a Dirichlet problem with boundary values given by the fundamental solution of the Laplace equation in $\mathbb{R}^n$.

Applying Green's second identity and using the fact that $\Delta\phi^x = 0$ in Ω we have

$$
\begin{aligned}
-\int_\Omega \phi^x(y)\Delta u(y)\,dy &= \int_\Omega \Bigg(u(y)\underbrace{\Delta\phi^x(y)}_{=0} - \phi^x(y)\Delta u(y)\Bigg)\,dy \\
&= \int_{\partial\Omega} \left(u(y)\frac{\partial\phi^x}{\partial\nu}(y) - \phi^x(y)\frac{\partial u}{\partial\nu}(y)\right) dS(y) \qquad (4.5)\\
&= \int_{\partial\Omega} \left(u(y)\frac{\partial\phi^x}{\partial\nu}(y) - \Phi(y-x)\frac{\partial u}{\partial\nu}(y)\right) dS(y).
\end{aligned}
$$

In the last equality we used the fact that $\phi^x(y) = \Phi(y-x)$ for $y \in \partial\Omega$.

Definition 4.25. The Green's function for Ω is

$$G(x,y) = \Phi(y-x) - \phi^x(y), \quad x,y \in \Omega, \quad x \neq y,$$

where $\phi^x = \phi^x(y)$ is a solution the Dirichlet problem

$$\begin{cases} \Delta_y \phi^x(y) = 0, & y \in \Omega, \\ \phi^x(y) = \Phi(y-x), & y \in \partial\Omega. \end{cases}$$

THE MORAL: For a fixed $x \in \Omega$, the function $y \mapsto G(x,y)$ differs from the function $y \mapsto \Phi(y-x)$ by a harmonic function. Thus Green's function is obtained by adding a harmonic function to the fundamental solution. The function $y \mapsto G(x,y)$ is zero on the boundary $\partial\Omega$ and has a singularity at x.

Remarks 4.26:

(1) Let us symbolically write δ_x for the generalized function (distribution) that has the property

$$\int_{\mathbb{R}^n} \phi(y)\delta_x(y)\,dy = \phi(x) \quad \text{for all} \quad \phi \in C_0^\infty(\mathbb{R}^n).$$

This generalized function is called Dirac's delta mass at x. Theorem 4.19 showed that $\Phi * f$ solves the Poisson equation $-\Delta u = f$ on $\mathbb{R}^n$. Formally we have $-\Delta\Phi = \delta_0$ and

$$-\Delta u(x) = -\int_{\mathbb{R}^n} \Delta\Phi(y) f(x-y)\,dy = \int_{\mathbb{R}^n} \delta_0(y) f(x-y)\,dy = f(x).$$

The generalized function δ_0, roughly speaking, can be interpreted as an object that has mass one at the point 0 and is zero everywhere else. Physically this means that we set a unit charge at the origin and the fundamental solution Φ is the induced potential.

(2) For a fixed $x \in \Omega$, consider the Green's function $G(x,y)$ as a function of y. Then

$$\begin{cases} -\Delta_y G(x,y) = \delta_x, & y \in \Omega, \\ G(x,y) = 0, & y \in \partial\Omega, \end{cases}$$

where δ_x a Dirac mass at $x \in \Omega$. In particular, the Green's function $G(x,y)$ is a harmonic function of y in $\Omega \setminus \{x\}$ and $G(x,y)$ has zero boundary values on $\partial\Omega$. The boundary condition follows from the fact that $\phi^x(y) = \Phi(y-x)$ for $y \in \partial\Omega$. Physically this means that we set a unit charge at the point x and require that the induced potential is zero on the boundary, that is, the boundary is grounded. Observe, that the Green's function depends only on the domain.

(3) The Green's function is unique, whenever it exists, since the difference of any two Green's functions is a harmonic function with zero boundary values.

With the definition of the Green's function and adding (4.3) to (4.5), we have

$$\begin{aligned} u(x) = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(y) \underbrace{(\Phi(y-x) - \Phi(y-x))}_{=0} \, dS(y) \\ - \int_{\partial\Omega} u(y) \left(\frac{\partial \Phi}{\partial \nu}(y-x) - \frac{\partial \phi^x}{\partial \nu}(y) \right) dS(y) \\ + \int_{\Omega} \Delta u(y)(\phi^x(y) - \Phi(y-x))\, dy \end{aligned}$$

$$= -\int_{\partial\Omega} u(y) \frac{\partial G}{\partial \nu}(x,y)\, dS(y) - \int_{\Omega} \Delta u(y) G(x,y)\, dy$$

where

$$\frac{\partial G}{\partial \nu}(x,y) = \nabla_y G(x,y) \cdot \nu(y)$$

and ν is the exterior unit normal on $\partial\Omega$.

Theorem 4.27. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set with a smooth boundary and $u \in C^2(\overline{\Omega})$. Let G be the Green's function for Ω. Then

$$u(x) = -\int_{\partial\Omega} u(y) \frac{\partial G}{\partial \nu}(x,y)\, dS(y) - \int_{\Omega} \Delta u(y) G(x,y)\, dy$$

for every $x \in \Omega$.

THE MORAL: This representation formula holds for any function $u \in C^2(\overline{\Omega})$. This allows us to determine u if we know the value of Δu in Ω, the value of u on the boundary $\partial\Omega$ and the Green's function for Ω. In contrast with (4.3), there is no normal derivative on the boundary.

The definition of the Green's function is based on solving a Dirichlet' problem with boundary values given by the fundamental solution, but the Green's function can be used to give a solution to a general Dirichlet problem.

Theorem 4.28 (Solution of the Poisson equation in a subdomain). Assume that u is a solution of the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega, \end{cases}$$

then

$$u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial \nu}(x,y)\,dS(y) + \int_{\Omega} f(y)G(x,y)\,dy,$$

where G is the Green's function for Ω.

THE MORAL: This is an explicit representation formula solution for the Poisson equation in a bounded subdomain. Note that $u = u_1 + u_2$, where u_1 is a solution to the corresponding homogeneous Dirichlet problem and u_2 is a solution to a nonhomogeneous problem with zero boundary values as in Remark 4.14 (3). The representation formula depends on the construction of the Green's function for Ω, which is a difficult task and depends heavily on the geometry of Ω. We shall derive the Green's functions of some relatively simple domains soon.

Remark 4.29. The solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega, \end{cases}$$

is

$$u(x) = \int_{\partial\Omega} H(x,y)g(y)\,dS(y),$$

where

$$H(x,y) = -\frac{\partial G}{\partial \nu}(x,y)$$

is called the Poisson kernel in Ω.

4.7 The Green's function for the upper half-space*

Consider the upper half-space $\mathbb{R}^n_+ = \{(x_1,\dots,x_n) : x_n > 0\}$. Although this domain is unbounded, and the arguments in the previous section do not directly apply, we shall determine its Green's function by the reflection method. After we have done that we have to check whether representation formula really gives a solution to the problem. In order to construct the Green's function, for every $x \in \mathbb{R}^n_+$, we need to construct a corrector function ϕ^x such that

$$\begin{cases} \Delta\phi^x(y) = 0, & y \in \mathbb{R}^n_+, \\ \phi^x(y) = \Phi(y-x), & y \in \partial\mathbb{R}^n_+. \end{cases}$$

The Green's function for $\mathbb{R}^n_+$ will then be $G(x,y) = \Phi(y-x) - \phi^x(y)$. Here Φ is the fundamental solution of the Laplace equation in $\mathbb{R}^n$, see Definition 4.16.

Let $x = (x_1,\dots,x_n) \in \mathbb{R}^n_+$. Reflecting the vector x across the boundary $\partial\mathbb{R}^n_+$ gives the point

$$x^* = (x_1,\dots,x_{n-1},-x_n).$$

Observe that x^* belongs to the lower half-space of $\mathbb{R}^n$. For x, $y \in \mathbb{R}^n_+$ we set

$$\phi^x(y) = \Phi(y-x^*).$$

The singularity at $x \in \mathbb{R}^n_+$ is reflected to x^*. Observe that, if $y \in \partial\mathbb{R}^n_+$, then

$$\phi^x(y) = \Phi(y-x^*) = \Phi(y-x)$$

since Φ is a radial function. Furthermore, since $x^* \notin \mathbb{R}^n_+$, we have $y-x^* \neq 0$ and thus ϕ^x is harmonic in $\mathbb{R}^n_+$. This shows that ϕ^x is a solution of the desired boundary value problem. This is called the reflection method.

Theorem 4.30. The Green's function for the upper half-space $\mathbb{R}^n_+$ is

$$G(x,y) = \Phi(y-x) - \Phi(y-x^*), \quad x,y \in \mathbb{R}^n_+, \quad x \neq y.$$

The we consider the solution of the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^n_+, \\ u = g & \text{on} \quad \partial\mathbb{R}^n_+. \end{cases} \tag{4.6}$$

The representation formula in Remark 4.29 gives

$$u(x) = -\int_{\partial\mathbb{R}^n_+} \frac{\partial G}{\partial \nu}(x,y) g(y)\, dS(y).$$

We derive an explicit expression for $\frac{\partial G}{\partial \nu}(x,y)$ in the upper half-space. By the definition of the fundamental solution for $n \geqslant 3$, see Definition 4.16, we have

$$G(x,y) = \frac{1}{n(n-2)\alpha(n)} \left(\frac{1}{|x-y|^{n-2}} - \frac{1}{|x^*-y|^{n-2}} \right).$$

Thus

$$\begin{aligned}
\frac{\partial G}{\partial \nu}(x,y) = \nabla G(x,y)\cdot \nu(y) &= -\frac{\partial G}{\partial y_n}(x,y) \\
&= -\frac{\partial \Phi}{\partial y_n}(y-x) + \frac{\partial \Phi}{\partial y_n}(y-x^*) \\
&= \frac{1}{n\alpha(n)} \left(\frac{y_n - x_n}{|x-y|^n} - \frac{y_n + x_n}{|x-y|^n} \right) \\
&= -\frac{2}{n\alpha(n)} \frac{x_n}{|x-y|^n}.
\end{aligned}$$

This holds also when $n = 2$ (exercise). By inserting this into the representation formula above, we obtain

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\,dy, \quad x \in \mathbb{R}^n_+, \quad y \in \partial\mathbb{R}^n_+.$$

The function

$$K(x,y) = \frac{2x_n}{n\alpha(n)} \frac{1}{|x-y|^n}, \quad x \in \mathbb{R}^n_+, \quad y \in \partial\mathbb{R}^n_+.$$

is the Poisson kernel of $\mathbb{R}^n_+$ and

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\,dy = \int_{\partial\mathbb{R}^n_+} K(x,y)g(y)\,dy, \quad x \in \mathbb{R}^n_+, \quad y \in \partial\mathbb{R}^n_+.$$

This is a convolution of g with the Poisson kernel. Previously, we derived the same formula using the Fourier transform, see Theorem 3.27. Observe that in Theorem 3.27 we consider the Poisson formula in $\mathbb{R}^{n+1}_+$, given by Lemma 3.24, instead of $\mathbb{R}^n_+$. Thus the formulas look different, but they are same (exercise).

Theorem 4.31 (Poisson formula in the upper half-space). The solution of the Dirichlet problem (4.6) is given by

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\,dy. \tag{4.7}$$

THE MORAL: This is an explicit representation formula for the solution of the Dirichlet problem in the upper half-space. The Poisson kernel for the upper half-space can be computed using the Green's function. We already know that the u given above solves the Dirichlet problem for the Laplace equation in the upper half-space with boundary data g by the approximation of the identity.

Example 4.32. For a harmonic function in the upper half plane $\mathbb{R}^2_+$, formula (4.7) gives

$$u(x,y) = \frac{y}{\pi} \int_{\mathbb{R}} \frac{g(z)}{(x-z)^2 + y^2}\,dz,$$

where $-\infty < x < \infty$ and $y > 0$.

4.8 The Green's function for a ball*

Let Ω to be the unit ball $B(0,1) \subset \mathbb{R}^n$. We shall again use the method of reflection to construct the Green's function. We need to find the corrector function ϕ^x, for every $x \in B(0,1)$, such that

$$\begin{cases} \Delta\phi^x(y) = 0, & y \in B(0,1), \\ \phi^x(y) = \Phi(y-x), & y \in \partial B(0,1). \end{cases}$$

the Green's function for $B(0,1)$ will then be $G(x,y) = \Phi(y-x) - \phi^x(y)$. For any point $x \in B(0,1)$ we will again reflect the point with across to the boundary $\partial B(0,1)$ as follows. For $x \in B(0,1)$, we define

$$x^* = \frac{x}{|x|^2}.$$

Observe that $|x| < 1$ implies that $|x^*| = \frac{1}{|x|} > 1$ and thus $x^* \notin B(0,1)$. The following calculations are essentially the same when $n = 2$ so we give the details only for the case $n \geqslant 3$. Set

$$\phi^x(y) = \Phi(|x|(y-x^*)).$$

Then

$$\phi^x(y) = \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}|y-x^*|^{n-2}} = \frac{1}{|x|^{n-2}} \Phi(y-x^*).$$

Since $y \neq x^*$ the function $\phi^x(y)$ is harmonic in $B(0,1)$ (as a function of y) as long as $x \neq 0$. Let us check what happens on the boundary. For $y \in \partial B(0,1)$ and $x \neq 0$ we have

$$\begin{aligned}
|x|^2|y-x^*|^2 &= |x|^2\left(y - \frac{x}{|x|^2}\right)\cdot\left(y - \frac{x}{|x|^2}\right) = |x|^2\left(y\cdot y + \frac{x}{|x|^2}\cdot\frac{x}{|x|^2} - \frac{2x\cdot y}{|x|^2}\right) \\
&\overset{(*)}{=} |x|^2\left(1 + \frac{1}{|x|^2} - \frac{2x\cdot y}{|x|^2}\right) = |x|^2 + 1 - 2x\cdot y \\
&= (x-y)\cdot(x-y) = |x-y|^2,
\end{aligned}$$

where in (*) we used that $y \cdot y = |y|^2 = 1$ since $y \in \partial B(0,1)$. The previous calculation implies $|x||y-x^*| = |y-x|$ when $y \in \partial B(0,1)$ so that

$$\phi^x(y) = \Phi(|x|(y-x^*)) = \frac{1}{n(n-2)\alpha(n)} \frac{1}{(|x||y-x^*|)^{n-2}} = \Phi(x-y)$$

when $y \in \partial B(0,1)$. Thus ϕ^x is the corrector function for $B(0,1)$ and the Green's function for $B(0,1)$ becomes

$$\begin{aligned}
G(x,y) &= \Phi(y-x) - \phi^x(y) \\
&= \Phi(y-x) - \Phi(|x|(y-x^*)), \quad x,y \in B(0,1), \quad x \neq y, \quad x \neq 0.
\end{aligned}$$

Theorem 4.33. The Green's function for the ball $B(0,1)$ is

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-x^*)), \quad x,y \in B(0,1), \quad x \neq y, \quad x \neq 0.$$

Let us now consider the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad B(0,1), \\ u = g & \text{on} \quad \partial B(0,1). \end{cases}$$

The general formula for the solution in Remark 4.29 is

$$u(x) = -\int_{\partial B(0,1)} \frac{\partial G}{\partial \nu}(x,y) g(y)\, dS(y)$$

where $\nu(y) = y/|y|$ is the exterior unit normal. Clearly

$$\frac{\partial G}{\partial \nu}(x,y) = \nabla_y G(x,y) \cdot \nu(y) = \nabla_y G(x,y) \cdot \frac{y}{|y|}.$$

We calculate the partial derivatives of G (again for $n \geqslant 3$), when $y \in \partial B(0,1)$, and have

$$\begin{aligned}
\frac{\partial G}{\partial y_j}(x,y) &= \frac{\partial \Phi}{\partial y_j}(y-x) - \frac{\partial}{\partial y_j}\big(\Phi(|x|(y-x^*)\big) \\
&= -\frac{1}{n\alpha(n)}\left(\frac{1}{|y-x|^{n-1}}\frac{y_j-x_j}{|y-x|} - \frac{1}{||x|(y-x^*)|^{n-1}}\frac{|x|(y_j-x_j^*)}{|x||y-x^*|}|x|\right) \\
&= -\frac{1}{n\alpha(n)}\left(\frac{y_j-x_j}{|y-x|^n} - \frac{|x|^2 y_j - x_j}{|y-x|^n}\right) \\
&= -\frac{1}{n\alpha(n)}\frac{1-|x|^2}{|x-y|^n}y_j, \quad j = 1,\dots,n.
\end{aligned}$$

Thus

$$\nabla_y G(x,y) = -\frac{1}{n\alpha(n)}\frac{1-|x|^2}{|x-y|^n}y$$

and

$$\frac{\partial G}{\partial \nu}(x,y) = \nabla_y G(x,y)\cdot\nu(y) = -\frac{1}{n\alpha(n)}\frac{1-|x|^2}{|x-y|^n}y\cdot\frac{y}{|y|} = -\frac{1}{n\alpha(n)}\frac{1-|x|^2}{|x-y|^n},$$

since $|y|^2 = 1$ when $y \in \partial B(0,1)$. Thus we conclude that

$$u(x) = \frac{1}{n\alpha(n)}\int_{\partial B(0,1)}\frac{1-|x|^2}{|x-y|^n}g(y)\,dS(y)$$

is the solution for the Dirichlet problem for the Laplace equation in the unit ball $B(0,1)$.

The Dirichlet problem for $B(0,r)$, $r > 0$, is

$$\begin{cases} \Delta u = 0 & \text{in} \quad B(0,r), \\ u = g & \text{on} \quad \partial B(0,r). \end{cases} \tag{4.8}$$

Then by a change of variables, we have

$$u(x) = \frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B(0,r)}\frac{g(y)}{|x-y|^n}\,dS(y), \quad x \in B(0,r).$$

The function

$$K(x,y) = \frac{r^2-|x|^2}{n\alpha(n)r}\frac{1}{|x-y|^n}, \quad x \in B(0,r), \quad y \in \partial B(0,r),$$

is called the Poisson kernel for the ball $B(0,r)$ and

$$u(x) = \frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B(0,r)}\frac{g(y)}{|x-y|^n}\,dS(y) = \int_{\partial B(0,r)}K(x,y)g(y)\,dS(y), \quad x \in B(0,r),$$

is the solution of the Dirichlet problem above.

Remark 4.34. The Poisson kernel has the following properties (exercise).

(1) $K(x,y)$ is a smooth function of $x \in B(0,r)$ for any fixed $y \in \partial B(0,r)$.

(2) $K(x,y) > 0$ for every $x \in B(0,r)$ and $y \in \partial B(0,r)$.

(3) For any fixed $x_0 \in \partial B(0,r)$ and $\delta > 0$,

$$\lim_{x \to x_0, x \in B(0,r)} K(x,y) = 0$$

for every $y \in \partial B(0,r) \setminus B(x_0,\delta)$.

(4) $\Delta_x K(x,y) = 0$ for every $x \in B(0,r)$ and $y \in \partial B(0,r)$.

(5) $\displaystyle\int_{\partial B(0,r)} K(x,y)\,dS(y) = 1$ for every $x \in B(0,r)$.

Note that these properties are analogous to the properties of an approximation of the identity, but now the kernel function is defined on the sphere.

Theorem 4.35 (Poisson formula in the ball). The solution of the Dirichlet problem (4.8) is given by

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\,dS(y).$$

THE MORAL: This is an explicit representation formula for the solution of the Dirichlet problem in a ball. The Poisson kernel for the ball can be computed using the Green's function for the corresponding ball.

Remark 4.36. In dimension $n = 2$ the unit ball $B(0,1)$ is a disc. Let us write $z = (r\cos\theta, r\sin\theta)$ in polar coordinates. Since $y \in \partial B(0,1)$ we can write $y = (\cos\phi, \sin\phi)$ and integrate in ϕ instead. We can calculate

$$\begin{aligned}
|z-y|^2 &= (r\cos\theta - \cos\phi)^2 + (r\sin\theta - \sin\phi)^2 \\
&= r^2 + 1 - 2r(\cos\phi\cos\theta + \sin\phi\sin\theta) \\
&= 1 - 2r\cos(\theta - \phi) + r^2
\end{aligned}$$

Thus the previous formula becomes

$$u(r,\theta) = \frac{1-r^2}{2\pi} \int_{-\pi}^{\pi} \frac{g(\cos\phi, \sin\phi)}{1 - 2r\cos(\theta-\phi) + r^2}\,d\phi,$$

where, in the last equality, we abuse notation and write $g(\phi)$ for $g(\cos\phi, \sin\phi)$. Again, we recover our familiar Poisson kernel for the disc in two dimensions, see Theorem 2.43.

4.9 Mean value formulas

Let $\Omega \subset \mathbb{R}^n$ be an open set and assume that u is a harmonic function in Ω. An important property of harmonic functions is that their value at every point $x \in \Omega$

equals the average of the function over balls $B(x,r)$ or spheres $\partial B(x,r)$ whenever $\overline{B(x,r)} \subset \Omega$. In one dimension, the Laplace equation is $u'' = 0$ and the harmonic functions are of the form $u(x) = ax + b$, where $a, b \in \mathbb{R}$. The value of this function at the midpoint of a finite interval is the arithmetic average of the values at the endpoints and the integral average over the interval. This is the one-dimensional version of the mean value formula.

Theorem 4.37 (Mean value formulas for harmonic functions). Let $u \in C^2(\Omega)$ be harmonic in Ω. Then for every ball $B(x,r)$ such that $\overline{B(x,r)} \subset \Omega$ we have

$$u(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\,dy = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\,dS(y).$$

THE MORAL: The value of a harmonic function at a point is the equal to the average of its values over any ball or sphere centered at that point. Recall that by approximation of the identity, see Example 3.23, for every function $u \in C(\Omega)$ we have

$$u(x) = \lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\,dy$$

and, as in the proof of Theorem 4.19, we obtain

$$u(x) = \lim_{r \to 0} \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\,dS(y)$$

for every $x \in \Omega$. For a harmonic function equalities hold without taking the limit.

Proof. Set

$$\begin{aligned}
\phi(r) &= \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\,dS(y) \\
&= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0,1)} u(x+rz) r^{n-1}\,dS(z) \\
&= \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} u(x+rz)\,dS(z),
\end{aligned}$$

where we used the change of variables $y = x + rz$, so that $z = \frac{1}{r}(y-x)$ and $dS(y) = r^{n-1}\,dS(z)$. Recall the formulas for the measure of the ball and the sphere from Remark 4.22. This change of variables takes the sphere $\partial B(x,r)$ to the sphere $\partial B(0,1)$. By differentiating with respect to r under the integral sign and using the chain rule, we get

$$\begin{aligned}
\phi'(r) &= \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} \nabla u(x+rz) \cdot z\,dS(z) \\
&= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} \nabla u(y) \cdot \frac{y-x}{r}\,dS(y) \\
&= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} \nabla u(y) \cdot \nu(y)\,dS(y),
\end{aligned}$$

where we used the change of variables $y = x + rz$, so that so that $z = \frac{1}{r}(y-x)$ and $dS(z) = r^{1-n}\,dS(y)$. Observe that this change of variables takes $\partial B(0,1)$ back to $\partial B(x,r)$. Note that $\nu(y) = (y-x)/r$ is the outer normal unit vector on the sphere $\partial B(x,r)$. By the Green's third identity

$$\begin{aligned}\frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(x,r)} \nabla u(y)\cdot\nu(y)\,dS(y) &= \frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(x,r)} \frac{\partial u}{\partial\nu}(y)\,dS(y)\\ &= \frac{1}{n\alpha(n)r^{n-1}}\int_{B(x,r)} \Delta u(y)\,dy = 0,\end{aligned}$$

since u is harmonic in $B(x,r)\subset\Omega$. Thus $\phi'(r) = 0$ which implies that ϕ is a constant function. Recall that

$$\lim_{t\to 0}\frac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} u(y)\,dS(y) = u(x)$$

whenever u is continuous, as in the proof of Theorem 4.19. Since ϕ is constant, we have

$$\phi(r) = \lim_{t\to 0}\phi(t) = \lim_{t\to 0}\frac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} u(y)\,dS(y) = u(x).$$

This shows that

$$u(x) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y)\,dS(y).$$

On the other hand,

$$\begin{aligned}\int_{B(x,r)} u(y)\,dy &= \int_0^r\int_{\partial B(x,s)} u(y)\,dS(y)\,ds = \int_0^r n\alpha(n)s^{n-1}u(x)\,ds\\ &= n\alpha(n)\frac{r^n}{n}u(x) = \alpha(n)r^n u(x) = |B(x,r)|u(x),\end{aligned}$$

which shows the mean value formula for balls as well. □

Remark 4.38. The mean value property of harmonic functions follows also from the Poisson formula for the ball in Theorem 4.35 (exercise).

We now state a converse to the mean value property. It states that the mean value property characterizes harmonic functions.

Theorem 4.39. If $u \in C(\Omega)$ satisfies

$$u(x) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y)\,dS(y)$$

for all balls $\overline{B(x,r)}\subset\Omega$, then u is harmonic in Ω.

THE MORAL: The mean value property is a special property of harmonic functions and it does not hold as such for other PDE.

Proof. We shall prove the claim under additional assumption that $u \in C^2(\Omega)$. A convolution approximation will give the general result, but this will be omitted here (exercise). If $\Delta u(x) \neq 0$ for some $x \in \Omega$ then there is a ball $B(x,r) \subset \Omega$ such that $\Delta u(y) > 0$ for every $y \in B(x,r)$. As in the previous proof we have $\phi'(r) = 0$, because of the hypothesis that the averages with respect to balls centered at x are constant and equal to $u(x)$. Thus we have

$$0 = \phi'(r) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y)\,dy > 0,$$

which is a contradiction. □

Remark 4.40. It follows from the proof above that a continuous function u is harmonic if and only if for every point in the domain of definition the mean value property holds true for small enough balls centered at the point.

4.10 Maximum principles

Recall, that a continuous function attains its maximum and minimum values on a closed and bounded set. The maximum principle asserts that if, in addition, the function is harmonic, then it must attain its maximum and minimum at the boundary of the set. Physically, the Laplace equation governs the steady state temperature distribution of a body. If the body is in thermal equilibrium, there cannot be no internal hot or cold spots, since otherwise the heat energy would flow from hot to cold. In other words, the temperature cannot have maximum or minimum inside the body unless the temperature is constant throughout the entire body. The maximum and minimum principles have several useful applications in PDE. Recall, that an open set Ω is connected, if every pair of points in Ω can be connected by a piecewise linear path in Ω.

Theorem 4.41. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set and assume that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a harmonic function in Ω.

(1) (Weak maximum principle) Then

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x).$$

(2) (Strong maximum principle) If Ω is a connected set and there exists $x_0 \in \Omega$ such that

$$u(x_0) = \max_{x \in \overline{\Omega}} u(x),$$

then u is constant in Ω.

Remark 4.42. By replacing u by $-u$ we get the minimum principles with min replacing max in the maximum principle (exercise).

THE MORAL: According to the weak maximum principle, a harmonic function cannot have a strict interior maximum or minimum point. It attains the minimum and maximum on the boundary. The strong maximum principle asserts that a harmonic function can attain an interior minimum or maximum only if it is a constant function. Observe that this result holds for all bounded domains without any regularity assumption on the boundary.

Proof. $\boxed{(2)}$ Suppose that there exists $x_0 \in \Omega$ such that

$$u(x_0) = \max_{x \in \overline{\Omega}} u(x) = M.$$

Let $0 < r < \operatorname{dist}(x_0, \partial\Omega)$. Then $B(x_0,r) \subset \Omega$ and the mean value property implies that

$$M = u(x_0) = \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} u(y)\,dy \leqslant \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} M\,dy = M.$$

It follows that an equality holds throughout and thus

$$\int_{B(x_0,r)} (M - u(y))\,dy = 0.$$

Since $M - u(y) \geqslant 0$, we conclude that $u(y) = M$ for every $y \in B(x_0,r)$.

Now consider any point $x \in \Omega$. Since Ω is connected, we can find a sequence of balls $B(x_0,r), B(x_1,r), \dots, B(x_N,r) \subset \Omega$ such that

$$x_{j+1} \in B(x_j,r) \quad \text{for every} \quad j = 0,1,\dots,N-1$$

and $x = x_N$.

Since $u(x) = M$ for every $x \in B(x_0,r)$ and $B(x_0,r)$ contains the center of $B(x_1,r)$ we conclude that $u(x_1) = M$. Repeating the same argument as above we have $u(x) = M$ for every $x \in B(x_1,r)$ as well. Continuing the same way we see that $u = M$ on all balls $B(x_0,r), B(x_1,r), \dots, B(x_N,r)$. Since $x \in B(x_N,r)$ we have $u(x) = M$. Since x was an arbitrary point in Ω, we have showed that $u(x) = M$ for every $x \in \Omega$.

$\boxed{(1)}$ We may assume that Ω is connected by considering every component separately. Observe that we always have

$$\max_{x \in \overline{\Omega}} u(x) \geqslant \max_{x \in \partial\Omega} u(x).$$

Let us assume (for contradiction) that this inequality is strict, that is,

$$\max_{x \in \overline{\Omega}} u(x) > \max_{x \in \partial\Omega} u(x).$$

Then there is $x_0 \in \Omega$ such that

$$u(x_0) = \max_{y \in \overline{\Omega}} u(y).$$

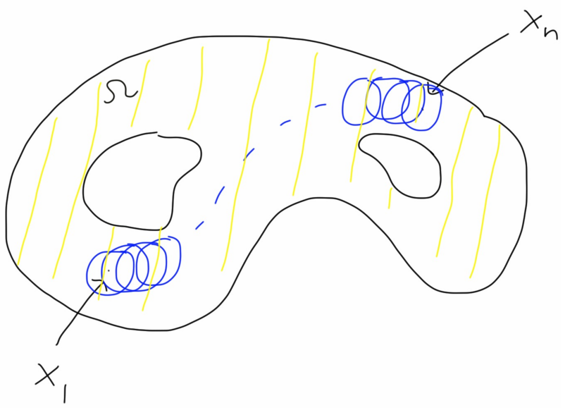


Figure 4.4: The chaining argument.

The claim (2) implies that u is constant in Ω and since $u \in C(\overline{\Omega})$ we conclude that u is constant in $\overline{\Omega}$ as well. Thus

$$\max_{x\in\partial\Omega} u(x) = \max_{x\in\overline{\Omega}} u(x),$$

which is a contradiction with our assumption. □

Remark 4.43. Let Ω be a bounded, open and connected set and $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Suppose that u is a solution to the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega. \end{cases}$$

Assume that $g \geqslant 0$. If $g(x) > 0$ for some $x \in \partial\Omega$, then the strong minimum principle implies that $u > 0$ everywhere in Ω. Thus u is positive everywhere in Ω if g is positive somewhere on $\partial\Omega$.

Reason. By the weak minimum principle

$$u(x_0) \geqslant \min_{x\in\overline{\Omega}} u(x) = \min_{x\in\partial\Omega} u(x) = \min_{x\in\partial\Omega} g(x) \geqslant 0$$

for every $x_0 \in \Omega$. If $u(x_0) = 0$ for some $x_0 \in \Omega$, then

$$u(x_0) = \min_{x\in\overline{\Omega}} u(x) = 0$$

and by the strong minimum principle $u = 0$ in Ω. Since $u \in C(\overline{\Omega})$, we have $u = 0$ in $\overline{\Omega}$. This implies that $u = g = 0$ on $\partial\Omega$, which is a contradiction. ■

Remark 4.44. The weak maximum principle can be also proved without the mean value property.

Reason. Assume first that $\Delta u(x) > 0$ for every $x \in \Omega$. If u has a maximum in at a point $x_0 \in \Omega$, then $\nabla u(x_0) = 0$ and

$$\frac{\partial^2 u}{\partial x_j^2}(x_0) \leqslant 0, \quad j = 1, \ldots, n,$$

(think of a function of one variable: if f has a local maximum at x_0 and f is twice differentiable, then $f''(x_0) \leqslant 0$). By summing up, we have $\Delta u(x_0) \leqslant 0$. This is impossible, thus u cannot have a maximum point in Ω. Since $u \in C(\Omega)$, we conclude

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x).$$

Then we consider the general case $\Delta u(x) \geqslant 0$ for every $x \in \Omega$. Consider the auxiliary function $v_\varepsilon(x) = u(x) + \varepsilon|x|^2$ with $\varepsilon > 0$. A direct calculation shows that

$$\Delta v_\varepsilon(x) = \Delta(u(x) + \varepsilon|x|^2) = \Delta u(x) + \varepsilon\Delta(|x|^2) > 0 \quad \text{for every} \quad x \in \Omega.$$

Note that the only property of the function $|x|^2$ that we use here is that its Laplace operator is strictly positive. Any other function with this property would do as well. Since $v_\varepsilon \in C^2(\Omega) \cap C(\overline{\Omega})$, by the beginning of the proof we have

$$\max_{x \in \overline{\Omega}} v_\varepsilon(x) = \max_{x \in \partial\Omega} v_\varepsilon(x).$$

This implies

$$\begin{aligned}
\max_{x \in \overline{\Omega}} u(x) + \varepsilon \min_{x \in \overline{\Omega}} |x|^2 &\leqslant \max_{x \in \overline{\Omega}} (u(x) + \varepsilon|x|^2) \\
&= \max_{x \in \partial\Omega} (u(x) + \varepsilon|x|^2) \\
&\leqslant \max_{x \in \partial\Omega} u(x) + \varepsilon \max_{x \in \partial\Omega} |x|^2.
\end{aligned}$$

By letting $\varepsilon \to 0$, we have

$$\max_{x \in \overline{\Omega}} u(x) \leqslant \max_{x \in \partial\Omega} u(x).$$

On the other hand, since $\partial\Omega \subset \overline{\Omega}$, we have

$$\max_{x \in \overline{\Omega}} u(x) \geqslant \max_{x \in \partial\Omega} u(x)$$

Thus

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x).$$

■

As immediate consequences we obtain a comparison principle, a stability and uniqueness results for the Dirichlet problem for the Poisson equation. The following results hold, in particular, for harmonic functions.

Theorem 4.45 (Comparison principle). Let $\Omega \subset \mathbb{R}^n$ be open and bounded and assume that $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ are solutions to the Poisson equation $-\Delta u = f$ in Ω. If $u \leqslant v$ on $\partial\Omega$, then $u \leqslant v$ in Ω.

T H E M O R A L : If the boundary values of two Dirichlet problems for the Poisson equation are ordered on the boundary, then the corresponding solutions are ordered inside.

Proof. Since $u \leqslant v$ on $\partial\Omega$ we have $u - v \leqslant 0$ on $\partial\Omega$. On the other hand,

$$\Delta(u - v) = \Delta u - \Delta v = -f + f = 0 \quad \text{in} \quad \Omega$$

The maximum principle implies that $u - v \leqslant 0$ in Ω and the claim follows. □

Theorem 4.46 (Stability). Let $\Omega \subset \mathbb{R}^n$ be open and bounded and assume that $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ are solutions to the Poisson equation $-\Delta u = f$ in Ω. If $|u - v| \leqslant \varepsilon$ on $\partial\Omega$, then $|u - v| \leqslant \varepsilon$ in Ω.

T H E M O R A L : The solution of the Dirichlet problem for the Poisson equation in any bounded open set depends continuously on the boundary data.

Proof. Since $|u - v| \leqslant \varepsilon$ on $\partial\Omega$ we have $-\varepsilon \leqslant u - v \leqslant \varepsilon$ on $\partial\Omega$. Since $u - v$ is harmonic in Ω, the minimum and maximum principles imply that $-\varepsilon \leqslant u - v \leqslant \varepsilon$ in Ω and the claim follows. □

Example 4.47. The following Hadamard's example shows that stability fails in unbounded domains. Consider the Cauchy problem

$$\begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad \text{in} \quad \mathbb{R}^2_+, \\ u(x,0) = 0, \quad x \in \mathbb{R}, \\ \dfrac{\partial u}{\partial y}(x,0) = \dfrac{1}{j}\sin(jx), \quad x \in \mathbb{R}, \quad j = 1, 2, \dots. \end{cases}$$

Then

$$u(x,y) = \frac{1}{j^2}\sinh(jy)\sin(jx)$$

is a solution of the problem. Recall that $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$. Observe that this function is a solution to the corresponding Dirichlet and Neumann problems. By taking j sufficiently large, the absolute value of the boundary data can be be everywhere arbitrarily small, while the solution takes arbitrarily large values even at points (x, y) with $|y|$ as small as we wish.

An important application of the maximum principle is uniqueness of the Dirichlet boundary value problem for the Poisson equation.

Theorem 4.48 (Uniqueness). Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then there exists at most one solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega. \end{cases}$$

T H E M O R A L : The solution of the Dirichlet problem for the Poisson equation in any bounded open set is unique.

Proof. Let u and v be two solutions for the Dirichlet problem. Then

$$\Delta(u - v) = \Delta u - \Delta v = -f + f = 0 \quad \text{in} \quad \Omega$$

and $u-v = g-g = 0$ on $\partial\Omega$. By the maximum principle we have $u-v \leqslant 0$ everywhere in Ω and by the minimum principle we have $u - v \geqslant 0$ everywhere in Ω. Thus $u - v = 0$ everywhere in Ω □

Remarks 4.49:

(1) Another way to show uniqueness in a smooth domain is to apply Gauss-Green formula as in Remark 4.8.

(2) The assumption that Ω is bounded is essential. Consider, for example, the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^n_+, \\ u = 0 & \text{on} \quad \partial\mathbb{R}^n_+, \end{cases}$$

in the unbounded open and connected unbounded set $\mathbb{R}^n_+$. Every function

$$u(x_1, \dots, x_{n-1}, x_n) = a x_n,$$

where $x_n > 0$ and $a \in \mathbb{R}$, is a solution to the problem. Thus the uniqueness of the boundary valued problem fails in this case. Indeed, there are infinitely many solutions to the Dirichlet problem with zero boundary value.

(3) Consider the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

If Ω is a bounded set, then $u = 0$ is the unique solution of the problem. However, we may have nontrivial solutions for an undounded Ω. Let $\Omega = \{x \in \mathbb{R}^n : |x| > 1\}$. Then

$$u(x) = \begin{cases} |x|^{2-n} - 1, & n \geqslant 3, \\ \log|x|, & n = 2, \end{cases}$$

is a nontrivial solution to the problem.

T H E M O R A L : A solution of the Dirichlet problem for the Poisson equation in an unbounded set is not necessarily is unique.

4.11 Harnack's inequality*

Harnack's inequality can be seen as a quantitative version of the maximum principle.

Theorem 4.50 (Harnack's inequality). Let $\Omega \subset \mathbb{R}^n$ be an open set and $V \subset \Omega$ be a connected bounded open set such that $\overline{V} \subset \Omega$. Then there exists a constant $c > 0$, depending only on V, such that

$$\sup_{x\in V} u(x) \leqslant c \inf_{x\in V} u(x)$$

for all nonnegative harmonic functions u in Ω.

Remark 4.51. Harnack's inequality gives the pointwise estimate

$$\frac{1}{c} u(y) \leqslant u(x) \leqslant c u(y)$$

for all points $x, y \in V$. This means that the values of nonnegative harmonic functions in V are comparable. Thus if u is small (or large) somewhere in V it is small (or large) everywhere in V. In particular, if $u(y) = 0$ for some $y \in \Omega$, then $u(x) = 0$ for every $x \in \Omega$. The assumption that $u \geqslant 0$ is essential in the result.

Proof. Since $\overline{V}$ is a compact set inside Ω, we have $\operatorname{dist}(V, \partial\Omega) > 0$. Let

$$r = \frac{1}{4} \operatorname{dist}(V, \partial\Omega).$$

Let $x, y \in V$ such that $|x - y| < r$. Then $B(y, r) \subset B(x, 2r)$. By the mean value property

$$\begin{aligned} u(x) &= \frac{1}{|B(x,2r)|} \int_{B(x,2r)} u(z)\,dz = \frac{1}{2^n \alpha(n) r^n} \int_{B(x,2r)} u(z)\,dz \\ &\geqslant \frac{1}{2^n \alpha(n) r^n} \int_{B(y,r)} u(z)\,dz = \frac{1}{2^n} \frac{1}{|B(y,r)|} \int_{B(y,r)} u(z)\,dz = \frac{1}{2^n} u(y). \end{aligned}$$

Now V is connected and $\overline{V}$ is compact so we can cover $\overline{V}$ by finitely many balls $\{B(x_j, r/2)\}_{j=1}^N$ such that

$$B\left(x_j, \frac{r}{2}\right) \cap B\left(x_{j+1}, \frac{r}{2}\right) \neq \emptyset \quad \text{for} \quad j = 1, \dots, N-1.$$

Then

$$u(x) \geqslant \frac{1}{(2^n)^N} u(y)$$

for any $x, y \in V$. By switching the roles of x and y, we have

$$u(y) \geqslant \frac{1}{(2^n)^N} u(x).$$

This implies that

$$\frac{1}{(2^n)^N} u(y) \leqslant u(x) \leqslant (2^n)^N u(y).$$

□

4.12 Energy methods

We now characterize the solution of the Dirichlet problem for the Poisson equation as a minimizer of an appropriate energy functional.

Definition 4.52. Let w be a function in the class

$$\mathscr{A} = \{w \in C^2(\Omega) \cap C(\overline{\Omega}) : w = g \text{ on } \partial\Omega\}.$$

This is the class of admissible functions for the Dirichlet problem with the given boundary values. The energy functional for the Poisson equation $-\Delta u = f$ is

$$I(w) = \int_\Omega \left(\frac{1}{2}|\nabla w|^2 - wf\right) dx.$$

Theorem 4.53 (Dirichlet's principle). Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ solves the Dirichlet problem for the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \quad \Omega, \\ u = g & \text{on } \quad \partial\Omega. \end{cases}$$

Then

$$I(u) = \min_{w \in \mathscr{A}} I(w).$$

Conversely, if $u \in \mathscr{A}$ satisfies $I(u) = \min_{w \in \mathscr{A}} I(w)$, then u is a solution of the Dirichlet problem above for the Poisson equation.

TERMINOLOGY: The Poisson equation is said to be the Euler-Lagrange equation for the energy (or variational) integral above.

THE MORAL: A solution of the Poisson (and Laplace) equation is a minimizer of an energy integral. Conversely, every minimizer of the energy integral above is a solution of the Poisson equation. Thus a function is a solution to the Poisson equation if and only if it is a minimizer of the energy integral. Observe that there are only first order partial derivatives in the energy functional, but the corresponding Poisson equation involves second order partial derivatives. For example, the finite element method to compute solutions numerically is based on this approach.

Proof. $\boxed{\Rightarrow}$ Let $w \in \mathscr{A}$. By Green's first identitty

$$\begin{aligned} 0 &= \int_\Omega \underbrace{(-\Delta u - f)}_{=0}(u - w)\,dx \\ &= -\int_{\partial\Omega} \frac{\partial u}{\partial \nu} \underbrace{(u - w)}_{=0}\,dx + \int_\Omega (\nabla u \cdot \nabla(u - w) - f(u - w))\,dx \\ &= \int_\Omega |\nabla u|^2\,dx - \int_\Omega \nabla u \cdot \nabla w\,dx - \int_\Omega f(u - w)\,dx, \end{aligned}$$

where we used the fact that $u - w = g - g = 0$ on $\partial\Omega$. Thus

$$\int_\Omega (|\nabla u|^2 - uf)\,dx = \int_\Omega (\nabla u \cdot \nabla w - fw)\,dx$$
$$\leqslant \int_\Omega \left(\frac{1}{2}|\nabla w|^2 + \frac{1}{2}|\nabla u|^2 - fw\right)dx,$$

where the last inequality follows from the Cauchy-Schwarz inequality since

$$|\nabla u \cdot \nabla w| \leqslant |\nabla u||\nabla w| \leqslant \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla w|^2.$$

In the last step we used the fact that $ab \leqslant \frac{1}{2}a^2 + \frac{1}{2}b^2$. Thus we have showed that $I(u) \leqslant I(w)$ for every $w \in \mathscr{A}$. Since $u \in \mathscr{A}$, we have

$$I(u) = \min_{w \in \mathscr{A}} I(w).$$

$\boxed{\Leftarrow}$ Assume that $I(u) = \min_{w\in\mathscr{A}} I(w)$. Let $v \in C_0^\infty(\Omega)$ and define

$$\phi(\tau) = I(u + \tau v), \quad \tau \in \mathbb{R}.$$

Observe that for every $\tau \in \mathbb{R}$ we have $u + \tau v \in \mathscr{A}$, because v vanishes on $\partial\Omega$. Since u minimizes the energy functional I we see that ϕ has a minimum at $\tau = 0$. Thus we must have $\phi'(0) = 0$ provided the derivative exists. However,

$$\phi(\tau) = \int_\Omega \left(\frac{1}{2}|\nabla u + \tau\nabla v|^2 - (u + \tau v)f\right)dx$$
$$= \int_\Omega \left(\frac{1}{2}(\nabla u + \tau\nabla v)\cdot(\nabla u + \tau\nabla v) - (u + \tau v)f\right)dx$$
$$= \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{\tau^2}{2}|\nabla v|^2 + \tau\nabla u \cdot \nabla v - uf - \tau vf\right)dx.$$

By differentiating under the integral, we obtain

$$\phi'(\tau) = \int_\Omega \left(\tau|\nabla v|^2 + \nabla u \cdot \nabla v - vf\right)dx.$$

By Greens' first identity

$$0 = \phi'(0) = \int_\Omega (\nabla u \cdot \nabla v - vf)\,dx$$
$$= \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v\,dS - \int_\Omega v(\Delta u + f)\,dx$$
$$= -\int_\Omega v(\Delta u + f)\,dx,$$

since $v = 0$ on $\partial\Omega$. We have thus proved that

$$\int_\Omega (\Delta u + f)v\,dx = 0 \quad \text{for every} \quad v \in C_0^\infty(\Omega).$$

Claim: This implies that $\Delta u + f = 0$ everywhere in Ω.

Reason. Let $x \in \Omega$ and $\phi \in C_0^\infty(B(0,1))$, which satisfies $\int_{\mathbb{R}^n} \phi(y)\,dy = 1$. Define

$$\phi_\varepsilon(y) = \frac{1}{\varepsilon^n}\phi\left(\frac{y}{\varepsilon}\right), \quad \varepsilon > 0.$$

If $\varepsilon > 0$ is small enough, then $\phi_\varepsilon(x-y)$ is supported in a small neighborhood of $x \in \Omega$ and thus $\phi_\varepsilon(x-\cdot) \in C_0^\infty(\Omega)$. Thus for all $\varepsilon > 0$ small enough

$$0 = \int_\Omega (\Delta u(y) + f(y))\phi_\varepsilon(x-y)\,dy = ((\Delta u + f) * \phi_\varepsilon)(x)$$

But $\{\phi_\varepsilon\}_{\varepsilon>0}$ is a family of good kernels so letting $\varepsilon \to 0$ we have

$$\Delta u(x) - f(x) = \lim_{\varepsilon\to 0}((\Delta u + f) * \phi_\varepsilon)(x) = 0$$

for all $x \in \Omega$ as long as f satisfies some mild assumptions, for example, f integrable, bounded, and continuous in Ω. ■

4.13 Weak solutions*

In this section we consider another point of view to the energy methods. If $u \in C^2(\Omega)$ is a solution to the Laplace equation $\Delta u = 0$ and $\varphi \in C_0^\infty(\Omega)$, that is, φ is a compactly supported smooth function on Ω. Then by Green's first identity

$$0 = \int_\Omega \varphi \Delta u\,dx = -\int_\Omega \nabla u \cdot \nabla \varphi\,dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \underbrace{\varphi}_{=0}\,dS,$$

so that

$$\int_\Omega \nabla u \cdot \nabla \varphi\,dx = 0 \tag{4.9}$$

for every $\varphi \in C_0^\infty(\Omega)$. On the other hand, if (4.9) holds, then the computation above shows that

$$\int_\Omega \varphi \Delta u\,dx = 0$$

for every $\varphi \in C_0^\infty(\Omega)$. This implies that $\Delta u = 0$ in Ω if and only if (4.9) holds for every $\varphi \in C_0^\infty(\Omega)$.

By Green's first identity

$$0 = \int_\Omega \nabla u \cdot \nabla \varphi\,dx = -\int_\Omega u \Delta \varphi\,dx + \int_{\partial\Omega} \underbrace{\frac{\partial \varphi}{\partial \nu}}_{=0} u\,dS,$$

so that

$$\int_\Omega u \Delta \varphi\,dx = 0 \tag{4.10}$$

for every $\varphi \in C_0^\infty(\Omega)$. As above,

$$\int_\Omega \varphi \Delta u\,dx = \int_\Omega u \Delta \varphi\,dx = 0$$

for every $\varphi \in C_0^\infty(\Omega)$. This shows that (4.10) holds for for every $\varphi \in C_0^\infty(\Omega)$ if and only if $\Delta u = 0$ in Ω. Formulas (4.9) and (4.10) give two possible ways to define a generalized solution to the Laplace equation. Observe, that in (4.9) we only need to assume that u has the fist order derivatives. In (4.9) it is enough to assume that u is integrable. With this interpretation, the function u does not have to have any derivatives at all. This is in contrast with the standard definition of the Laplacian, where u has to be twice differentiable.

4.14 The Laplace equation in other coordinates*

In this section we consider the three-dimensional space $\mathbb{R}^3$. We have solved the two-dimensional Laplace equation over rectangular domains and on the disc by switching to polar coordinates. Among the most useful alternatives to Cartesian coordinates in $\mathbb{R}^3$ there are two coordinate systems that generalize polar coordinates in $\mathbb{R}^2$.

Cylindrical coordinates

Cylindrical coordinates (r,θ,z) in $\mathbb{R}^3$ are defined by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

The 3-dimensional Laplace equation takes the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

in cylindrical coordinates.

Reason. By a direct computation of the derivatives, we have

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

By the chain rule

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y},$$

and again by the chain rule

$$\begin{aligned}
\frac{\partial^2 u}{\partial r^2} &= \frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial r}\left(\sin\theta\frac{\partial u}{\partial y}\right) \\
&= \cos\theta\left(\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial r} + \left(\frac{\partial}{\partial y}\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial r}\right) + \sin\theta\left(\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial r} + \left(\frac{\partial}{\partial y}\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial r}\right) \\
&= \cos^2\theta\frac{\partial^2 u}{\partial x^2} + \sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y} + \sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2}.
\end{aligned}$$

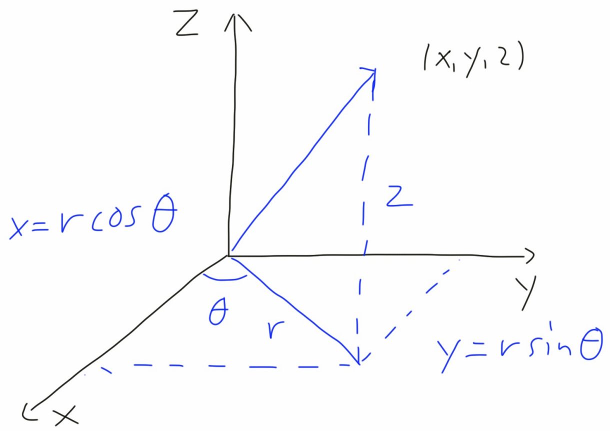


Figure 4.5: Cylindrical coordinates.

With respect to θ, we have

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}.$$

By the product and the chain rules for derivatives

$$\begin{aligned}\frac{\partial^2 u}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left(-r\sin\theta\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial \theta}\left(r\cos\theta\frac{\partial u}{\partial y}\right)\\ &= -r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\left(\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial \theta} + \left(\frac{\partial}{\partial y}\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial \theta}\right)\\ &\quad - r\sin\theta\frac{\partial u}{\partial y} + r\cos\theta\left(\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial \theta} + \left(\frac{\partial}{\partial y}\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial \theta}\right)\\ &= -r\cos\theta\frac{\partial u}{\partial x} + r^2\sin^2\theta\frac{\partial^2 u}{\partial x^2} - r^2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y}\\ &\quad - r\sin\theta\frac{\partial u}{\partial y} - r^2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y} + r^2\cos^2\theta\frac{\partial^2 u}{\partial y^2}.\end{aligned}$$

Thus

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\\
&=\left(\cos^2\theta\frac{\partial^2 u}{\partial x^2}+2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y}+\sin^2\theta\frac{\partial^2 u}{\partial y^2}\right)+\left(\frac{\cos\theta}{r}\frac{\partial u}{\partial x}+\frac{\sin\theta}{r}\frac{\partial u}{\partial y}\right)\\
&\quad+\left(-\frac{\cos\theta}{r}\frac{\partial u}{\partial x}+\sin^2\theta\frac{\partial^2 u}{\partial x^2}-2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y}-\frac{\sin\theta}{r}\frac{\partial u}{\partial y}+\cos^2\theta\frac{\partial^2 u}{\partial y^2}\right)\\
&=\left(\cos^2\theta+\sin^2\theta\right)\frac{\partial^2 u}{\partial x^2}+\left(\sin^2\theta+\cos^2\theta\right)\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2},
\end{aligned}
$$

which gives

$$
\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\Delta u=0.
$$

■

Spherical coordinates

Spherical coordinates (r,θ,ϕ) in $\mathbb{R}^3$ are defined by

$$
x=r\cos\theta\sin\phi,\quad y=r\sin\theta\sin\phi,\quad z=r\cos\phi.
$$

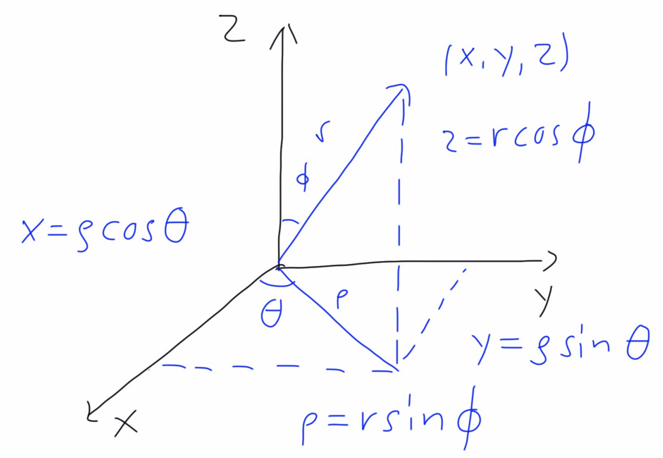


Figure 4.6: Spherical coordinates.

The 3-dimensional Laplace operator takes the form

$$
\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2(\sin\phi)^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2}+\frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi}=0
$$

or equivalently

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left(\frac{1}{(\sin\phi)^2}u_{\theta\theta} + \frac{1}{\sin\phi}(u_\phi \sin\phi)_\phi\right) = 0$$

in spherical coordinates.

Reason. A direct differentiation gives

$$\begin{aligned}
&\frac{\partial x}{\partial r} = \cos\theta\sin\phi, && \frac{\partial x}{\partial\theta} = -r\sin\theta\sin\phi, && \frac{\partial x}{\partial\phi} = r\cos\theta\cos\phi,\\
&\frac{\partial y}{\partial r} = \sin\theta\sin\phi, && \frac{\partial}{y}\partial\theta = r\cos\theta\sin\phi, && \frac{\partial y}{\partial\phi} = r\sin\theta\cos\phi,\\
&\frac{\partial z}{\partial r} = \cos\phi, && \frac{\partial z}{\partial\theta} = 0, && \frac{\partial z}{\partial\phi} = -r\sin\phi.
\end{aligned}$$

By the chain rule

$$\begin{aligned}
\frac{\partial u}{\partial u} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r}\\
&= \cos\theta\sin\phi\frac{\partial u}{\partial x} + \sin\theta\sin\phi\frac{\partial u}{\partial y} + \cos\phi\frac{\partial u}{\partial z},
\end{aligned}$$

from which, again by the chain rule, we obtain

$$\begin{aligned}
\frac{\partial^2 u}{\partial r^2} &= \frac{\partial}{\partial r}\left(\cos\theta\sin\phi\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial r}\left(\sin\theta\sin\phi\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial r}\left(\cos\phi\frac{\partial u}{\partial z}\right)\\
&= \cos\theta\sin\phi\left(\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial r} + \left(\frac{\partial}{\partial y}\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial r} + \left(\frac{\partial}{\partial z}\frac{\partial u}{\partial x}\right)\frac{\partial z}{\partial r}\right)\\
&\quad + \sin\theta\sin\phi\left(\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial r} + \left(\frac{\partial}{\partial y}\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial r} + \left(\frac{\partial}{\partial z}\frac{\partial u}{\partial y}\right)\frac{\partial z}{\partial r}\right)\\
&\quad + \cos\phi\left(\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial z}\right)\frac{\partial x}{\partial r} + \left(\frac{\partial}{\partial y}\frac{\partial u}{\partial z}\right)\frac{\partial y}{\partial r} + \left(\frac{\partial}{\partial z}\frac{\partial u}{\partial z}\right)\frac{\partial z}{\partial r}\right)\\
&= \cos^2\theta\sin^2\phi\frac{\partial^2 u}{\partial x^2} + \sin\theta\cos\theta\sin^2\phi\frac{\partial^2 u}{\partial x\partial y} + \cos\theta\sin\phi\cos\phi\frac{\partial^2 u}{\partial x\partial z}\\
&\quad + \sin\theta\cos\theta\sin^2\phi\frac{\partial^2 u}{\partial x\partial y} + \sin^2\theta\sin^2\phi\frac{\partial^2 u}{\partial y^2} + \sin\theta\sin\phi\cos\phi\frac{\partial^2 u}{\partial y\partial z}\\
&\quad + \cos\theta\sin\phi\cos\phi\frac{\partial^2 u}{\partial x\partial z} + \sin\theta\sin\phi\cos\phi\frac{\partial^2 u}{\partial y\partial z} + \cos^2\frac{\partial^2 u}{\partial z^2}.
\end{aligned}$$

With respect to θ, we have

$$\frac{\partial u}{\partial\theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial\theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial\theta} = -r\sin\theta\sin\phi\frac{\partial u}{\partial x} + r\cos\theta\sin\phi\frac{\partial u}{\partial y}uy + 0,$$

and thus

$$
\begin{aligned}
\frac{\partial^2 u}{\partial\theta^2} &= \frac{\partial}{\partial\theta}\left(-r\sin\theta\sin\phi\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial\theta}\left(r\cos\theta\sin\phi\frac{\partial u}{\partial y}uy\right)\\
&= -r\cos\theta\sin\phi\frac{\partial u}{\partial x}-r\sin\theta\sin\phi\left(\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial\theta}+\left(\frac{\partial}{\partial y}\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial\theta}+\left(\frac{\partial}{\partial z}\frac{\partial u}{\partial x}\right)\frac{\partial z}{\partial\theta}\right)\\
&\quad -r\sin\theta\sin\phi\frac{\partial u}{\partial y}+r\cos\theta\sin\phi\left(\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial\theta}+\left(\frac{\partial}{\partial y}\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial\theta}+\left(\frac{\partial}{\partial z}\frac{\partial u}{\partial y}\right)\frac{\partial z}{\partial\theta}\right)\\
&= -r\cos\theta\sin\phi\frac{\partial u}{\partial x}+r^2\sin^2\theta\sin^2\phi\frac{\partial^2 u}{\partial x^2}-r^2\sin\theta\cos\theta\sin^2\phi\frac{\partial^2 u}{\partial x\partial y}+0\\
&\quad -r\sin\theta\sin\phi\frac{\partial u}{\partial y}-r^2\sin\theta\cos\theta\sin^2\phi\frac{\partial^2 u}{\partial x\partial y}+r^2\cos^2\theta\sin^2\phi\frac{\partial^2 u}{\partial y^2}+0.
\end{aligned}
$$

With respect to ϕ, we have

$$
\begin{aligned}
\frac{\partial u}{\partial\phi} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial\phi}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial\phi}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial\phi}\\
&= r\cos\theta\cos\phi\frac{\partial u}{\partial x}+r\sin\theta\cos\phi\frac{\partial u}{\partial y}-r\sin\phi\frac{\partial u}{\partial z},
\end{aligned}
$$

from which we have

$$
\begin{aligned}
\frac{\partial^2 u}{\partial\phi^2} &= \frac{\partial}{\partial\phi}\left(r\cos\theta\cos\phi\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial\phi}\phi\left(r\sin\theta\cos\phi\frac{\partial u}{\partial y}\right)-\frac{\partial}{\partial\phi}\phi\left(r\sin\phi\frac{\partial u}{\partial z}\right)\\
&= -r\cos\theta\sin\phi\frac{\partial u}{\partial x}+r\cos\theta\cos\phi\left(\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial\phi}+\left(\frac{\partial}{\partial y}\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial\phi}+\left(\frac{\partial}{\partial z}\frac{\partial u}{\partial x}\right)\frac{\partial z}{\partial\phi}\right)\\
&\quad -r\sin\theta\sin\phi+r\sin\theta\cos\phi\left(\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial\phi}+\left(\frac{\partial}{\partial y}\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial\phi}+\left(\frac{\partial}{\partial z}\frac{\partial u}{\partial y}\right)\frac{\partial z}{\partial\phi}\right)\\
&\quad -r\cos\phi\frac{\partial u}{\partial z}-r\sin\phi\left(\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial z}\right)\frac{\partial x}{\partial\phi}+\left(\frac{\partial}{\partial y}\frac{\partial u}{\partial z}\right)\frac{\partial y}{\partial\phi}+\left(\frac{\partial}{\partial z}\frac{\partial u}{\partial z}\right)\frac{\partial z}{\partial\phi}\right)\\
&= -r\cos\theta\sin\phi\frac{\partial u}{\partial x}+r^2\cos^2\theta\cos^2\phi\frac{\partial^2 u}{\partial x^2}+r^2\sin\theta\cos\theta\cos^2\phi\frac{\partial^2 u}{\partial x\partial y}\\
&\quad -r^2\cos\theta\sin\phi\cos\phi\frac{\partial^2 u}{\partial x\partial z}\\
&\quad -r\sin\theta\sin\phi\frac{\partial u}{\partial y}+r^2\sin\theta\cos\theta\cos^2\phi\frac{\partial^2 u}{\partial x\partial y}+r^2\sin^2\theta\cos^2\phi\frac{\partial^2 u}{\partial y^2}\\
&\quad -r^2\sin\theta\sin\phi\cos\phi\frac{\partial^2 u}{\partial y\partial z}\\
&\quad -r\cos\phi\frac{\partial u}{\partial z}-r^2\cos\theta\sin\phi\cos\phi\frac{\partial^2 u}{\partial x\partial z}-r^2\sin\theta\sin\phi\cos\phi\frac{\partial^2 u}{\partial y\partial z}+r^2\sin^2\phi\frac{\partial^2 u}{\partial z^2}.
\end{aligned}
$$

Using the obtained expressions for the partial derivatives, we collect form the formula

$$
\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2(\sin\phi)^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\phi^2}+\frac{\cot\phi}{r^2}\frac{\partial u}{\partial\phi}
$$

partial derivatives with respect to x, y and z. For the $\frac{\partial u}{\partial x}$-terms, we have

$$\begin{aligned}
&0+\frac{2}{r}(\cos\theta\sin\phi)+\frac{1}{r^2(\sin\phi)^2}(-r\cos\theta\sin\phi)+\frac{1}{r^2}(-r\cos\theta\sin\phi)\\
&\quad+\frac{\cot\phi}{r^2}(r\cos\theta\cos\phi)\\
&=\frac{\cos\theta}{r\sin\phi}(2\sin^2\phi-1-\sin^2\phi+\cos^2\phi)=0.
\end{aligned}$$

Similarly, for the $\frac{\partial u}{\partial y}$-terms, we have

$$\begin{aligned}
&0+\frac{2}{r}(\sin\theta\sin\phi)+\frac{1}{r^2(\sin\phi)^2}(-r\sin\theta\sin\phi)+\frac{1}{r^2}(-r\sin\theta\sin\phi)\\
&\quad+\frac{\cot\phi}{r^2}(r\sin\theta\cos\phi)\\
&=\frac{\sin\theta}{r\sin\phi}(2\sin^2\phi-1-\sin^2\phi+\cos^2\phi)=0
\end{aligned}$$

and for the $\frac{\partial u}{\partial z}$-terms we have

$$\begin{aligned}
&0+\frac{2}{r}\cos\phi+\frac{1}{r^2(\sin\phi)^2}\cdot 0+\frac{1}{r^2}(-r\cos\phi)+\frac{\cot\phi}{r^2}(-r\sin\phi)\\
&=\frac{1}{r}(2\cos\phi-\cos\phi-\cos\phi)=0.
\end{aligned}$$

For the $\frac{\partial^2 u}{\partial x^2}$-terms, we have

$$\begin{aligned}
&\cos^2\theta\sin^2\phi+\frac{2}{r}\cdot 0+\frac{1}{r^2(\sin\phi)^2}(r^2\sin^2\theta\sin^2\phi)+\frac{1}{r^2}(r^2\cos^2\theta\cos^2\phi)+\frac{\cot\phi}{r^2}\cdot 0\\
&=\cos^2\theta(\sin^2\phi+\cos^2\phi)+\sin^2\theta=1.
\end{aligned}$$

Similarly, for the $\frac{\partial^2 u}{\partial y^2}$-terms, we have

$$\begin{aligned}
&\sin^2\theta\sin^2\phi+\frac{2}{r}\cdot 0+\frac{1}{r^2(\sin\phi)^2}(r^2\cos^2\theta\sin^2\phi)+\frac{1}{r^2}(r^2\sin^2\theta\cos^2\phi)+\frac{\cot\phi}{r^2}\cdot 0\\
&=\sin^2\theta(\sin^2\phi+\cos^2\phi)+\cos^2\theta=1
\end{aligned}$$

and for the $\frac{\partial^2 u}{\partial z^2}$-terms, we have

$$\cos^2\phi+\frac{2}{r}\cdot 0+\frac{1}{r^2(\sin\phi)^2}\cdot 0+\frac{1}{r^2}(r^2\sin^2\phi)+\frac{\cot\phi}{r^2}\cdot 0=1.$$

Moreover, for the $\frac{\partial^2 u}{\partial x\partial y}$-terms, we have

$$\begin{aligned}
&2\sin\theta\cos\theta\sin^2\phi+\frac{2}{r}\cdot 0+\frac{1}{r^2(\sin\phi)^2}(-2r^2\sin\theta\cos\theta\sin^2\phi)\\
&\quad+\frac{1}{r^2}(2r^2\sin\theta\cos\theta\cos^2\phi)+\frac{\cot\phi}{r^2}\cdot 0\\
&=2\sin\theta\cos\theta(\sin^2\phi-1+\cos^2\phi)=0.
\end{aligned}$$

Similarly, for the $\frac{\partial^2 u}{\partial x \partial z}$-terms

$$2\cos\theta\sin\phi\cos\phi+\frac{2}{r}\cdot 0+\frac{1}{r^2(\sin\phi)^2}\cdot 0+\frac{1}{r^2}(-2r^2\cos\theta\sin\phi\cos\phi)+\frac{\cot\phi}{r^2}\cdot 0=0$$

and for the $\frac{\partial^2 u}{\partial y \partial z}$-terms

$$2\sin\theta\sin\phi\cos\phi+\frac{2}{r}\cdot 0+\frac{1}{r^2(\sin\phi)^2}\cdot 0+\frac{1}{r^2}(-2r^2\sin\theta\sin\phi\cos\phi)+\frac{\cot\phi}{r^2}\cdot 0=0.$$

Collecting everything together gives

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2(\sin\phi)^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\phi^2}+\frac{\cot\phi}{r^2}\frac{\partial u}{\partial\phi}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0. \qquad \blacksquare$$

Remark 4.54. Separation of variables can be also applied in spherical coordinates in $\mathbb{R}^3$. In this case we look for harmonic functions

$$u(r,\theta,\phi)=A(\theta,\phi)B(r).$$

By substituting this into the Laplace equation in spherical coordinates, we obtain

$$\frac{r^2B_{rr}+2rB_r}{B}+\frac{(\sin\phi)^{-2}A_{\theta\theta}+(\sin\phi)^{-1}(A_\phi\sin\phi)_\phi}{A}=0$$

and equivalently

$$\frac{r^2B_{rr}+2rB_r}{B}=-\frac{(\sin\phi)^{-2}A_{\theta\theta}+(\sin\phi)^{-1}(A_\phi\sin\phi)_\phi}{A}$$

for every r, θ and ϕ. Thus

$$\frac{r^2B_{rr}+2rB_r}{B}=\mu=-\frac{(\sin\phi)^{-2}A_{\theta\theta}+(\sin\phi)^{-1}(A_\phi\sin\phi)_\phi}{A}$$

for every r, θ and ϕ, where μ is the separation constant. Consequently, we may rewrite the separated equations as two differential equations

$$\begin{cases}\frac{1}{(\sin\phi)^2}A_{\theta\theta}+\frac{1}{\sin\phi}(A_\phi\sin\phi)_\phi+\mu A=0,\\ r^2B_{rr}+2rB_r-\mu B=0.\end{cases}$$

The equation for B is similar to the the corresponding ODE in the two-dimensional case, see (2.15), but the PDE for A is more challenging. Solutions to the PDE for A are special functions called the spherical harmonics, see pages 271–277 in [10] and pages 98–108 in [5] for the n-dimensional theory. The methods of separation of variables can be used to find solutions to other PDEs than the Laplace equation. It may also be used for other coordinate systems than polar or spherical coordinates.

4.15 Summary

We have considered four aspects of the PDE theory for the Laplace operator.

(1) **Existence.** We have derived several representation formulas for the solutions. The Fourier series and the Fourier transform can be used to solve boundary value problems in special cases. The solution to the Poisson equation in the whole space can be represented as a convolution of the source term with the fundamental solution by Theorem 4.19. Remark 4.21 shows how this can be used to solve a Dirichlet problem in a subdomain. Theorem 4.28 gives a solution of a boundary value problems for the Poisson equation in a subdomain in terms of the Green's function for Ω.

(2) **Uniqueness.** Remark 4.8 shows how we can conclude uniqueness of a solution to the Dirichlet problem from Green's formulas. On the other hand, uniqueness can be shown using the maximum principle as in Theorem 4.48.

(3) **Stability.** Theorem 4.46 shows that the solution of the Dirichlet problem depends continuously on the boundary data. This can be concluded also from the representation formulas whenever they are available.

(4) **Regularity.** Representation formulas for solutions are integrals which can be used to show that the solutions are smooth. This can be also concluded from the mean value property. It is possible to consider generalized or energy solutions for the Laplace equations to obtain existence under very general conditions. It can be shown that the generalized solutions are smooth as well.

The heat equation governs the diffusion of heat in a body. The heat kernel will be the fundamental solution of the heat equation. The fundamental solution can be used to derive representation formulas for solutions of nonhomogeneous problems. For subdomains the solutions can be constructed through separation of variables, which leads to an eigenvalue problem for the Laplace operator. We also discuss the maximum principle to study uniqueness of the solution.

5 Heat equation

In this chapter we study the heat equation

$$u_t - \Delta u = 0$$

and the nonhomogeneous heat equation

$$u_t - \Delta u = f$$

with appropriate initial and boundary conditions. Here

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2},$$

that is, the Laplace is taken with respect to the spatial variable x. Let Ω be an open subset of $\mathbb{R}^n$ and $T > 0$. The problem is to find a function $u = u(x,t)$ such that it is a solution to the heat equation in $\Omega \times (0,T)$.

Physically, a solution $u = u(x,t)$ of the heat equation represents the temperature of the body Ω at the point x and time t. Observe that any solution $v = v(x)$ of the Laplace equation induces a time independent solution $u = u(x,t) = v(x)$ of the heat equation. In practice this suggests that for every claim about solutions to the Laplace equation there should be a corresponding claim for the solutions of the heat equation. However, the dependence in time leads to new phenomena and challenges which are not visible in the stationary case.

The appropriate initial condition is

$$u(x,0) = g(x), \quad x \in \Omega.$$

This describes the initial temperature distribution at the time $t = 0$. In addition, we may have a Dirichlet type boundary condition

$$u(x,t) = g(x,t), \quad x \in \partial\Omega, \quad t > 0,$$

which describes the temperature on the boundary, or a Neumann type boundary condition

$$\frac{\partial u}{\partial \nu}(x,t) = \nabla_x u(x,t)\cdot \nu(x) = h(x,t), \quad x \in \partial\Omega, \quad t > 0,$$

which describes the heat flow through the boundary. Here, as usual, $\nu(x) = (\nu_1(x),\dots,\nu_n(x))$ is the unit outer normal on $\partial\Omega$.

Remark 5.1. The more general equation

$$u_t - a\Delta u = 0,$$

where $a \in \mathbb{R}$ is the heat conductivity, can be reduced to the standard heat equation by the change of variables $t \mapsto at$ (exercise). This means that it is enough to consider the case $a = 1$.

5.1 Physical interpretation

The heat equation, also known as the diffusion equation, governs the propagation or diffusion in time of the density of some quantity such as heat or chemical concentration. If V is any smooth subdomain of Ω, the rate of change of the total quantity of heat in V equals the negative of the net flux through the boundary ∂V

$$\frac{\partial}{\partial t}\int_V u(x,t)\,dx = -\int_{\partial V} F(x,t)\cdot \nu(x)\,dS(x) = 0,$$

where $F = (F_1,\dots,F_n)$ is the flux density and ν is the unit outer normal of ∂V. By the Gauss-Green theorem we have

$$\begin{aligned}\int_V \operatorname{div}_x F(x,t)\,dx &= \int_{\partial V} F(x,t)\cdot\nu(x)\,dS(x)\\ &= -\frac{\partial}{\partial t}\int_V u(x,t)\,dx = -\int_V \frac{\partial u}{\partial t}(x,t)\,dx.\end{aligned}$$

The last equality follows by switching the order of the derivative and the integral. Since this holds for every subdomain V of Ω, we have

$$\operatorname{div}_x F(x,t) = -\frac{\partial u}{\partial t}(x,t).$$

It is physically reasonable to assume that the flux F is proportional to the gradient ∇u but in the opposite direction, since the flow is usually from regions of high temperature to regions of low temperature or high concentration to low concentration. Thus

$$F(x,t) = -a\nabla u(x,t), \quad a > 0.$$

This gives

$$u_t(x,t) = \frac{\partial u}{\partial t}(x,t) = -\operatorname{div}_x F(x,t) = a\operatorname{div}_x \nabla u(x,t) = a\Delta u(x,t),$$

which implies that u is a solution of the general heat equation.

5.2 The fundamental solution

In section 3.10 we have already derived a formula for the solution of the Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = g & \text{on} \quad \mathbb{R}^n \times \{t=0\}, \end{cases}$$

where $g \in C_0^\infty(\mathbb{R}^n)$. Indeed, by Theorem 3.28

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\,dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy \tag{5.1}$$

is a solution to the heat equation in $\mathbb{R}^n \times (0,\infty)$. Here

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, & x \in \mathbb{R}^n, \quad t>0, \\ 0, \quad x \in \mathbb{R}^n, & t \leqslant 0, \end{cases}$$

is the fundamental solution of the heat equation (or the heat kernel). Note that Φ is a solution to the heat equation in the upper half-space and that Φ is unbounded in any neighbourhood of $(0,0)$. The fact that the solution attains the initial values g in the sense

$$\lim_{t\to 0} u(x,t) = g(x) \quad \text{for every} \quad x \in \mathbb{R}^n,$$

is a consequence of Theorem 3.20, which is the general result about approximations of the identity and the fact that the fundamental solution induces a family of good kernels, see Example 3.21 and the proof of the Fourier inversion theorem 3.14. This gives the existence of a solution to the Cauchy problem, but the uniqueness may fail in unbounded domains, as we shall see later. Note also that the solution given by (5.1) is smooth, since the fundamental solution is smooth in the upper half-space and the convolution inherits smoothness.

Remarks 5.2:

(1) We can also write

$$\begin{cases} \frac{\partial \Phi}{\partial t} - \Delta \Phi = 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ \Phi = \delta_0 & \text{in} \quad \mathbb{R}^n \times \{t=0\}, \end{cases}$$

where δ_0 is the Dirac measure in $\mathbb{R}^n$ giving unit mass to the point 0. In particular, Φ is a solution to the heat equation in $\mathbb{R}^n \times (0,\infty)$.

(2) The heat equation has a natural symmetry, which may be used to derive the formula for the fundamental solution. We may use the similarity method to find $\alpha, \beta > 0$ and $v : \mathbb{R}^n \to \mathbb{R}$ so that the function

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right), \quad x \in \mathbb{R}^n, \quad t>0,$$

is a solution to the heat equation. We can insert this in the heat equation and assume that v is a radial function. This gives an ODE for v which can be solved. This is an alternative way to derive the formula for the fundamental solution, which applies to other PDEs as well.

(3) If $g \in C(\mathbb{R}^n)$ is bounded, $g \geqslant 0$ and there is a point $y \in \mathbb{R}^n$ such that $g(y) > 0$, then $u > 0$ everywhere in the upper half-space. This means that the heat equation has infinite speed of propagation of disturbances. If the initial temperature is nonnegative and positive somewhere, the temperature is positive at any moment of time (no matter how small). Note that $u(x,t)$ for any $t > 0$ depends on the initial values of $g(y)$ for all $y \in \mathbb{R}^n$.

(4) By (3.28), we have

$$\begin{aligned}|u(x,t)| &= \left|\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy\right| \\ &\leqslant \frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} \left|e^{-\frac{|x-y|^2}{4t}} g(y)\right| dy \\ &\leqslant \frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} |g(y)|\,dy\end{aligned}$$

for every $x \in \mathbb{R}^n$ and $t > 0$. This gives the decay estimate

$$\sup_{x\in\mathbb{R}^n} |u(x,t)| \leqslant \frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} |g(y)|\,dy, \quad t > 0.$$

In particular, this implies that the solution tends to zero as time goes to infinity. The same argument can also be used to study stability of solutions in the upper half-space.

5.3 The nonhomogeneous problem

Consider the nonhomogeneous Cauchy problem

$$\begin{cases} u_t - \Delta u = f & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = 0 \quad \text{in} & \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{5.2}$$

By (5.1), for every fixed s, with $0 < s < t$, the function

$$u(x,t;s) = \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,dy$$

solves the translated initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t;s) - \Delta u(x,t;s) = 0, & x \in \mathbb{R}^n, \quad t > s, \\ u(x,s;s) = f(x,s), & x \in \mathbb{R}^n. \end{cases}$$

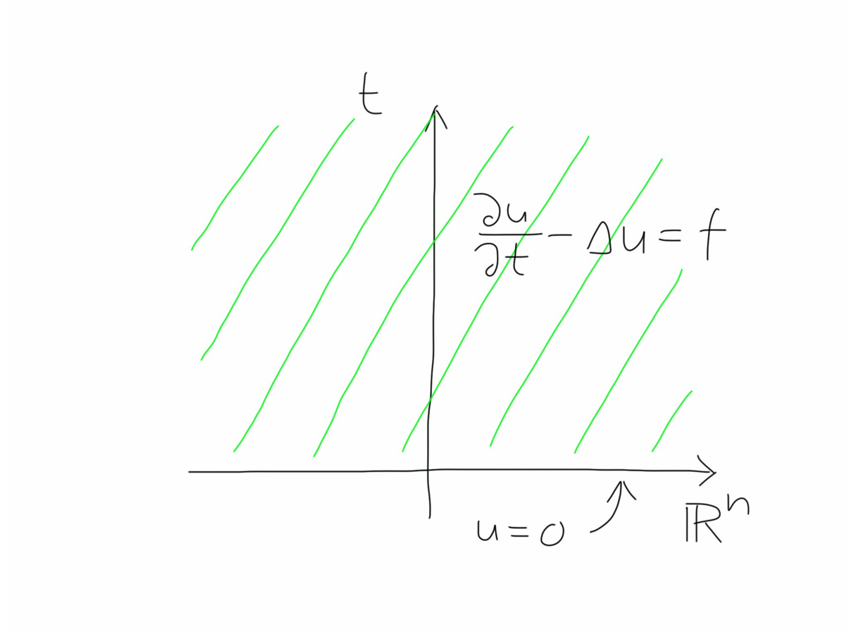


Figure 5.1: Nonhomogeneous problem for the heat equation.

Observe that this is a initial value problem for the homogeneous heat equation with the time $t = 0$ replaced by the time $t = s$. This is called the translation principle.

So called Duhamel's principle, see Remark 3.29, suggests that we can construct a solution to the nonhomogeneous problem by integrating solutions $u(x,t;s)$ over $s \in (0,t)$ and have

$$u(x,t) = \int_0^t u(x,t;s)\,ds, \quad x \in \mathbb{R}^n, \quad t > 0.$$

THE MORAL: Duhamel's principle is a process of expressing the solution of a nonhomogeneous problem as an integral of the solutions to the homogeneous problem in the way that the source term is interpreted as the initial condition.

To verify that this function is a solution to the heat equation, we have to differentiate a function defined by an integral, as illustrated by the following lemma.

Lemma 5.3.

$$\frac{\partial u}{\partial t}(x,t) = u(x,t;t) + \int_0^t \frac{\partial u}{\partial t}(x,t;s)\,ds.$$

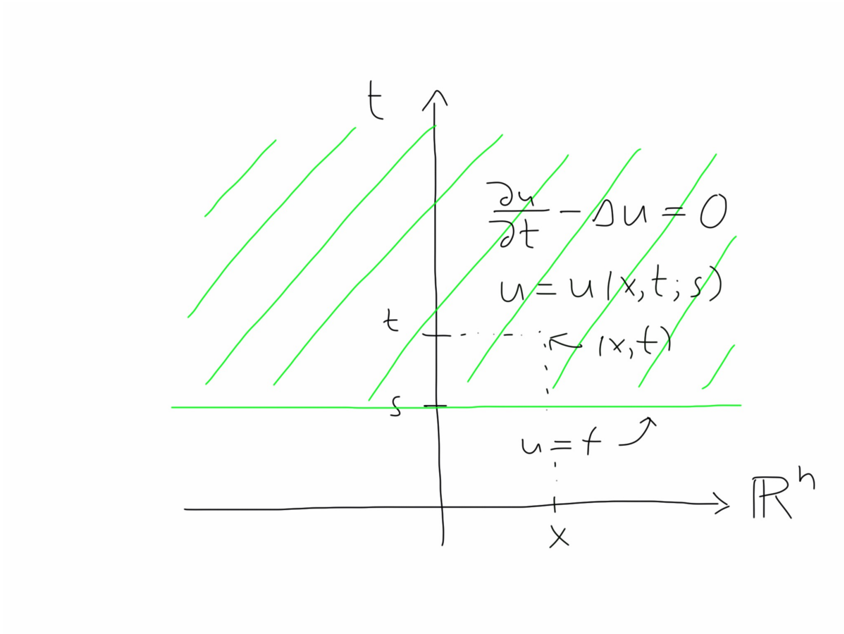


Figure 5.2: Duhamel's principle.

Proof. A direct formal calculation shows that

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,t) &= \lim_{h\to 0}\frac{1}{h}\left(\int_0^{t+h} u(x,t+h;s)\,ds - \int_0^t u(x,t;s)\,ds\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\int_0^{t+h} u(x,t+h;s)\,ds - \int_0^{t+h} u(x,t;s)\,ds\right)\\
&\quad + \lim_{h\to 0}\frac{1}{h}\left(\int_0^{t+h} u(x,t;s)\,ds - \int_0^t u(x,t;s)\,ds\right)\\
&= \lim_{h\to 0}\int_0^{t+h}\frac{1}{h}(u(x,t+h;s)-u(x,t;s))\,ds + \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} u(x,t;s)\,ds\\
&= u(x,t;t) + \int_0^t \frac{\partial u}{\partial t}(x,t;s)\,ds. \qquad \square
\end{aligned}$$

The following theorem claims that this gives a solution to the initial value problem for the nonhomogeneous heat equation.

Theorem 5.4. Let $f \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$. The solution of the nonhomogeneous Cauchy problem (5.2) is

$$\begin{aligned}
u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,dy\,ds\\
&= \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}}\left(\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\,dy\right) ds. \qquad (5.3)
\end{aligned}$$

Proof. We give a formal argument here. By the previous lemma, we have

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) &= u(x,t;t) + \int_0^t \frac{\partial u}{\partial t}(x,t;s)\,ds \\ &= f(x,t) + \int_0^t \Delta u(x,t;s)\,ds \\ &= f(x,t) + \Delta\left(\int_0^t u(x,t;s)\,ds\right) \\ &= f(x,t) + \Delta u(x,t).\end{aligned}$$

Moreover, by letting $t \to 0$ in (5.3), we see that the boundary condition is satisfied using the approximation of the identity. The detailed proof is rather similar to the proof of Theorem 4.19 and it will be omitted here. □

Remarks 5.5:

(1) To solve the general problem

$$\begin{cases} u_t - \Delta u = f & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = g & \text{in} \quad \mathbb{R}^n \times \{t=0\}, \end{cases}$$

we can use the same approach is in Remark 4.14 (3) for the Laplace equation and write $u = u_1 + u_2$ with

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \Delta u_1 = f & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u_1 = 0 & \text{in} \quad \mathbb{R}^n \times \{t=0\}, \end{cases}$$

and

$$\begin{cases} \dfrac{\partial u_2}{\partial t} - \Delta u_2 = 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u_2 = g & \text{in} \quad \mathbb{R}^n \times \{t=0\}. \end{cases}$$

We can combine (5.1) and (5.3) to find that

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)\,dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)\,dy\,ds.$$

(2) Duhamel's principle does not depend on the specific structure of the equation and it applies to other linear ODEs and PDE as well.

5.4 Separation of variables in $\mathbb{R}^n$

In the previous section we derived a solution to an initial value problem for the heat equation in the whole space $\mathbb{R}^n$. The goal of this section is to derive corresponding solutions in a subdomain. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a smooth boundary. Consider the initial and boundary value problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \Omega \times (0,\infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u = g & \text{on} \quad \Omega \times \{t=0\}. \end{cases}$$

Step 1 (Separation of variables): We separate variables and look for a solution in the form

$$u(x,t) = v(t)w(x), \quad x \in \Omega, \quad t > 0. \tag{5.4}$$

Then

$$u_t(x,t) = v'(t)w(x) \quad \text{and} \quad \Delta u(x,t) = v(t)\Delta w(x)$$

and

$$0 = u_t(x,t) - \Delta u(x,t) = v'(t)w(x) - v(t)\Delta w(x).$$

It follows that

$$\frac{v'(t)}{v(t)} = \frac{\Delta w(x)}{w(x)}$$

for every $x \in \Omega$ and $t > 0$ such that $w(x) \neq 0$ and $v(t) \neq 0$. Since the left hand side depends only on t and the right hand side depends only on x, both sides have to be the same constant. Thus we arrive at

$$\begin{cases} \dfrac{v'(t)}{v(t)} = -\lambda, & t > 0, \\ \dfrac{\Delta w(x)}{w(x)} = -\lambda, & x \in \Omega. \end{cases}$$

Here the negative sign is a convention that will be apparent soon.

THE MORAL: The problem for the heat equation has been transformed to an ODE and an eigenvalue problem for the Laplace equation.

Step 2 (Solution of the separated equations): The general solution of $v' = -\lambda v$ is

$$v(t) = ce^{-\lambda t},$$

where c is a constant.

Consider then the other equation $-\Delta w = \lambda w$. We say that λ is an eigenvalue of the (negative) Laplacian in Ω, if there exists a solution w of the problem

$$\begin{cases} -\Delta w = \lambda w & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

so that w is not identically zero in Ω. The function w is the eigenfunction corresponding to the eigenvalue λ. Observe that Green's first identity gives

$$\begin{aligned} \int_\Omega |\nabla w|^2 \, dx &= \int_\Omega \nabla w \cdot \nabla w \, dx \\ &= -\int_\Omega w \Delta w \, dx + \int_{\partial\Omega} \frac{\partial w}{\partial \nu} \underbrace{w}_{=0} \, dS \\ &= \lambda \int_\Omega w^2 \, dx. \end{aligned}$$

This implies that $\lambda \geqslant 0$ and this is explains the choice of the negative sign above. In fact, we have $\lambda > 0$, since if $\lambda = 0$, then the Dirichlet problem above only has the trivial solution $w = 0$. From (5.4) we conclude that

$$u(x,t) = ce^{-\lambda t}w(x)$$

is a solution to

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \Omega \times (0,\infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \end{cases} \tag{5.5}$$

with the initial condition $u(x,0) = cw(x)$ for $x \in \Omega$.

Step 3 (Solution of the entire problem): If λ_j, $j = 1,2,\dots$, is an eigenvalue and w_j, $j = 1,2,\dots$, is the corresponding eigenfunction, then the linear combination

$$u(x,t) = \sum_{j=1}^{\infty} c_j e^{-\lambda_j t} w_j(x)$$

should be a solution to (5.5) with the initial condition

$$u(x,0) = \sum_{j=1}^{\infty} c_j w_j(x).$$

If we can determine the coefficients c_j, $j = 1,2,\dots$, so that

$$\sum_{j=1}^{\infty} c_j w_j(x) = g(x), \tag{5.6}$$

then u is a solution of the original problem.

It is known (but out of the scope of this course) that there are infinite number of eigenvalues $\lambda_j > 0$, $j = 1,2,\dots$, with $\lambda_j \to \infty$ as $j \to \infty$. We arrange the eigenvalues in the increasing order so that $0 < \lambda_1 < \lambda_2 < \dots$. Moreover, the corresponding eigenfunctions $\{w_j\}_{j=1}^{\infty}$can be chosen to be an orthonormal basis in $L^2(\Omega)$. By results in Chapter 1, this means that if $g \in L^2(\Omega)$, then

$$c_j = \langle g, w_j \rangle = \int_{\Omega} g(y) w_j(y)\,dy, \quad j = 1,2,\dots.$$

Here $\langle \cdot,\cdot \rangle$ denotes the standard inner product in $L^2(\Omega)$. The coefficients c_j, $j = 1,2,\dots$, can be seen as the Fourier coefficients of $g \in L^2(\Omega)$ and the series

$$g = \sum_{j=1}^{\infty} c_j w_j = \sum_{j=1}^{\infty} \langle g, w_j \rangle w_j$$

converges in $L^2(\Omega)$, as in Chapter 1. In addition, the series

$$u(x,t) = \sum_{j=1}^{\infty} \langle g, w_j \rangle e^{-\lambda_j t} w_j(x) \tag{5.7}$$

gives the solution to the original problem in $L^2(\Omega)$. Thus we have the representation formula

$$\begin{aligned}
u(x,t) &= \sum_{j=1}^{\infty} \langle g, w_j\rangle e^{-\lambda_j t} w_j(x) \\
&= \sum_{j=1}^{\infty} \left(\int_\Omega g(y) w_j(y)\, dy \right) e^{-\lambda_j t} w_j(x) \\
&= \int_\Omega \left(\sum_{j=1}^{\infty} e^{-\lambda_j t} w_j(x) w_j(y) \right) g(y)\, dy \\
&= \int_\Omega K(x,y,t) g(y)\, dy,
\end{aligned}$$

where

$$K(x,y,t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} w_j(x) w_j(y)$$

is the heat kernel in Ω.

THE MORAL: Existence of a solution can be proved using separation of variables and eigenfunction expansions in bounded domains.

This is precisely the same strategy as in the separation of variables in Chapter 1. Note that this approach depends on whether we are able to find many enough eigenvalues and eigenfunctions so that we can represent the initial value as an infinite linear combination of the eigenfunctions. Moreover, we have to verify that the series above converges in an appropriate sense and that the representation formula (5.7) above really is a solution to the original problem.

Remark 5.6. For time dependent problems it is also relevant to study the behaviour of solutions as $t \to \infty$.

Claim:

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leqslant e^{-\lambda_1 t} \|g\|_{L^2(\Omega)}, \quad t > 0,$$

where $\lambda_1 > 0$ is the first (and the smallest) eigenvalue of Laplacian.

Reason. Since

$$\begin{aligned}
|u(x,t)| &= \left| \sum_{j=1}^{\infty} \langle g, w_j\rangle e^{-\lambda_j t} w_j(x) \right| \\
&\leqslant \sum_{j=1}^{\infty} |\langle g, w_j\rangle| e^{-\lambda_j t} |w_j(x)| \\
&\leqslant e^{-\lambda_1 t} \sum_{j=1}^{\infty} |\langle g, w_j\rangle| |w_j(x)|, \quad (0 < \lambda_1 < \lambda_2 < \dots)
\end{aligned}$$

by Parseval's equality we have

$$\begin{aligned}
\|u(\cdot,t)\|_{L^2(\Omega)} &\leqslant e^{-\lambda_1 t}\left\|\sum_{j=1}^{\infty}\langle g,w_j\rangle w_j(\cdot)\right\|_{L^2(\Omega)} \\
&= e^{-\lambda_1 t}\left(\sum_{j=1}^{\infty}|\langle g,w_j\rangle|^2\right)^{1/2} \\
&= e^{-\lambda_1 t}\|g\|_{L^2(\Omega)}.
\end{aligned}$$

■

In particular, the solution tends to zero as time goes to infinity. This method can be used to study stability as well.

Remark 5.7. Similar argument also applies, for example, to the Neumann problem

$$\begin{cases}
u_t - \Delta u = 0 & \text{in} \quad \Omega \times (0,\infty), \\
\dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\
u = g & \text{on} \quad \Omega \times \{t=0\}.
\end{cases}$$

Physically this models an insulated boundary problem. This leads to the construction of orthonormal eigenfunctions for the Laplacian with the zero Neumann boundary conditions.

5.5 Maximum principle

The heat equation resembles the Laplace equation in the sense that its solutions satisfy a maximum principle. To formulate this principle, let us introduce the space-time cylinder

$$\Omega_T = \Omega \times (0,T),$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded and $0 < T < \infty$. By the maximum principle for the solutions of the Laplace equation $\Delta u = 0$ we know that harmonic functions achieve their maximum on the boundary $\partial\Omega$. For the heat equation, the result states that the maximum is achieved on certain part of the boundary, which is called the parabolic boundary

$$\Gamma_T = (\partial\Omega \times [0,T]) \cup (\Omega \times \{t=0\}).$$

Observe that Γ_T consists of the base and the lateral sides of the cylinder Ω_T.

Theorem 5.8 (Weak maximum principle). Let $\Omega \subset \mathbb{R}^n$ be open and bounded and assume that $u \in C^2(\Omega_T) \cap C(\overline{\Omega_T})$ is a solution to the heat equation in Ω_T. Then

$$\max_{(x,t)\in\overline{\Omega_T}} u(x,t) = \max_{(x,t)\in\Gamma_T} u(x,t).$$

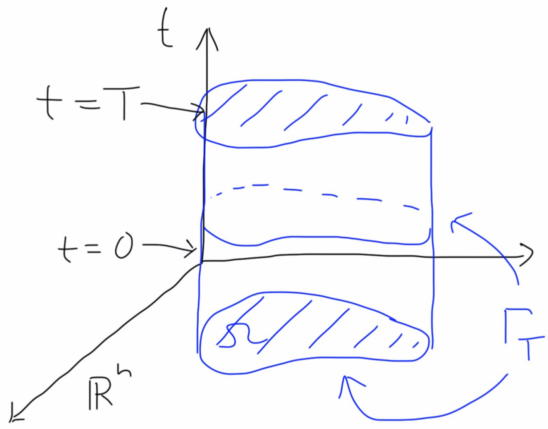


Figure 5.3: The maximum principle for the heat operator.

Remark 5.9. If we replace u by $-u$ we get the corresponding statement with min replacing max. Observe that $-u$ solves the heat equation whenever u solves the heat equation.

THE MORAL: A solution to the heat equation attains maximum and minimum in Ω_T on the parabolic boundary Γ_T. This result holds for any bounded domain without regularity assumptions on the boundary. Observe carefully that at a fixed time $t > 0$ the function $x \mapsto u(x,t)$ does not need to have to have maximum or minimum at $\partial\Omega$.

Proof. We shall prove the claim in two steps.

Step 1: Fix $\varepsilon > 0$ and set $v(x,t) = u(x,t) - \varepsilon t$. We will first show that

$$\max_{(x,t)\in\overline{\Omega_T}} v(x,t) = \max_{(x,t)\in\Gamma_T} v(x,t)$$

and then conclude the claim by letting $\varepsilon \to 0$.

Now let us assume, to the contrary, that there is a maximum point in $\Omega \times (0,T]$, that is,

$$\max_{(x,t)\in\overline{\Omega_T}} v(x,t) > \max_{(x,t)\in\Gamma_T} v(x,t).$$

Then there exists $(x_0,t_0) \in \overline{\Omega_T} \setminus \Gamma_T = \Omega \times (0,T]$ such that

$$v(x_0,t_0) = \max_{(x,t)\in\overline{\Omega_T}} v(x,t).$$

In particular, we have $v(x_0,t) \leqslant v(x_0,t_0)$ for all $t < t_0$. This implies that

$$\frac{\partial v}{\partial t}(x_0,t_0) \geqslant 0,$$

since otherwise the maximum would not be achieved at $t = t_0$ as a function of t. On the other hand, $v(x,t_0) \leqslant v(x_0,t_0)$ for all $x \in \Omega$, since (x_0,t_0) is a maximum of v as a function of x. This implies that $\nabla v(x_0,t_0) = 0$ and

$$\frac{\partial^2 v}{\partial x_j^2}(x_0,t_0) \leqslant 0, \quad j = 1,\dots,n,$$

(think of a function of one variable: if f has a local maximum at x_0 and f is twice differentiable, then $f''(x_0) \leqslant 0$). By summing up, we have $\Delta v(x_0,t_0) \leqslant 0$. Thus

$$\frac{\partial v}{\partial t}(x_0,t_0) - \Delta v(x_0,t_0) \geqslant 0.$$

This is a contradiction with

$$\frac{\partial v}{\partial t} - \Delta v = u_t - \varepsilon - \Delta u = -\varepsilon < 0.$$

Note that the auxiliary function v was introduced only to get a strict inequality here.

Step 2:

$$\begin{aligned}
\max_{(x,t)\in\overline{\Omega_T}} u(x,t) &= \max_{(x,t)\in\overline{\Omega_T}} (v(x,t) + \varepsilon t) \quad (v = u - \varepsilon t) \\
&\leqslant \max_{(x,t)\in\overline{\Omega_T}} v(x,t) + \varepsilon T \\
&= \max_{(x,t)\in\Gamma_T} v(x,t) + \varepsilon T \quad \text{(Step 1)} \\
&\leqslant \max_{(x,t)\in\Gamma_T} u(x,t) + \varepsilon T. \quad (v \leqslant u)
\end{aligned}$$

Letting $\varepsilon \to 0$ we have

$$\max_{(x,t)\in\overline{\Omega_T}} u(x,t) \leqslant \max_{(x,t)\in\Gamma_T} u(x,t).$$

On the other hand, we always have

$$\max_{(x,t)\in\overline{\Omega_T}} u(x,t) \geqslant \max_{(x,t)\in\Gamma_T} u(x,t)$$

and thus

$$\max_{(x,t)\in\overline{\Omega_T}} u(x,t) = \max_{(x,t)\in\Gamma_T} u(x,t).$$

□

Remark 5.10. The maximum principle gives a comparison principle and a stability result as in the case of the Laplace equation (exercises).

Theorem 5.11 (Uniqueness for bounded domains). Assume that Ω_T is bounded, $g \in C(\Gamma_T)$ and $f \in C(\Omega_T)$. Then there exists at most one solution $u \in C^2(\Omega_T) \cap C(\overline{\Omega_T})$ of the initial and boundary value problem

$$\begin{cases} u_t - \Delta u = f & \text{in} \quad \Omega_T, \\ u = g & \text{on} \quad \Gamma_T. \end{cases}$$

THE MORAL: The initial and boundary values can be given only on the parabolic boundary Γ_T. The equation determines the values uniquely inside Ω_T and on the top $\Omega \times \{t = T\}$, as in the representation formula (5.1).

Proof. Let u, v be two solutions to the problem. Then $w = u - v$ satisfies

$$\begin{cases} w_t - \Delta w = 0 & \text{in} \quad \Omega_T, \\ w = 0 & \text{on} \quad \Gamma_T. \end{cases}$$

By the weak maximum principle applied to w we see that $w \leqslant 0$ in $\overline{\Omega_T}$. Now applying the weak maximum principle to $-w$ we get the opposite inequality $w \geqslant 0$ in $\overline{\Omega_T}$. Thus $w = 0$ on $\overline{\Omega_T}$, which implies $u = v$ in $\overline{\Omega_T}$. □

Example 5.12. Let $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$,

$$u(x,t) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} \frac{\partial^j}{\partial t^j}(e^{-1/t^2}), & x \in \mathbb{R}, \quad t > 0, \\ 0, \quad t = 0. \end{cases}$$

Then

$$\begin{aligned} \lim_{t \to 0} u(x,t) &= \lim_{t \to 0} \sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} \frac{\partial^j}{\partial t^j}(e^{-1/t^2}) \\ &= \sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} \underbrace{\lim_{t \to 0} \frac{\partial^j}{\partial t^j}(e^{-1/t^2})}_{=0} = 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(x,t) &= \sum_{j=1}^{\infty} 2j(2j-1) \frac{x^{2j-2}}{(2j)!} \frac{\partial^j}{\partial t^j}(e^{-1/t^2}) \\ &= \sum_{j=1}^{\infty} \frac{x^{2j-2}}{(2j-2)!} \frac{\partial^j}{\partial t^j}(e^{-1/t^2}) \end{aligned}$$

and

$$\begin{aligned} u_t(x,t) &= \sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} \frac{\partial^{j+1}}{\partial t^{j+1}}(e^{-1/t^2}) \\ &= \sum_{j=1}^{\infty} \frac{x^{2(j-1)}}{(2(j-1))!} \frac{\partial^j}{\partial t^j}(e^{-1/t^2}). \end{aligned}$$

Thus u is a solution to

$$\begin{cases} u_t - u_{xx} = 0 & \text{in} \quad \mathbb{R} \times (0,\infty), \\ u = 0 & \text{on} \quad \mathbb{R} \times \{t = 0\}. \end{cases}$$

Since au is a solution to the same problem for every $a \in \mathbb{R}$, we see that the uniqueness fails for unbounded domains without extra assumptions. In this case, the problem has infinitely many solutions. Observe that $u = 0$ is the physically reasonable solution, but there are many nonphysical solutions as well. The growth condition in the previous theorem excludes these nonphysical solutions. The nonphysical solutions grow extremely fast as $|x| \to \infty$.

Note that the definition of $u(x,t)$ also works for $t < 0$, so that the backward Cauchy problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{in} \quad \mathbb{R} \times (-\infty,0), \\ u = 0 & \text{on} \quad \mathbb{R} \times \{t = 0\}. \end{cases}$$

does not have a unique solution in general.

Remarks 5.13:

(1) There is also a version of the strong maximum principle for solutions of the heat equation. Let Ω be an open, bounded and connected set in $\mathbb{R}^n$. If there exists a point $(x_0,t_0) \in \Omega \times (0,T]$ such that

$$u(x_0,t_0) = \max_{(x,t)\in\overline{\Omega}_T} u(x,t)$$

then u is constant in $\overline{\Omega}_{t_0}$. The proof of the strong maximum principle is beyond the scope of this course.

(2) By applying the strong maximum principle to the function $-u$ we get a similar statement with min replacing max. The strong maximum (or minimum) principle asserts that if u attains its maximum or minimum at an interior point, then u is constant at all earlier times. This is physically intuitive, since the solution will be constant on the time interval $[0,t_0]$ if the initial and boundary conditions are constant. However, the solution may change at later times $t > t_0$ if the boundary conditions change. This happens, for example, if we start heating the boundary.

(3) Changing t to $-t$ does not preserve the heat equation and the PDE distinguishes between solutions forward and backward in time. This corresponds to the physical fact that heat conduction is, in general, irreversible, given an initial temperature we may predict future temperatures, but we cannot in general determine the thermal status that generated that particular temperature distribution. The backward in time problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \mathbb{R}^n \times (-\infty,0), \\ u = g & \text{on} \quad \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

is illposed in the sense that, unlike the forward in time problem, it is not solvable, in general, within the class of bounded solutions. Indeed, if a bounded, continuous solution did exist for every choice $g \in C(\mathbb{R}^n)$, then the representation formula (5.1) gives

$$\begin{aligned} g(x) = u(x,0) &= \int_{\mathbb{R}^n} \Phi(x-y,T)u(y,-T)\,dy \\ &= \frac{1}{(4\pi T)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4T}} u(y,-T)\,dy. \end{aligned}$$

This implies that g is smooth, which cannot be the case if g is only assumed to be continuous.

Moreover, the backward in time problem is not stable. To see this, let $\varepsilon > 0$ and define

$$u(x,t) = \varepsilon e^{-t/\varepsilon^2} \sin\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}, \quad t < 0.$$

Then u is a solution to the problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{in} \quad \mathbb{R} \times (-\infty, 0), \\ u = g & \text{on} \quad \mathbb{R} \times \{t = 0\}, \end{cases}$$

with

$$g(x) = \varepsilon \sin\left(\frac{x}{\varepsilon}\right).$$

Then

$$\max_{x\in\mathbb{R}} |g(x)| \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

but

$$\max_{x\in\mathbb{R}} |u(x,t)| \to \infty \quad \text{as} \quad \varepsilon \to 0.$$

This means that the solution does not depend continuously on the boundary values.

5.6 Energy methods for the heat equation

In this section we discuss how one can use energy methods to show, for example, uniqueness of solutions to the heat equation. This discussion is analogous to the energy methods applications for harmonic functions.

Let $\Omega \subset \mathbb{R}^n$ be open and bounded. We consider the initial and boundary value problem for the heat equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \Omega_T, \\ u = g & \text{on} \quad \Gamma_T. \end{cases} \tag{5.8}$$

Theorem 5.14 (Uniqueness by energy methods). There exists at most one solution for (5.8).

Proof. Let u, v be two solutions to (5.8). Then $w = u - v$ satisfies

$$\begin{cases} w_t - \Delta w = 0 \quad \text{in} \quad \Omega_T, \\ w = 0 \quad \text{on} \quad \Gamma_T. \end{cases}$$

For $0 \leqslant t \leqslant T$ we define the energy

$$e(t) = \int_\Omega w(x,t)^2\,dx.$$

Then

$$\begin{aligned} e'(t) &= \frac{\partial}{\partial t}\int_\Omega w(x,t)^2\,dx = \int_\Omega \frac{\partial}{\partial t}(w(x,t)^2)\,dx \\ &= \int_\Omega 2w\frac{\partial w}{\partial t}\,dx = \int_\Omega 2w\Delta w\,dx \quad (w \text{ satisfies the heat equation}) \\ &= 2\int_{\partial\Omega}\frac{\partial w}{\partial \nu}w\,dS - 2\int_\Omega |\nabla w|^2\,dx \quad (\text{Green's first identity}) \\ &= -2\int_\Omega |\nabla w|^2\,dx \leqslant 0. \quad (w = 0 \text{ on } \partial\Omega) \end{aligned}$$

We then conclude that $e'(t) \leqslant 0$, which implies that $e(t)$ is a decreasing function, so that $0 \leqslant e(t) \leqslant e(0)$ for all $0 \leqslant t \leqslant T$. However $e(0) = 0$ since $w = 0$ on Γ_T. Thus

$$e(t) = \int_\Omega w(x,t)^2\,dx = 0 \quad \text{for all} \quad 0 \leqslant t \leqslant T$$

and thus $w = 0$ on Ω_T. □

5.7 Summary

(1) We have derived several representation formulas for the solutions. The Fourier series and the Fourier transform can be used to solve boundary value problems in special cases. The solution to the Cauchy problem for heat equation in the whole space can be represented as a convolution of the source term with the fundamental solution.

(2) The nonhomogeneous Cauchy problem can be solved using Duhamel's principle by Theorem 5.4.

(3) Separation of variables and eigenvalue problems can be used to derive a representation formula for the solutions in a subdomain, see Section 5.4.

(4) The heat equation has infinite propagation speed for disturbances.

(5) The heat equation smoothens the boundary data.

(6) The boundary and initial values can be given only on the parabolic boundary of a bounded space time cylinder. The same applies to the maximum principle, uniqueness and stability results.

(7) The uniqueness fails for the Cauchy problem without extra growth assumptions.

(8) Energy decreases and solutions decay to zero as $t \to \infty$.

(9) Energy methods can be used to give a short proof of the uniqueness in a bounded space time cylinder.

(10) If the direction of time changed, then the obtained backward in time heat equation does not have a unique solution and the solution does not depend continuously on the boundary data.

We study the wave equation in all dimensions, but with particular focus on the physically relevant cases of dimensions one, two and three. d'Alembert's formula gives a solution of the Cauchy problem in the one-dimensional case. The three-dimensional Cauchy problem is solved by a spherical mean reduction to the one-dimensional case. Finally, the two-dimensional problem is solved by Hadamard's decent method from three dimensions. Duhamel's principle applies to the nonhomogeneous problem. Energy methods can be used to prove the existence.

6 Wave equation

In this chapter we study the n-dimensional wave equation with particular attention to the physically relevant cases of dimensions one, two and three. It turns out that the properties of the solutions depend on the dimension. We can think of the wave equation as describing the displacement of a vibrating string (in dimension $n=1$), a vibrating membrane (in dimension $n=2$) or an elastic solid (dimension $n=3$). In dimension $n=3$ this equation also determines the behaviour of electromagnetic waves in vacuum and the propagation of sound waves.

The $n-$dimensional wave equation is

$$u_{tt}-\Delta u=0$$

and the nonhomogeneous wave equation is

$$u_{tt}-\Delta u=f.$$

Here

$$\Delta u=\sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_j^2},$$

that is, the Laplace operator is taken with respect to the spatial variable x.

Remark 6.1. The general equation

$$u_{tt}-c^2\Delta u=0,$$

where $c\in\mathbb{R}$, can be reduced to the standard wave equation by the change of variables (exercise).

6.1 Physical interpretation

The function $u = u(x,t)$ represents the displacement of the point x at the moment t. If V is any smooth subdomain of Ω, the acceleration in V is

$$\frac{\partial^2}{\partial t^2}\int_V u(x,t)\,dx$$

and the net contact force is

$$-\int_{\partial V} F(x,t)\cdot\nu(x)\,dS(x),$$

where $F = (F_1,\dots,F_n)$ is the force acting on V through ∂V and ν is the unit outer normal of ∂V. This follows from Newton's law, which asserts that the mass times the acceleration is the force. The mass density is assumed to be one. By the Gauss-Green theorem we have

$$\begin{aligned}\int_V \operatorname{div}_x F(x,t)\,dx &= \int_{\partial V} F(x,t)\cdot\nu(x)\,dS(x)\\ &= -\frac{\partial^2}{\partial t^2}\int_V u(x,t)\,dx = -\int_V \frac{\partial^2 u}{\partial t^2}(x,t)\,dx.\end{aligned}$$

The last equality follows by switching the order of the derivative and the integral. Since this holds for every subdomain V of Ω, we have

$$\operatorname{div}_x F(x,t) = -\frac{\partial^2 u}{\partial t^2}(x,t).$$

It is physically reasonable to assume that the force F is proportional to the gradient ∇u but in the opposite direction. Thus

$$F(x,t) = -c^2\nabla u(x,t), \quad c\in\mathbb{R}.$$

This gives

$$u_{tt}(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t) = -\operatorname{div}_x F(x,t) = c^2\operatorname{div}_x\nabla u(x,t) = c^2\Delta u(x,t),$$

which implies that u is a solution of the general wave equation.

6.2 The one-dimensional wave equation

We have showed using the Fourier transform that the solution of the Cauchy problem

$$\begin{cases} u_{tt}-\Delta u = 0 \quad \text{in} \quad \mathbb{R}^n\times(0,\infty),\\ u = g \quad \text{in} \quad \mathbb{R}^n\times\{t=0\},\\ \dfrac{\partial u}{\partial t} = h \quad \text{in} \quad \mathbb{R}^n\times\{t=0\}. \end{cases} \tag{6.1}$$

for the n-dimensional wave equation is given by formula (3.5), which states that

$$u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi)\cos(|\xi|t) + \widehat{h}(\xi)\frac{\sin(|\xi|t)}{|\xi|} \right) e^{ix\cdot\xi}\,d\xi.$$

Remark 6.2. To consider initial value problems for the wave equation in a bounded domain $\Omega_T = \Omega \times (0,\infty)$, we can use eigenfunction expansions as in section 5.4.

In the one-dimensional case with $n = 1$, we have

$$\begin{aligned}
u(x,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} \left(\widehat{g}(\xi)\cos(|\xi|t) + \widehat{h}(\xi)\frac{\sin(|\xi|t)}{|\xi|} \right) e^{ix\xi}\,d\xi \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \left(\widehat{g}(\xi)\cos(\xi t) + \widehat{h}(\xi)\frac{\sin(\xi t)}{\xi} \right) e^{ix\xi}\,d\xi \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(\xi)\frac{1}{2}(e^{i\xi t} + e^{-i\xi t})e^{ix\xi}\,d\xi \\
&\qquad + \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{h}(\xi)\frac{1}{2i\xi}(e^{i\xi t} - e^{-i\xi t})e^{ix\xi}\,d\xi.
\end{aligned}$$

Here we used the symmetry of the trigonometric functions and trigonometric identities (2.11). Theorem 3.4 (6) and the Fourier inversion theorem 3.14 imply

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(\xi)\frac{1}{2}(e^{i\xi t} + e^{-i\xi t})e^{ix\xi}\,d\xi \\
&\quad = \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi t}\widehat{g}(\xi)e^{ix\xi}\,d\xi + \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi t}\widehat{g}(\xi)e^{ix\xi}\,d\xi \\
&\quad = \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g(x+t)}(\xi)e^{ix\xi}\,d\xi + \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g(x-t)}(\xi)e^{ix\xi}\,d\xi \\
&\quad = \frac{1}{2}(g(x+t) + g(x-t)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{h}(\xi)\frac{1}{2i\xi}(e^{i\xi t} - e^{-i\xi t})e^{ix\xi}\,d\xi \\
&\quad = \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi t}\widehat{H}(\xi)e^{ix\xi}\,d\xi - \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi t}\widehat{H}(\xi)e^{ix\xi}\,d\xi \\
&\quad = \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{H(x+t)}(\xi)e^{ix\xi}\,d\xi - \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{H(x-t)}(\xi)e^{ix\xi}\,d\xi \\
&\quad = \frac{1}{2}(H(x+t) - H(x-t)),
\end{aligned}$$

where

$$\widehat{H}(\xi) = \frac{\widehat{h}(\xi)}{i\xi}.$$

Claim: $H'(x) = h(x)$ and

$$H(x) = \int_{-\infty}^{x} H'(y)\,dy = \int_{-\infty}^{x} h(y)\,dy$$

Reason. Theorem 3.5 implies

$$\widehat{h}(\xi) = i\xi\widehat{H}(\xi) = \widehat{H'}(\xi)$$

and thus by the Fourier inversion theorem 3.14, we obtain $H'(x) = h(x)$. ■

We conclude that

$$\begin{aligned} u(x,t) &= \frac{1}{2}(g(x+t)+g(x-t)) + \frac{1}{2}(H(x+t)-H(x-t)) \\ &= \frac{1}{2}(g(x+t)+g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy. \end{aligned}$$

This is d'Alembert's formula for the one-dimensional wave equation. Thus we have shown that every solution of the one-dimensional Cauchy problem satisfies this formula. The next result shows that converse holds as well.

Theorem 6.3 (d'Alembert's formula). Assume that $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, and define u by

$$u(x,t) = \frac{1}{2}(g(x+t)+g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$

Then u is a solution to the Cauchy problem (6.1) in the one-dimensional case.

Proof. The proof is a straightforward calculation (exercise). □

THE MORAL: A function is a solution of the one-dimensional Cauchy problem if and only if it satisfies d'Alembert's formula.

Remarks 6.4:

(1) d'Alembert's formula can be derived without Fourier analysis, see for example [4] and [3].

(2) d'Alembert's formula gives existence. The solution u is of the form

$$u(x,t) = F(x+t) + G(x-t)$$

for appropriate functions F and G. Conversely, every function of this form is a solution to $u_{tt} - u_{xx} = 0$ in $\mathbb{R} \times (0,\infty)$ (exercise). The solution is a superposition of traveling waves. The function $F(x+t)$ can be thought of as a wave traveling in time with the speed one. In other words, think of the graph of the function $F(x)$ as a wave at time $t = 0$. Then at time t the wave has moved to left with speed one and this is the graph of $F(x+t)$. On the other hand, $G(x-t)$ is a wave traveling to right with the speed one.

(3) d'Alembert's formula gives uniqueness.

Reason. Assume that u and v are solutions with the same initial values. Then $u - v$ is a solution with the zero initial values and d'Alembert's formula implies $u - v = 0$. ■

(4) d'Alembert's formula gives stability.

Reason. Assume that u is a solution with the initial values g_1 and h_1 and that v is a solution with the initial values g_2 and h_2. d'Alembert's formula

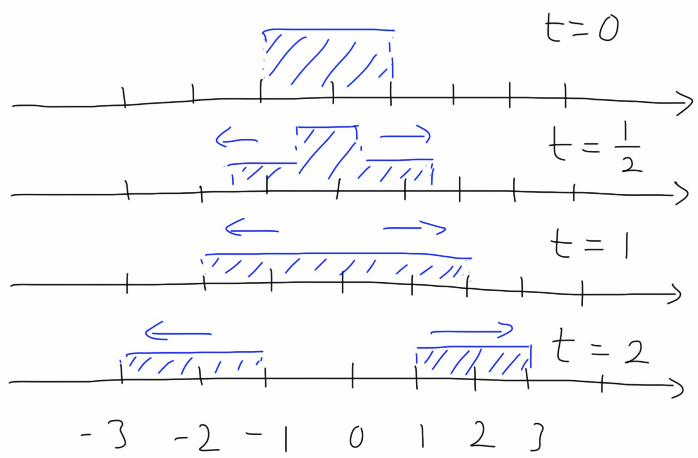


Figure 6.1: d'Alembert's solution with $h = 0$.

gives

$$\begin{aligned}|u(x,t)-v(x,t)| &\leq \frac{1}{2}|g_1(x+t)-g_2(x+t)| + \frac{1}{2}|g_1(x-t)-g_2(x-t)| \\ &\quad + \frac{1}{2}\int_{x-t}^{x+t} |h_1(y)-h_2(y)|\,dy \\ &\leq \sup_{y\in\mathbb{R}} |g_1(y)-g_2(y)| + t \sup_{y\in\mathbb{R}} |h_1(y)-h_2(y)| \\ &\leq \varepsilon + t\varepsilon = (1+t)\varepsilon,\end{aligned}$$

if

$$\sup_{y\in\mathbb{R}} |g_1(y)-g_2(y)| \leq \varepsilon \quad \text{and} \quad \sup_{y\in\mathbb{R}} |h_1(y)-h_2(y)| \leq \varepsilon.$$

∎

This means that a small change in the initial data g and h will affect the solution u only a small amount. Thus the solution depends continously on the boundary data.

(5) If $g \in C^k(\mathbb{R})$ and $h \in C^{k-1}(\mathbb{R})$, then $u \in C^k(\mathbb{R}\times(0,\infty))$, but is not in general smoother. Thus the wave equation does not smoothen the solution as in the case of the Laplace or the heat equation.

(6) The solution at the point (x,t) depends only on the values of g and h on the interval $[x-t, x+t]$. This is called the domain of dependence of (x,t). Thus the value $u(x,t)$ is not affected by the choice of g and h outside that interval. Conversely, for every $x_0 \in \mathbb{R}$, there is a conical region called the

range of influence of x_0. Physically this means that the disturbances or signals propagate with a finite speed. Initial disturbance at x_0 will not be felt at a point x until time $t = |x - x_0|$.

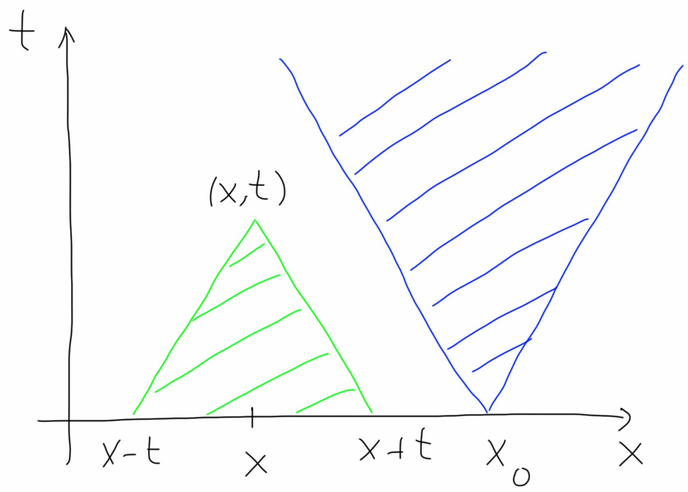


Figure 6.2: The domain of dependence and the range of influence.

(7) d'Alembert's formula makes sense also with initial values that are not necessary even continuous. This is analogous to the mean value principle for the harmonic functions, see section 4.9. Then the function $u(x,t)$ is no longer differentiable and it may not be a solution of the wave equation in the classical sense. However, it is possible to consider so-called weak solutions, but this is out of the scope of this course.

THE MORAL: d'Alembert's formula gives a complete answer to the Cauchy problem in the one-dimensional case. However, at this point it is not clear what happens in higher dimensions.

Example 6.5. If

$$g(x) = \sin\left(\frac{j\pi x}{L}\right), \quad j = 1,2,\dots, \quad \text{and} \quad h(x) = 0,$$

then Theorem 2.50 gives

$$u(x,t) = \sin\left(\frac{j\pi x}{L}\right)\cos\left(\frac{j\pi x}{L}\right).$$

On the other hand, d'Alembert's formula above gives

$$u(x,t) = \frac{1}{2}\left(\sin\left(\frac{j\pi(x-t)}{L}\right) + \sin\left(\frac{j\pi(x-t)}{L}\right)\right).$$

Since $\sin a \cos b = \frac{1}{2}(\sin(a+b)+\sin(a-b))$ the two solutions are the same.

Example 6.6. Assume that $g(x)=0$ and $h(x)=x$, $0<x<1$. We take the odd 2-periodic extension of the function $h(x)=x$, $-1<x<1$, to the whole real line $\mathbb{R}$. By d'Alembert's formula

$$u(x,t)=\frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy=\frac{1}{2}(H(x+t)-H(x-t)),$$

where H is an antiderivative of h. For example, we can take

$$H(x)=\int_{-1}^{x} h(y)\,dy=\int_{-1}^{x} y\,dy=\frac{1}{2}x^2-\frac{1}{2},\quad -1<x<1.$$

Recall, that the antiderivative is unique up to an additive constant and, in this case, we can choose the constant as we like, since they cancel in the formula for the solution. The solution of the problem is

$$u(x,t)=\frac{1}{4}((x+t)^2-(x-t)^2)=xt,\quad -1<x<1.$$

We can take the 2-periodic extension of this function to the whole $\mathbb{R}$.

Remark 6.7. Consider the initial and boundary value problem on the first quadrant

$$\begin{cases} u_{tt}-u_{xx}=0 & \text{in} \quad \mathbb{R}_+\times(0,\infty),\\ u=g, \quad u_t=h & \text{on} \quad \mathbb{R}_+\times\{t=0\},\\ u=0 & \text{on} \quad \{x=0\}\times(0,\infty),\end{cases}$$

where $g(0)=h(0)=0$. We solve this problem by the method of reflection and extend u, g and h to $\mathbb{R}$ by an odd reflection

$$\widetilde{u}(x,t)=\begin{cases} u(x,t), & x\geqslant 0, \quad t\geqslant 0,\\ -u(-x,t), & x\leqslant 0, \quad t\geqslant 0,\end{cases}$$

$$\widetilde{g}(x)=\begin{cases} g(x), & x\geqslant 0,\\ -g(-x), & x\leqslant 0,\end{cases}$$

and

$$\widetilde{h}(x)=\begin{cases} h(x), & x\geqslant 0,\\ -h(-x), & x\leqslant 0.\end{cases}$$

Then the problem becomes

$$\begin{cases} \widetilde{u}_{tt}-\widetilde{u}_{xx}=0 & \text{in} \quad \mathbb{R}\times(0,\infty),\\ \widetilde{u}=\widetilde{g}, \quad \widetilde{u}_t=\widetilde{h} & \text{on} \quad \mathbb{R}\times\{t=0\}.\end{cases}$$

d'Alembert's formula gives

$$\widetilde{u}(x,t)=\frac{1}{2}(\widetilde{g}(x+t)+\widetilde{g}(x-t))+\frac{1}{2}\int_{x-t}^{x+t}\widetilde{h}(y)\,dy.$$

By the definitions of the odd reflections, we have

$$u(x,t)=\begin{cases}\dfrac{1}{2}(g(x+t)+g(x-t))+\dfrac{1}{2}\displaystyle\int_{x-t}^{x+t}h(y)\,dy, & x\geqslant t\geqslant 0,\\ \dfrac{1}{2}(g(x+t)-g(t-x))+\dfrac{1}{2}\displaystyle\int_{-x+t}^{x+t}h(y)\,dy, & 0\leqslant x\leqslant t.\end{cases}\tag{6.2}$$

This formula means that the initial displacement g splits into two parts, one moving to right with speed one and one moving left with speed one. The latter reflects off the point $x=0$, where the string is held fixed.

Example 6.8. By Hadamard's Example 4.47, the function

$$u(x_1,x_2)=\frac{1}{j^2}\sin(jx_1)\sinh(jx_2),\quad j=1,2,\dots,$$

is a solution to the Dirichlet problem

$$\begin{cases}\Delta u(x_1,x_2)=0 \quad \text{in} \quad (x_1,x_2)\in\mathbb{R}^2,\\ u(x_1,0)=0,\quad x_1\in\mathbb{R},\\ \dfrac{\partial u}{\partial x_2}(x_1,0)=\dfrac{1}{j}\sin(jx_1),\quad x_1\in\mathbb{R}.\end{cases}$$

Observe that for $x_2>0$, we have

$$\lim_{j\to\infty}\frac{1}{j^2}\sinh(jx_2)=\infty,$$

which shows that the solutions blows up even if the boundary values tend to zero as $j\to\infty$. Recall that $\sinh(x)=\frac{1}{2}(e^x-e^{-x})$. This shows that stability fails for this problem.

Next we modify Hadamard's example so that it applies to the wave equation. Consider u as a function on $\mathbb{R}^n\times(0,\infty)$ that is independent of variables $x_3,\dots,x_n$ and t. This is the trivial extension of u to the upper half-space. Then

$$u_{tt}-\Delta u=0 \quad \text{in} \quad \mathbb{R}^n\times(0,\infty)$$

with the initial conditions

$$u(x_1,0,x_3,\dots,x_n,t)=0$$

and

$$\frac{\partial u}{\partial x_2}(x_1,0,x_3,\dots,x_n,t)=\frac{1}{j}\sin(jx_1).$$

By taking j sufficiently large, the absolute value of the boundary data can be be everywhere arbitrarily small, while the solution takes arbitrarily large values even at points with $x_2\neq 0$ as small as we wish.

THE MORAL: This shows that stability fails for the Cauchy problem for to the wave operator when $n\geqslant 2$.

Remark 6.9. It is enough to solve the Cauchy problem only in the upper half-space $t\geqslant 0$. The reason is that the problem for the lower half-space $t\leqslant 0$ can be reduced to the problem in the upper half-space by switching t to $-t$. Note that the wave equation remains unchanged in this change of variables (exercise).

6.3 The Euler-Poisson-Darboux equation

In the higher dimensional case, the solutions of the wave equation do not have as simple expression as d'Alembert's formula above. In this section we consider the method of spherical averages and use it to solve the Cauchy problem (6.1).

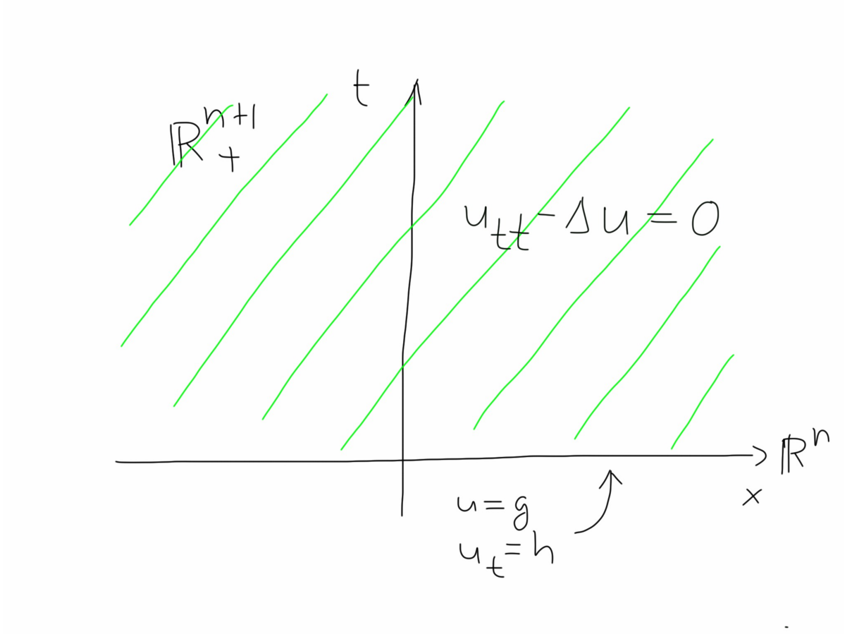


Figure 6.3: The Cauchy problem for the wave equation.

We shall use the method of spherical means and define the appropriate averages over spheres

$$
\begin{aligned}
U(x;r,t) &= \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y,t)\,dS(y), \\
G(x;r) &= \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} g(y)\,dS(y), \\
H(x;r) &= \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} h(y)\,dS(y).
\end{aligned}
$$

THE MORAL: The pointwise values of the functions are replaced by integral averages over spheres.

Remember that these averages converge to the corresponding function as $r \to 0$, when the function is continuous, that is,

$$
\lim_{r\to 0} U(x;r,t) = \lim_{r\to 0} \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y,t)\,dS(y) = u(x,t), \tag{6.3}
$$

if u is continuous, see the proof of Theorem 4.19. Here $|\partial B(x,r)| = n\alpha(n)r^{n-1}$ is the $(n-1)$-dimensional volume of the sphere $\partial B(x,r)$, see Remark 4.22. Note that these integral averages are continuous functions of x and $r \geqslant 0$. In this sense the integral averages above can be seen as approximations of the functions u, g and h. It can be shown by differentiating under the integral, that if u is a C^k as a function of x, then U has the same property. In other words, smoothness is preserved in the integral averages.

It turns out that $U(x;r,t)$ satisfies a one-dimensional equation which can be further transformed to the one-dimensional wave equation.

Lemma 6.10 (Euler-Poisson-Darboux equation). Let u be a solution of the Cauchy problem (6.1). Then for every fixed $x \in \mathbb{R}^n$, we have

$$\begin{cases} U_{tt} - U_{rr} - \dfrac{n-1}{r} U_r = 0 & \text{in} \quad \mathbb{R}_+ \times (0,\infty), \\ U = G, \quad U_t = H & \text{in} \quad \mathbb{R}_+ \times \{t = 0\}. \end{cases} \tag{6.4}$$

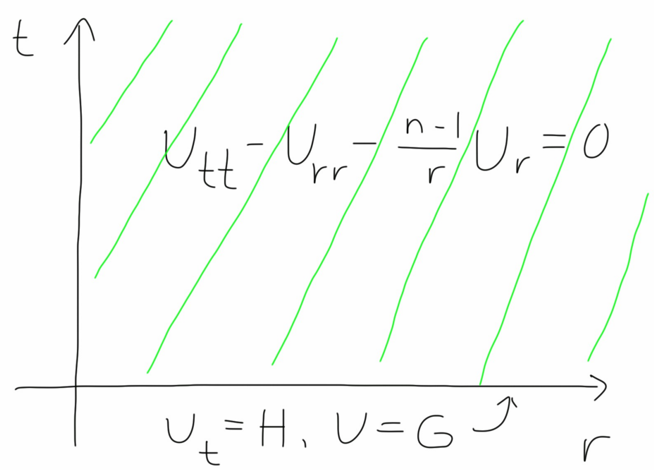


Figure 6.4: The Euler-Poisson-Darboux equation.

THE MORAL: The original Cauchy problem for the n-dimensional wave equation is transformed to a PDE in two variables r and t. If we can solve (6.4) at every point x, then we can obtain the solution of (6.1) by letting $r \to 0$ as in (6.3).

Remark 6.11. Recall from Section 4.4 that the Laplace operator applied to a radial function $u(x) = v(|x|) = v(r)$ is

$$\Delta u(x) = v''(r) + \frac{n-1}{r} v'(r).$$

Observe that this term also appears in the Euler-Poisson-Darboux equation above. For a radial function $u(x,t) = v(|x|,t) = v(r,t)$, we have $U(x;r,t) = v(r,t)$ and we have the Laplace equation in the spherical coordinates in the Euler-Poisson-Darboux equation.

Proof. As in the proof of the mean value property for harmonic functions, see Theorem 4.37, we have

$$\begin{aligned} U_r(x;r,t) &= \frac{r}{n}\frac{1}{|B(x,r)|}\int_{B(x,r)} \Delta u(y,t)\,dy \\ &= \frac{1}{n\alpha(n)r^{n-1}}\int_{B(x,r)} \Delta u(y,t)\,dy \\ &= \frac{1}{n\alpha(n)r^{n-1}}\int_{B(x,r)} u_{tt}(y,t)\,dy. \qquad (\Delta u = u_{tt}) \end{aligned}$$

This implies

$$\begin{aligned} r^{n-1}U_r(x;r,t) &= \frac{r^n}{n|B(x,r)|}\int_{B(x,r)} u_{tt}(y,t)\,dy \\ &= \frac{1}{n\alpha(n)}\int_{B(x,r)} u_{tt}(y,t)\,dy \\ &= \frac{1}{n\alpha(n)}\int_0^r\int_{\partial B(x,\rho)} u_{tt}(y,t)\,dS(y)\,d\rho. \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial}{\partial r}(r^{n-1}U_r(x;r,t)) &= \frac{1}{n\alpha(n)}\int_{\partial B(x,r)} u_{tt}(y,t)\,dS(y) \\ &= \frac{r^{n-1}}{|\partial B(x,r)|}\int_{\partial B(x,r)} u_{tt}(y,t)\,dS(y) \\ &= r^{n-1}U_{tt}(x;r,t), \end{aligned}$$

where in the last equality we switched the order of the integral and the derivative. On the other hand, by the product rule

$$\frac{\partial}{\partial r}(r^{n-1}U_r(x;r,t)) = (n-1)r^{n-2}U_r(x;r,t) + r^{n-1}U_{rr}(x;r,t).$$

This gives

$$r^{n-1}U_{tt}(x;r,t) = (n-1)r^{n-2}U_r(x;r,t) + r^{n-1}U_{rr}(x;r,t),$$

which implies the Euler-Poisson-Darboux equation.

The claims for the initial values are relatively clear. Indeed,

$$\begin{aligned} U(x;r,0) &= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y,0)\,dS(y) \\ &= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} g(y)\,dS(y) = G(x;r) \end{aligned}$$

and

$$\begin{aligned} U_t(x;r,t) &= \frac{\partial}{\partial t}\left(\frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y,t)\,dS(y)\right) \\ &= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} \frac{\partial u}{\partial t}(y,t)\,dS(y), \end{aligned}$$

from which it follows that

$$\begin{aligned} U_t(x;r,0) &= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} \frac{\partial u}{\partial t}(y,0)\,dS(y) \\ &= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} h(y)\,dS(y) = H(x;r). \end{aligned} \qquad \square$$

6.4 The three-dimensional wave equation

In this section we assume that $n = 3$ and that u is a solution to the Cauchy problem (6.1). We recall the definitions of the integral averages U, G and H from the previous section and denote

$$\widetilde{U} = rU, \quad \widetilde{G} = rG \quad \text{and} \quad \widetilde{H} = rH.$$

THE MORAL: These are modified integral averages of the functions over spheres.

Then

$$\begin{aligned} \widetilde{U}_{tt} - \widetilde{U}_{rr} &= rU_{tt} - (U + rU_r)_r \\ &= rU_{tt} - (U_r + U_r + rU_{rr}) \\ &= rU_{tt} - 2U_r - rU_{rr} \\ &= r(U_{tt} - U_{rr} - \frac{2}{r}U_r) = 0 \end{aligned}$$

in $\mathbb{R}_+ \times (0,\infty)$ by the Euler-Darboux-Poisson equation (6.4) with $n = 3$. Furthermore the initial conditions become

$$\begin{aligned} \widetilde{U}(x;r,0) = rU(x;r,0) &= \frac{r}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y,0)\,dS(y) \\ &= \frac{r}{|\partial B(x,r)|}\int_{\partial B(x,r)} g(y)\,dS(y) = rG(x;r) = \widetilde{G}(x;r) \end{aligned}$$

and, in the same way,

$$\begin{aligned} \widetilde{U}_t(x;r,t) = rU_t(x;r,t) &= \frac{\partial}{\partial t}\left(\frac{r}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y,t)\,dS(y)\right) \\ &= \frac{r}{|\partial B(x,r)|}\int_{\partial B(x,r)} \frac{\partial u}{\partial t}(y,t)\,dS(y), \end{aligned}$$

from which it follows that

$$\begin{aligned} \widetilde{U}_t(x;r,0) &= \frac{r}{|\partial B(x,r)|}\int_{\partial B(x,r)} \frac{\partial u}{\partial t}(y,0)\,dS(y) \\ &= \frac{r}{|\partial B(x,r)|}\int_{\partial B(x,r)} h(y)\,dS(y) = rH(x;r) = \widetilde{H}(x;r). \end{aligned}$$

Moreover,

$$\widetilde{U}(x;0,t) = \lim_{r\to 0} \frac{r}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y,t)\,dS(y) = u(x,t) \underbrace{\lim_{r\to 0} r}_{=0} = 0.$$

Thus we have reduced the problem to the wave equation in the first quadrant

$$\begin{cases} \widetilde{U}_{tt} - \widetilde{U}_{rr} = 0 & \text{in} \quad \mathbb{R}_+ \times (0,\infty), \\ \widetilde{U} = \widetilde{G}, \quad \widetilde{U}_t = \widetilde{H} & \text{on} \quad \mathbb{R}_+ \times \{t=0\}, \\ \widetilde{U} = 0 \quad \text{on} \quad \{r=0\} \times (0,\infty). \end{cases}$$

THE MORAL: The original Cauchy problem for the three-dimensional wave equation is transformed to a problem for the one-dimensional wave equation. We can solve this problem by d'Alemberts formula at every point x and we can obtain the solution to the original problem by considering

$$u(x,t) = \lim_{r\to 0} U(x;r,t) = \lim_{r\to 0} \frac{rU(x;r,t)}{r} = \lim_{r\to 0} \frac{\widetilde{U}(x;r,t)}{r}.$$

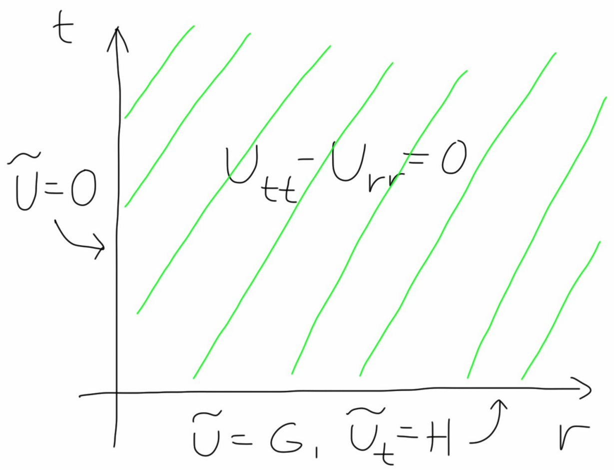


Figure 6.5: The one-dimensional problem for $\widetilde{U}$.

Applying the formula (6.2), we obtain

$$\widetilde{U}(x;r,t) = \begin{cases} \frac{1}{2}(\widetilde{G}(x;r+t) + \widetilde{G}(x;r-t)) + \frac{1}{2}\int_{r-t}^{r+t} \widetilde{H}(x;y)\,dy, & r \geqslant t \geqslant 0, \\ \frac{1}{2}(\widetilde{G}(x;r+t) - \widetilde{G}(x;t-r)) + \frac{1}{2}\int_{-r+t}^{r+t} \widetilde{H}(x;y)\,dy, & 0 \leqslant r \leqslant t. \end{cases}$$

Here we are interested in the solution for $0 \leqslant r \leqslant t$ since we want to eventually pass $r \to 0$. Thus we have

$$
\begin{aligned}
u(x,t) &= \lim_{r\to 0} \frac{\widetilde{U}(x;r,t)}{r} \\
&= \lim_{r\to 0}\left(\frac{\widetilde{G}(x;r+t)-\widetilde{G}(x;t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r} \widetilde{H}(x;y)\,dy\right) \\
&= \widetilde{G}_t(x;t) + \widetilde{H}(x;t) \\
&= \frac{\partial}{\partial t}\left(\frac{t}{|\partial B(x,t)|}\int_{\partial B(x,t)} g(y)\,dS(y)\right) + \frac{t}{|\partial B(x,t)|}\int_{\partial B(x,t)} h(y)\,dS(y).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\frac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} g(y)\,dS(y) &= \frac{1}{4\pi t^2}\int_{\partial B(x,t)} g(y)\,dS(y) \qquad (|\partial B(x,t)| = 4\pi t^2) \\
&= \frac{1}{4\pi t^2}\int_{\partial B(0,1)} g(x+tz)t^2\,dS(z) \\
&\qquad (y = x+tz,\ dS(y) = t^2\,dS(z)) \\
&= \frac{1}{|\partial B(0,1)|}\int_{\partial B(0,1)} g(x+tz)\,dS(z) \qquad (|\partial B(0,1)| = 4\pi)
\end{aligned}
$$

and thus

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} g(y)\,dS(y)\right) &= \frac{1}{|\partial B(0,1)|}\int_{\partial B(0,1)} \frac{\partial}{\partial t}(g(x+tz))\,dS(z) \\
&= \frac{1}{|\partial B(0,1)|}\int_{\partial B(0,1)} \nabla g(x+tz)\cdot z\,dS(z) \\
&= \frac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} \nabla g(y)\cdot \frac{y-x}{t}\,dS(y). \\
&\qquad (y = x+tz,\ dS(z) = t^{-2}\,dS(y))
\end{aligned}
$$

This gives

$$
\begin{aligned}
u(x,t) &= \frac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} g(y)\,dS(y) + t\frac{\partial}{\partial t}\left(\frac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} g(y)\,dS(y)\right) \\
&\qquad + \frac{t}{|\partial B(x,t)|}\int_{\partial B(x,t)} h(y)\,dS(y) \qquad (6.5) \\
&= \frac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} (th(y) + g(y) + \nabla g(y)\cdot(y-x))\,dS(y), \quad x\in\mathbb{R}^3, \quad t>0.
\end{aligned}
$$

This is Kirchhoff's formula for the solution of the Cauchy problem (6.1) for the three-dimensional wave equation.

Remarks 6.12:

(1) To compute $u(x,t)$ we only need information on the data on the sphere $\partial B(x,r)$, not the entire ball $B(x,r)$. In other words, the domain of dependence of a point (x,t) is the surface of the sphere $\partial B(x,t)$ in $\mathbb{R}^3$. Similarly, the range of influence of a point $x_0 \in \mathbb{R}^3$ is the surface of the (light) cone

$$\{(x,t)\in\mathbb{R}^3\times(0,\infty) : |x-x_0| = t\}.$$

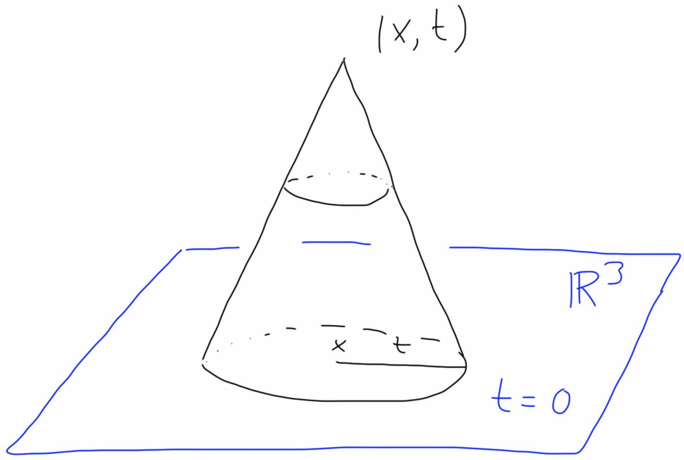


Figure 6.6: The domain of dependence in the three-dimensional case.

Physically this is a finite propagation speed of the disturbances, and more specifically, the existence of sharp signals for three-dimensional waves such as light or sound. For example, an initial disturbance near $x = 0$ will be observed at $x_0 \in \mathbb{R}^3$ only at the time $t = |x_0|$ and not after that. The information propagates at exactly the unit speed, no faster and no slower. This is called Huygens' principle. This is on contrast with the one-dimensional case and also with the two-dimensional case, as we shall see in the next section.

(2) One way to express the finite speed of propagation us that the solution at a given point is determined by the initial values in a bounded subset. An important consequence is that the process of solving initial value problems for the wave equation can be localized in space. This is in a strict contrast with the heat equation, since it has infinite speed of propagation.

(3) Another difference to the one-dimensional case is the loss of regularity. Indeed, if $g \in C^2(\mathbb{R}^3)$ and $h \in C^1(\mathbb{R}^3)$, then Kirchhoff's formula gives only that u is C^1-function with respect to the space variable. This is due to the focusing of the initial irregularities to a smaller set.

Remark 6.13. Consider solid vibrations in the unit ball $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$. Its small transverse vibrations satisfy the three-dimensional wave equation in $\Omega \times$

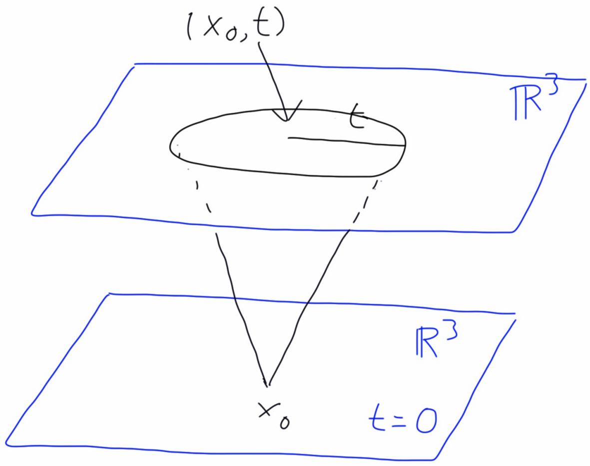


Figure 6.7: The range of influence in the three-dimensional case.

$(0,\infty)$ with Dirichlet boundary conditions. Thus we consider the problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in} \quad \Omega \times (0,\infty), \\ u = 0 & \text{in} \quad \partial\Omega \times (0,\infty), \\ u = g & \text{in} \quad \Omega \times \{t=0\}, \\ \dfrac{\partial u}{\partial t} = h & \text{in} \quad \Omega \times \{t=0\}. \end{cases}$$

As we have seen Section 5.4 for the heat equation, the separation variables $u(x,t) = v(t)w(x)$, $x \in \Omega$, $t > 0$, leads to the eigenvalue problem

$$\begin{cases} -\Delta w = \lambda w & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We apply separation of variables can be also applied in spherical coordinates as in Remark 4.54. In this case we look for solutions to $-\Delta w = \lambda w$ in the form

$$w(r,\theta,\phi) = A(\theta,\phi)B(r).$$

By substituting this into the PDE in spherical coordinates, we obtain

$$\lambda r^2 + \frac{r^2 B_{rr} + 2rB_r}{B} + \frac{(\sin\phi)^{-2}A_{\theta\theta} + (\sin\phi)^{-1}(A_\phi \sin\phi)_\phi}{A} = 0$$

and equivalently

$$\lambda r^2 + \frac{r^2 B_{rr} + 2rB_r}{B} = -\frac{(\sin\phi)^{-2}A_{\theta\theta} + (\sin\phi)^{-1}(A_\phi \sin\phi)_\phi}{A}$$

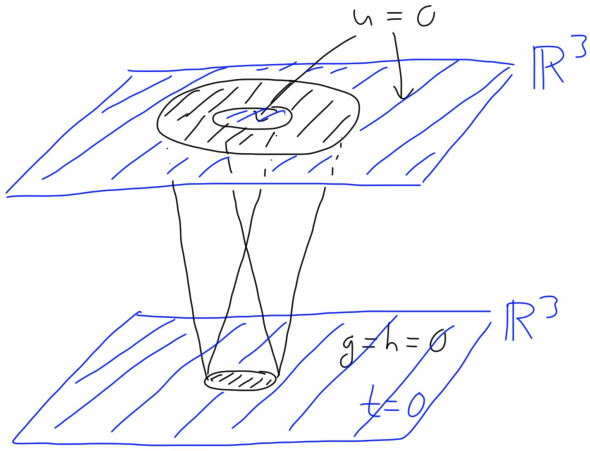


Figure 6.8: Propagation of disturbances near the origin in the three-dimensional case.

for every r, θ and ϕ. Thus

$$\lambda r^2 + \frac{r^2 B_{rr} + 2rB_r}{B} = \mu = -\frac{(\sin\phi)^{-2} A_{\theta\theta} + (\sin\phi)^{-1}(A_\phi \sin\phi)_\phi}{A}$$

for every r, θ and ϕ, where μ is the separation constant. Consequently, we may rewrite the separated equations as two differential equations

$$\begin{cases} \frac{1}{(\sin\phi)^2} A_{\theta\theta} + \frac{1}{\sin\phi}(A_\phi \sin\phi)_\phi + \mu A = 0, \\ r^2 B_{rr} + 2rB_r + (\lambda r^2 - \mu) B = 0. \end{cases}$$

The equation for B is similar to the the corresponding ODE in the two-dimensional case, see (2.15), but the PDE for A is more challenging. Solutions to the PDE for A are special functions called the spherical harmonics, see pages 271–277 in [10] and pages 98–108 in [5] for the n-dimensional theory. This approach may also be used for other coordinate systems than polar or spherical coordinates.

6.5 The two-dimensional wave equation

In this section we consider the Cauchy problem (6.1) for the wave equation when $n = 2$. We shall use Hadamard's method of descent and view the two-dimensional wave equation as a special case of the three-dimensional problem in which the third spatial variable x_3 does not appear.

To set this up, let us assume that u is solution of the initial value problem (6.1) when $n = 2$. Let us write

$$\widetilde{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t), \quad (x_1, x_2, x_3, t) \in \mathbb{R}^3 \times (0, \infty).$$

We also define $\widetilde{g}(x_1, x_2, x_3) = g(x_1, x_2)$ and $\widetilde{h}(x_1, x_2, x_3) = h(x_1, x_2)$ in the same fashion. Since u satisfies (6.1), we see that $\widetilde{u}$ satisfies

$$\begin{cases} \widetilde{u}_{tt} - \Delta \widetilde{u} = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ \widetilde{u} = \widetilde{g}, \quad \widetilde{u}_t = \widetilde{h} & \text{on} \quad \mathbb{R}^3 \times \{t = 0\}. \end{cases} \tag{6.6}$$

THE MORAL: We add a dummy variable to the two-dimensional problem to obtain a three-dimensional problem so that we may use Kirchhoff's formula.

For $x = (x_1, x_2) \in \mathbb{R}^2$ let us write $\widetilde{x} = (x_1, x_2, 0) \in \mathbb{R}^3$. Then, Kirchhoff's formula (6.5) for the three-dimensional problem gives

$$u(x,t) = \widetilde{u}(\widetilde{x},t) = \frac{\partial}{\partial t}\left(t \frac{1}{|\partial \widetilde{B}(\widetilde{x},t)|} \int_{\partial \widetilde{B}(\widetilde{x},t)} \widetilde{g}(\widetilde{y})\, dS(\widetilde{y}) \right) + \frac{t}{|\partial \widetilde{B}(\widetilde{x},t)|} \int_{\partial \widetilde{B}(\widetilde{x},t)} \widetilde{h}(\widetilde{y})\, dS(\widetilde{y}).$$

Here $\widetilde{B}(\widetilde{x},t)$ denotes the ball in $\mathbb{R}^3$, centered at $\widetilde{x} \in \mathbb{R}^3$ and having radius t. We can simplify the previous formula by observing that

$$\frac{1}{|\partial \widetilde{B}(\widetilde{x},t)|} \int_{\partial \widetilde{B}(\widetilde{x},t)} \widetilde{g}(\widetilde{y})\, dS(\widetilde{y}) = \frac{1}{4\pi t^2} \int_{\partial \widetilde{B}(\widetilde{x},t)} \widetilde{g}(\widetilde{y})\, dS(\widetilde{y})$$

Now if $\widetilde{y} \in \widetilde{B}(\widetilde{x},t)$ and $\widetilde{y} = (y, y_3)$ with $y \in \mathbb{R}^2$, $y_3 \in \mathbb{R}$, this means that

$$|\widetilde{y} - \widetilde{x}| = t \iff |y - x|^2 + y_3^2 = t^2 \iff y_3^2 = t^2 - |x - y|^2.$$

Consider the upper hemisphere of $\partial \widetilde{B}(\widetilde{x},t)$ in $\mathbb{R}^3$ and let us denote it by $\partial \widetilde{B}(\widetilde{x},t)^+$. Then every point $\widetilde{y}$ in the hemisphere can be given in the form

$$\widetilde{y} = (y_1, y_2, \gamma(y_1, y_2)), \quad \text{where} \quad \gamma(y_1, y_2) = (t^2 - |x - y|^2)^{1/2}.$$

Setting $y = (y_1, y_2) \in \mathbb{R}^2$ as before the previous display can be written as $\widetilde{y} = (y, \gamma(y))$ when $\widetilde{y} \in \partial \widetilde{B}(\widetilde{x},t)^+$. Then we have the following (general) formula for change of variables

$$\int_{\partial \widetilde{B}(\widetilde{x},t)^+} F(\widetilde{y})\, dS(\widetilde{y}) = \int_{B(x,t)} F(y, \gamma(y))(1 + |\nabla \gamma(y)|^2)^{1/2}\, dy$$

Applying this to our function $\widetilde{g}$ we get

$$\begin{aligned} \frac{1}{4\pi t^2} \int_{\partial \widetilde{B}(\widetilde{x},t)^+} \widetilde{g}(\widetilde{y})\, dS(\widetilde{y}) &= \frac{1}{4\pi t^2} \int_{B(x,t)} \widetilde{g}(y, \gamma(y))(1 + |\nabla \gamma(y)|^2)^{1/2}\, dy. \\ &= \frac{1}{4\pi t^2} \int_{B(x,t)} g(y)(1 + |\nabla \gamma(y)|^2)^{1/2}\, dy, \end{aligned}$$

since $\widetilde{g}$ is independent of the third variable. For the lower hemisphere we have that $y_3 = -(t^2 - |x - y|^2)^{1/2} = -\gamma(y)$. Since $|\nabla \gamma(y)| = |\nabla(-\gamma(y))|$, we obtain

$$\frac{1}{4\pi t^2} \int_{\partial \widetilde{B}(\widetilde{x},t)} \widetilde{g}(\widetilde{y})\, dS(\widetilde{y}) = \frac{2}{4\pi t^2} \int_{B(x,t)} g(y)(1 + |\nabla \gamma(y)|^2)^{1/2}\, dy.$$

The factor two enters since $\partial\widetilde{B}(\widetilde{x},t)$ consists of two hemispheres.

We have
$$\frac{\partial\gamma}{\partial y_j}(y)=\frac{x_j-y_j}{(t^2-|x-y|^2)^{1/2}},\quad j=1,2,$$
so that
$$|\nabla\gamma(y)|=\frac{|x-y|}{(t^2-|x-y|^2)^{1/2}}$$
and
$$(1+|\nabla\gamma(y)|^2)^{1/2}=\left(1+\frac{|x-y|^2}{t^2-|x-y|^2}\right)^{1/2}=\frac{t}{(t^2-|x-y|^2)^{1/2}}.$$
From this we conclude that
$$\begin{aligned}\frac{1}{|\partial\widetilde{B}(\widetilde{x},t)|}\int_{\partial\widetilde{B}(\widetilde{x},t)}\widetilde{g}(\widetilde{y})\,dS(\widetilde{y})&=\frac{1}{2\pi t}\int_{B(x,t)}\frac{g(y)}{(t^2-|x-y|^2)^{1/2}}\,dy\\&=\frac{t}{2}\frac{1}{|B(x,t)|}\int_{B(x,t)}\frac{g(y)}{(t^2-|x-y|^2)^{1/2}}\,dy.\end{aligned}$$
In the same way, we have
$$\frac{1}{|\partial\widetilde{B}(\widetilde{x},t)|}\int_{\partial\widetilde{B}(\widetilde{x},t)}\widetilde{h}(\widetilde{y})\,dS(\widetilde{y})=\frac{t}{2}\frac{1}{|B(x,t)|}\int_{B(x,t)}\frac{h(y)}{(t^2-|x-y|^2)^{1/2}}\,dy.$$
Thus the formula for the solution becomes
$$\begin{aligned}u(x,t)&=\frac{1}{2}\frac{\partial}{\partial t}\left(t^2\frac{1}{|B(x,t)|}\int_{B(x,t)}\frac{g(y)}{(t^2-|x-y|^2)^{1/2}}\,dy\right)\\&\qquad+\frac{t^2}{2}\frac{1}{|B(x,t)|}\int_{B(x,t)}\frac{h(y)}{(t^2-|x-y|^2)^{1/2}}\,dy\\&=I_1+I_2.\end{aligned}$$

A final calculation will allow us to further simplify the formula given above. For I_1 we change variables $y=x+tz$, $dy=t^2\,dz$, to get
$$\begin{aligned}t^2\frac{1}{|B(x,t)|}\int_{B(x,t)}\frac{g(y)}{(t^2-|x-y|^2)^{1/2}}\,dy&=\frac{t^2}{\pi t^2}\int_{B(0,1)}\frac{g(x+tz)}{t(1-|z|^2)^{1/2}}t^2\,dz\\&=\frac{t}{\pi}\int_{B(0,1)}\frac{g(x+tz)}{(1-|z|^2)^{1/2}}\,dz.\end{aligned}$$
The product and chain rules together with a change of variables give
$$\begin{aligned}I_1&=\frac{1}{2\pi}\int_{B(0,1)}\frac{g(x+tz)}{(1-|z|^2)^{1/2}}\,dz+\frac{t}{2\pi}\int_{B(0,1)}\frac{\nabla g(x+tz)\cdot z}{(1-|z|^2)^{1/2}}\,dz\\&=\frac{t}{2|B(x,t)|}\int_{B(x,t)}\frac{g(y)}{(t^2-|x-y|^2)^{1/2}}\,dy+\frac{t}{2|B(x,t)|}\int_{B(x,t)}\frac{\nabla g(y)\cdot(y-x)}{(t^2-|x-y|^2)^{1/2}}\,dy.\end{aligned}$$
We thus arrive at the formula
$$u(x,t)=\frac{1}{2}\frac{1}{|B(x,t)|}\int_{B(x,t)}\frac{tg(y)+t^2h(y)+t\nabla g(y)\cdot(y-x)}{(t^2-|x-y|^2)^{1/2}}\,dy,\quad x\in\mathbb{R}^2,\quad t>0.$$
This is Poisson's formula for the solution of the two-dimensional wave equation.

Remark 6.14. To compute $u(x,t)$ we need information on the data on the entire ball $B(x,r)$, not just on the sphere $\partial B(x,r)$ as in the three-dimensional case. The domain of dependence of a point (x,t) is the disk $B(x,t)$ in $\mathbb{R}^2$. Similarly, the range of influence of a point $x_0 \in \mathbb{R}^2$ is the interior of the cone

$$\{(x,t) \in \mathbb{R}^2 \times (0,\infty) : |x-x_0| \leqslant t\}.$$

Physically this is a finite propagation speed, and more specifically, the absence of sharp signals for two-dimensional waves such as water waves. For example, an initial disturbance near $x=0$ will be felt at $x_0 \in \mathbb{R}^2$ after the time $t=|x_0|$ forever.

Remark 6.15. Consider a membrane stretched across the top of a circular drum $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$ of radius one. Its small transverse vibrations satisfy the two-dimensional wave equation in $\Omega \times (0,\infty)$ with Dirichlet boundary conditions. Thus we consider the problem

$$\begin{cases} u_{tt} - \Delta u = 0 \quad \text{in} \quad \Omega \times (0,\infty), \\ u = 0 \quad \text{in} \quad \partial\Omega \times (0,\infty), \\ u = g \quad \text{in} \quad \Omega \times \{t=0\}, \\ \dfrac{\partial u}{\partial t} = h \quad \text{in} \quad \Omega \times \{t=0\}. \end{cases}$$

By switching to polar coordinates (see Section 2.11), we obtain

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = u_{tt}, \quad 0<r<1, \quad -\pi \leqslant \theta < \pi, \quad t>0,$$

for $u = u(r,\theta,t)$. We separate variables as in Section 2.11 and look for a product solution of the form

$$u(r,\theta) = A(\theta)B(r)C(t).$$

Inserting this into the PDE and multiplying by r^2, we obtain

$$r^2A(\theta)B''(r)C(t) + rA(\theta)B'(r)C(t) + A''(\theta)B(r)C(t) = A(\theta)B(r)C''(t),$$

and so we have

$$\frac{r^2B''(r) + rB'(r) + \lambda r^2 B(r)}{B(r)} + \frac{A''(\theta)}{A(\theta)} = \frac{C''(t)}{C(t)}.$$

Consequently, we may rewrite the separated equations as three ODEs

$$\begin{cases} A''(\theta) + \mu A(\theta) = 0, \\ r^2B''(r) + rB'(r) + \lambda r^2 B(r) - \mu B(r) = 0, \\ C''(t) + \mu C(t) = 0, \end{cases}$$

where μ is the separation constant. The second equation is the parametric version of the modified Bessel equation of order zero and its solutions are called Bessel functions, see pages 264–270 in [10].

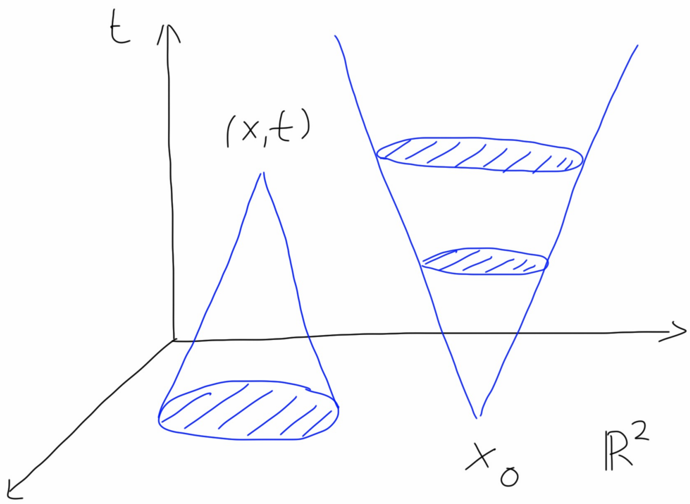


Figure 6.9: The domain of dependence and the range of influence in the two-dimensional case.

6.6 The nonhomogeneous problem

In this section we study the nonhomogeneous Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = f & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = 0, \quad u_t = 0 & \text{in} \quad \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{6.7}$$

The goal is to apply Duhamel's principle as in Section 5.3 for the heat equation. Then, for every fixed s, let the function $u(x,t;s)$ be the solution of the Cauchy problem

$$\begin{cases} u_{tt}(x,t;s) - \Delta u(x,t;s) = 0, & x \in \mathbb{R}^n, \quad t > s, \\ u(x,s;s) = 0, \quad u_t(x,s;s) = f(x,s), & x \in \mathbb{R}^n. \end{cases}$$

Observe that this is just the homogeneous initial value problem for the wave equation with the time $t=0$ replaced by the time $t=s$. Duhamel's principle suggests that we can build a solution to the nonhomogeneous problem by integrating solutions $u(x,t;s)$ over $s \in (0,t)$

$$u(x,t) = \int_0^t u(x,t;s)\,ds, \quad x \in \mathbb{R}^n, \quad t > 0.$$

A formal calculation shows that

$$u_t(x,t) = \underbrace{u(x,t;t)}_{=0} + \int_0^t u_t(x,t;s)\,ds = \int_0^t u_t(x,t;s)\,ds$$

and

$$\begin{aligned}
u_{tt}(x,t) &= u_t(x,t;t) + \int_0^t u_{tt}(x,t;s)\,ds \\
&= f(x,t) + \int_0^t u_{tt}(x,t;s)\,ds \\
&= f(x,t) + \int_0^t \Delta u(x,t;s)\,ds \\
&= f(x,t) + \Delta\left(\int_0^t u(x,t;s)\,ds\right) \\
&= f(x,t) + \Delta u(x,t).
\end{aligned}$$

Moreover, inserting $t=0$ in the formula for u, we obtain $u(x,0)=u_t(x,0)=0$. Thus u defined as above is a solution to (6.7).

Remark 6.16. The nonhomogeneous Cauchy problem with more general boundary values can be solved by summing up the solutions to (6.1) and (6.7) as in Remark 5.5 (1) for the heat equation and Remark 4.14 (3) for the Laplace equation.

THE MORAL: Duhamel's principle is a process of expressing the solution of a nonhomogeneous problem as an integral of the solutions to the homogeneous problem in the way that the source term is interpreted as the initial condition. This gives a method to solve a nonhomogeneous PDE with nonzero boundary values as a sum of the nonhomogeneous PDE with zero boundary values and the homogeneous PDE with nonzero boundary values.

Examples 6.17:

(1) For $n=1$ d'Alembert's formula in Theorem 6.3 gives

$$u(x,t;s) = \frac{1}{2}\int_{x-(t-s)}^{x+(t-s)} f(y,s)\,dy.$$

Thus

$$\begin{aligned}
u(x,t) &= \int_0^t u(x,t;s)\,ds \\
&= \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(y,s)\,dy\,ds \\
&= \frac{1}{2}\int_0^t \int_{x-r}^{x+r} f(y,t-r)\,dy\,dr \qquad (s=t-r) \\
&= \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y,t-s)\,dy\,ds.
\end{aligned}$$

(2) For $n=3$ Kirchhoff's formula (6.5) gives

$$u(x,t;s) = \frac{t-s}{|\partial B(x,t-s)|}\int_{\partial B(x,t-s)} f(y,s)\,dS(y).$$

Thus

$$
\begin{aligned}
u(x,t) &= \int_0^t u(x,t;s)\,ds \\
&= \int_0^t \frac{t-s}{|\partial B(x,t-s)|}\int_{\partial B(x,t-s)} f(y,s)\,dS(y)\,ds \\
&= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,t-s)} \frac{f(y,s)}{t-s}\,dS(y)\,ds \qquad (|\partial B(x,t-s)| = 4\pi(t-s)^2) \\
&= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,r)} \frac{f(y,t-r)}{r}\,dS(y)\,dr \qquad (r = t-s) \\
&= \frac{1}{4\pi}\int_{B(x,t)} \frac{f(y,t-|y-x|)}{|y-x|}\,dy.
\end{aligned}
$$

Example 6.18. We solve

$$
\begin{cases}
u_t - u_{xx} = e^{-t}\cos x, & x\in\mathbb{R}, \quad t>0, \\
u(x,0) = 0, \quad u_t(x,0) = 0, & x \in \mathbb{R}.
\end{cases}
$$

By Duhamel's principle, we consider solution $u = u(x,t;s)$ to the auxiliary problem

$$
\begin{cases}
u_{tt} - u_{xx} = 0, & x\in\mathbb{R}, \quad t>0, \\
u(x,s;s) = 0, & x\in\mathbb{R}, \\
u_t(x,s;s) = e^{-s}\cos x, & x\in\mathbb{R}.
\end{cases}
$$

where $s>0$ is fixed and $t>s$. By d'Alembert's formula

$$
\begin{aligned}
u(x,t;s) &= \frac{e^{-s}}{2}\int_{x-(t-s)}^{x+(t-s)} \cos y\,dy \\
&= \frac{e^{-s}}{2}(\sin(x+(t-s)) - \sin(x-(t-s)) \\
&= e^{-s}\sin(t-s)\cos x.
\end{aligned}
$$

The solution to the original problem is

$$
\begin{aligned}
u(x,t) &= \int_0^t e^{-s}\sin(t-s)\cos x\,ds \\
&= \cos x \int_0^t e^{-s}\sin(t-s)\,ds \\
&= \frac{1}{2}\cos x(e^{-t} - \cos t + \sin t).
\end{aligned}
$$

6.7 Energy methods

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with a smooth boundary and $0 < T < \infty$. Recall that we denote $\Omega_T = \Omega \times (0,T)$ and $\Gamma_T = (\partial\Omega \times [0,T]) \cup (\Omega \times \{t=0\})$. Define the energy

$$
e(t) = \frac{1}{2}\int_\Omega \left((u_t)^2 + |\nabla u|^2\right) dx, \quad 0 \leqslant t \leqslant T.
$$

This energy measures the first order regularity of a function. If the solution develops a singularity so that the first order derivatives become unbounded, we might expect the energy to become unbounded. On the other hand, if the energy is constant, then such singularities become concentrated to a smaller and smaller sets. We shall show that the wave equation conserves energy.

By differenting the energy, we have

$$\begin{aligned}
e'(t) &= \frac{1}{2}\int_\Omega \frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial t}\right)^2 + |\nabla u|^2\right) dx \\
&= \frac{1}{2}\int_\Omega \left(2\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + 2\nabla u \cdot \frac{\partial}{\partial t}\nabla u\right) dx \\
&= \int_\Omega \left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}\Delta u\right) dx \qquad \text{(Green's first identity)} \\
&= \int_\Omega u_t(u_{tt} - \Delta u)\, dx = 0,
\end{aligned}$$

provided

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in} \quad \Omega_T, \\ u = 0 & \text{on} \quad \Gamma_T. \end{cases}$$

Thus $e'(t) = 0$, $0 \leqslant t \leqslant T$, which implies that $e(t) = e(0)$, $0 \leqslant t \leqslant T$.

THE MORAL: The energy is preserved in the wave equation.

Theorem 6.19. There exists at most one solution to the problem

$$\begin{cases} u_{tt} - \Delta u = f & \text{in} \quad \Omega_T, \\ u = g & \text{on} \quad \Gamma_T, \\ u_t = h & \text{in} \quad \Omega \times \{t = 0\}. \end{cases}$$

Proof. If u and v are solutions, then $w = u - v$ is a solution to

$$\begin{cases} w_{tt} - \Delta w = 0 & \text{in} \quad \Omega_T, \\ w = 0 & \text{on} \quad \Gamma_T, \\ w_t = 0 & \text{in} \quad \Omega \times \{t = 0\}. \end{cases}$$

Then

$$e(t) = \frac{1}{2}\int_\Omega \left((w_t)^2 + |\nabla w|^2\right) dx = e(0) = 0,$$

from which it follows that

$$\frac{\partial w}{\partial t} = 0 \quad \text{and} \quad \nabla w = 0 \quad \text{in} \quad \Omega_T.$$

This shows that w is constant in every component of Ω_T. Since the $w = 0$ on $\Omega \times \{t = 0\}$ we conclude $w = u - v = 0$ in Ω_T. This proves the claim. □

6.8 Epilogue

We have seen that nonhomogeneous linear PDEs can be solved by Duhamel's principle if we can solve the corresponding homogeneous PDEs. Thus we may first focus on homogeneous problems only. Let us return to homogeneous Maxwell's equations

$$\begin{cases} \operatorname{div} E = 0, \\ \operatorname{div} B = 0, \\ \operatorname{curl} E = -\dfrac{\partial B}{\partial t}, \\ \operatorname{curl} B = c^{-2}\dfrac{\partial E}{\partial t}, \quad c = \dfrac{1}{\sqrt{\mu_0 \epsilon_0}}, \end{cases}$$

discussed in Introduction. We have seen that each component of electric field $E = (E_1, E_2.E_3)$ satisfies the wave equation

$$\frac{\partial^2 E}{\partial t^2} - c^2 \Delta E = 0.$$

Similarly, magnetic field B satisfies the same wave equation

$$\frac{\partial^2 B}{\partial t^2} - c^2 \Delta B = 0.$$

Consider initial conditions $E(x,0) = E^0(x)$ and $B(x,0) = B^0(x)$, where the vector fields $E^0(x)$ and $B^0(x)$ are otherwise arbitrary except for the conditions

$$\operatorname{div} E^0 = 0 \quad \text{and} \quad \operatorname{div} B^0 = 0,$$

which are the first two equations in Maxwell's equations.

We shall derive a solution to homogeneous Maxwell's equations in the three-dimensional space by using Kirchhoff's formula for the solution of the wave equation. Indeed, in the first set of problems we have seen that Maxwell's equations can be solved by the wave equation. By Maxwell's equations, electric field E satisfies the initial conditions

$$E(x,0) = E^0(x) \quad \text{and} \quad \frac{\partial E}{\partial t}(x,0) = c^2 \operatorname{curl} B^0(x)$$

and magnetic field B satisfies the initial conditions

$$B(x,0) = B^0(x) \quad \text{and} \quad \frac{\partial B}{\partial t}(x,0) = -\operatorname{curl} E^0(x).$$

Thus we have the Cauchy problems

$$\begin{cases} \dfrac{\partial^2 E}{\partial t^2} - c^2 \Delta E = 0 \quad \text{in} \quad \mathbb{R}^3 \times (0,\infty), \\ E = E^0 \quad \text{in} \quad \mathbb{R}^3 \times \{t = 0\}, \\ \dfrac{\partial E}{\partial t} = c^2 \operatorname{curl} B^0 \quad \text{in} \quad \mathbb{R}^3 \times \{t = 0\}. \end{cases}$$

and

$$\begin{cases} \dfrac{\partial^2 B}{\partial t^2} - c^2 \Delta B = 0 \quad \text{in} \quad \mathbb{R}^3 \times (0,\infty), \\ B = B^0 \quad \text{in} \quad \mathbb{R}^3 \times \{t=0\}, \\ \dfrac{\partial B}{\partial t} = -\operatorname{curl} E^0 \quad \text{in} \quad \mathbb{R}^3 \times \{t=0\}. \end{cases}$$

By a change of variables and Kirchhoff's formula (6.5) we see that

$$u(x,t) = \frac{1}{|\partial B(x,ct)|} \int_{\partial B(x,ct)} (ct\,h(y) + g(y) + \nabla g(y)\cdot(y-x))\,dS(y),$$

$x \in \mathbb{R}^3$, $t > 0$ and $c > 0$, is a solution to

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 \quad \text{in} \quad \mathbb{R}^3 \times (0,\infty), \\ u = g, \quad u_t = h \quad \text{in} \quad \mathbb{R}^3 \times \{t=0\}. \end{cases}$$

Thus we have

$$E(x,t) = \frac{1}{|\partial B(x,ct)|} \int_{\partial B(x,ct)} \left(tc^3 \operatorname{curl} B^0(y) + E^0(y) + \nabla E^0(y)\cdot(y-x)\right) dS(y)$$

for $x \in \mathbb{R}^3$ and $t > 0$. Similarly

$$B(x,t) = \frac{1}{|\partial B(x,ct)|} \int_{\partial B(x,ct)} \left(-tc \operatorname{curl} E^0(y) + B^0(y) + \nabla B^0(y)\cdot(y-x)\right) dS(y)$$

for $x \in \mathbb{R}^3$ and $t > 0$. These formulas give a solution to Maxwell's equations in $\mathbb{R}^3$.

6.9 Summary

(1) We have derived several representation formulas for the solutions. The Fourier series and the Fourier transform can be used to solve boundary value problems in special cases.

(2) The behaviour of the solutions to the wave equation is different in dimensions one, two and three.

(3) The nonhomogenerous Cauchy problem can be solved using Duhamel's principle.

(4) The wave equation has finite propagation speed for disturbances.

(5) There is no maximum principle for the wave equation.

(6) The wave equation does not smoothen the boundary data.

(7) The direction of time can be changed in the wave equation.

(8) The energy is preserved and thus there is no decay as $t \to \infty$.

(9) The energy is preserved in the wave equation and this can be used to give a short proof of the uniqueness in a bounded space time cylinder.

7

Notation and tools

(1) $\mathbb{R}^n = \{x = (x_1, \dots, x_n) : x_j \in \mathbb{R}, j = 1, 2, \dots, n\}$ is the n-dimensional Euclidean space and $\mathbb{R} = \mathbb{R}^1$ is the real line.

(2) $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$ is the complex plane. Here i is the imaginary unit for which $i^2 = -1$. The real part of $z = x + iy \in \mathbb{C}$ is $\mathrm{Re}(z) = y$ and the imaginary part is $\mathrm{Im}(z) = y$. The complex conjugate of $z = x + iy$ is $\overline{z} = x - iy$. Observe that $z\overline{z} = |z|^2$.

(3) $x \cdot y = \sum_{j=1}^{n} x_j y_j$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ is the standard inner product in $\mathbb{R}^n$.

(4) $\|x\| = (x \cdot x)^{\frac{1}{2}} = \sqrt{\sum_{j=1}^{n} x_j^2}$ is the norm of $x \in \mathbb{R}^n$.

(5) $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ is the jth standard basis vector of $\mathbb{R}^n$.

(6) $\mathbb{R}^{n+1}_+ = \{(x, t) : x \in \mathbb{R}^n, t > 0\}$ is the upper half space in $\mathbb{R}^{n+1}$.

(7) $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ is the open ball with the center $x \in \mathbb{R}^n$ and radius $r > 0$.

(8) Ω is an open subset of $\mathbb{R}^n$, if for every $x \in \Omega$ there is $r > 0$ such that $B(x, r) \subset \Omega$, that is, every point x in Ω is an interior point of Ω.

(9) A point x belongs to the boundary of Ω, denoted $\partial\Omega$, if $B(x, r) \cap \Omega \neq \emptyset$ and $B(x, r) \cap (\mathbb{R}^n \setminus \Omega) \neq \emptyset$ for every $r > 0$, that is, every ball with a positive radius centered at the boundary intersects both the set and its complement.

(10) $\partial\Omega$ is smooth, if the boundary can be locally represented as a graph of a smooth function.

(11) $\overline{\Omega} = \Omega \cup \partial\Omega$ is the closure of Ω, that is, the closure of an open set is the union of the set and its boundary.

(12) An open set Ω is connected, if every pair of points in Ω can be connected by a piecewise linear path in Ω.

(13) $\Omega \subset \mathbb{R}^n$ is bounded if there is $M < \infty$ such that $\|x\| \leqslant M$ for every $x \in \Omega$.

Equivalently, there is $r>0$ such that $\Omega\subset B(0,r)$, that is, Ω is bounded if it is contained in a ball with a finite radius centered at the origin.

(14) The jth partial derivative of u is

$$\frac{\partial u}{\partial x_j}(x)=\lim_{h\to 0}\frac{u(x+he_j)-u(x)}{h},\quad j=1,2,\dots,n,$$

provided the limit exists.

(15) $\nabla u(x)=\left(\frac{\partial u}{\partial x_1}(x),\dots,\frac{\partial u}{\partial x_n}(x)\right)$ is the gradient of u at point x.

(16) Higher order derivatives are denoted by

$$\frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}(x),\quad |\alpha|=\alpha_1+\dots+\alpha_n,$$

where each component α_j is a nonnegative integer.

(17) If $\nu(x)=(\nu_1(x),\dots,\nu_n(x))$ is a unit vector, then

$$\frac{\partial u}{\partial \nu}(x)=\nabla u(x)\cdot\nu(x)$$

is the derivative of u in the direction ν at point x. Note that

$$\frac{\partial u}{\partial x_j}(x)=\nabla u(x)\cdot e_j,\quad j=1,2,\dots,n.$$

(18) $C(\Omega)=\{u:\Omega\to\mathbb{R}: u \text{ is continuous}\}$.

(19) $C(\overline{\Omega})$ is the class of continuous functions in Ω, which can be extended continuously to $\overline{\Omega}$.

(20) $C^k(\Omega)=\{u:\Omega\to\mathbb{R}: u \text{ is } k \text{ times continuously differentiable}\}$.

(21) $C^k(\overline{\Omega})$ the class of is k times continuously differentiable functions in Ω, whose all partial derivatives up to order k can be extended continuously to $\overline{\Omega}$.

(22) $C^\infty(\Omega)=\bigcap_{k=0}^\infty C^k(\Omega)$ is the class of smooth functions.

(23) The support of a function $u:\Omega\to\mathbb{R}$ is the set $\overline{\{x\in\Omega: u(x)\neq 0\}}$.

(24) A function $u:\Omega\to\mathbb{R}$ is compactly supported in Ω, if the support of u is a bounded (and thus compact) subset of Ω. Equivalently, u is compactly supported in Ω, if there exists an open an bounded open set Ω' such that $\overline{\Omega'}\subset\Omega$ and $u(x)=0$ for every $x\in\Omega\setminus\overline{\Omega'}$.

(25) $C_0(\Omega)$, $C_0^k(\Omega)$ and $C_0^\infty(\Omega)$ denote functions in the corresponding classes with compact support.

(26) $u\in C_0^\infty(\mathbb{R}^n)$ is called a compactly supported smooth function in $\mathbb{R}^n$. $u\in C_0^\infty(\mathbb{R}^n)$, if $u\in C^\infty(\mathbb{R}^n)$ and there is $r>0$ such that $u(x)=0$ for every $x\in\mathbb{R}^n\setminus B(0,r)$, that is, the function is are zero outside a ball with a finite radius centered at the origin.

(27) For $1\leqslant p<\infty$, the space $L^p(\Omega)$ consists of functions $f:\Omega\to\mathbb{C}$ such that

$$\|f\|_{L^p(\Omega)}=\left(\int_\Omega |f(x)|^p\,dx\right)^{\frac{1}{p}}<\infty.$$

For a continuous function f we set $\|f\|_{L^\infty(\Omega)} = \sup_{x\in\Omega}|f(x)|$.

(28) A sequence of functions $f_j : \Omega \to \mathbb{C}$, $j = 1,2,\ldots$, converges to f in $L^p(\Omega)$ $1 \leqslant p < \infty$, if $\|f_i - f\|_{L^p(\Omega)} \to 0$ as $j \to \infty$, that is,

$$\lim_{j\to\infty}\int_\Omega |f_j(x) - f(x)|^p\,dx = 0.$$

(29) A sequence of continuous functions $f_j : \Omega \to \mathbb{C}$, $j = 1,2,\ldots$, converges uniformly in Ω to f, if $\|f_j - f\|_{L^\infty(\Omega)} \to 0$ as $j \to \infty$, that is,

$$\lim_{j\to\infty}\sup_{x\in\Omega}|f_i(x) - f(x)| = 0.$$

(30) Let $f_j : \Omega \to \mathbb{C}$, $j = 1,2,\ldots$, be a sequence of integrable functions such that $f_i(x) \to f(x)$ for every $x \in \Omega$ as $j \to \infty$. If there exists an integrable function g such that $|f_j(x)| \leqslant g(x)$ for every $x \in \Omega$ and for every $j = 1,2,\ldots$, then the Lebesgue dominated convergence theorem asserts that f is integrable and

$$\int_\Omega f(x)\,dx = \int_\Omega \lim_{j\to\infty} f_j(x)\,dx = \lim_{j\to\infty}\int_\Omega f_j(x)\,dx.$$

Thes means that the order of taking limits and integral can be switched if there is an integrable majorant function g. Observe that the same g has to do for all functions f_j.

(31) The Lebesgue dominated convergence theorem can be used to show that in certain ceases we can switch the order of integrals and limits. Assume thar $I \subset \mathbb{R}$ is an interval. Suppose that for every fixed $t \in I$ there exists an integrable function on Ω. Thus we have $f : \Omega \times I \to \mathbb{R}$, $f = f(x,t)$. For each $t \in I$ we denote

$$F(t) = \int_\Omega f(x,t)\,dx.$$

Assume that for every $y \in I$, the function $x \mapsto f(x,t)$ is integrable in Ω, the function $t \mapsto f(x,t)$ is continuous for every $x \in \Omega$ at $t_0 \in I$ and there exists $g \in L^1(\Omega)$ such that $|f(x,t)| \leqslant g(x)$ for every $(x,t) \in \Omega \times I$. Then F is continuous at t_0, that is,

$$\begin{aligned}\lim_{t\to t_0} F(t) &= \lim_{t\to t_0}\int_\Omega f(x,t)\,dx \\ &= \int_\Omega \lim_{t\to t_0} f(x,t)\,dx = \int_\Omega f(x,t_0)\,dx.\end{aligned}$$

(32) The Lebesgue dominated convergence theorem can be used to show that in certain ceases we can switch the order of integrals and derivatives. We use the same notation as in the previous item. Assume that for every $t \in I$, the function $x \mapsto f(x,t)$ is integrable in Ω, the function $t \mapsto f(x,t)$ is differentiable for every $x \in \Omega$ at every point $t \in I$ and there exists $h \in L^1(\Omega)$ such that $\left|\frac{\partial f}{\partial t}(x,t)\right| \leqslant h(x)$ for every $(x,t) \in \Omega \times I$. Then F is differentiable at every point $t \in I$ and

$$F'(t) = \frac{\partial}{\partial t}\left(\int_\Omega f(x,t)\,dx\right) = \int_\Omega \frac{\partial}{\partial t} f(x,t)\,dx.$$

(33) $\langle f,g\rangle = \int_\Omega f(x)\overline{g(x)}\,dx$ is the standard inner product in $L^2(\Omega)$.

(34) $\|f\|_{L^2(\Omega)} = \langle f,f\rangle^{\frac{1}{2}} = \left(\int_\Omega f(x)\overline{f(x)}\,dx\right)^{\frac{1}{2}} = \left(\int_\Omega |f(x)|^2\,dx\right)^{\frac{1}{2}}$ is the L^2 norm in Ω.

(35) $|B(x,r)| = |B(0,r)| = r^n|B(0,1)| = \alpha(n)r^n$, where $\alpha(n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ is the n-dimensional volume of the unit ball $B(0,1)$. Here $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\,dx$ is the gamma function

(36) The boundary of the ball $B(x,r)$ is the sphere $\partial B(x,r) = \{y \in \mathbb{R}^n : |x-y| = r\}$.

(37) $|\partial B(x,r)| = |\partial B(0,r)| = r^{n-1}|\partial B(0,1)| = \beta(n)r^{n-1}$, where $\beta(n) = n\alpha(n)$ is the $(n-1)$-dimensional volume of the unit sphere $\partial B(0,1)$.

(38) Let $U \subset \mathbb{R}^n$ be an open set and suppose that $\Phi : U \to \mathbb{R}^n$, $\Phi = (\phi_1,\dots,\phi_n)$ is a C^1 diffeomorphism. We denote by $D\Phi$ the derivative matrix with entries $D_j\phi_i$, $i,j = 1,\dots,n$. The mapping Φ is called C^1 diffeomorphism if it is injective and $D\Phi(x)$ is invertible at every $x \in U$. In this case the inverse function theorem guarantees that the inverse map $\Phi^{-1} : \Phi(U) \to U$ is also a C^1 diffeomorphism. This means that all component functions ϕ_i, $i = 1,\dots,n$ have continuous first order partial derivatives and

$$D\Phi^{-1}(y) = (D\Phi(\Phi^{-1}(y)))^{-1}$$

for every $y \in \Phi(U)$. If f is a Lebesgue measurable function on $\Phi(U)$, the $f \circ \Phi$ is a Lebesgue measurable function on U. If f is nonnegative or integrable on $\Phi(U)$, then

$$\int_{\Phi(U)} f(y)\,dy = \int_U f(\Phi(x))|\det D\Phi(x)|\,dx.$$

Moreover, if $A \subset U$ is a Lebesgue measurable set, then $\Phi(A)$ is a Lebesgue measurable set and

$$|\Phi(A)| = \int_A |\det D\Phi|\,dx.$$

This is a change of variables formula for differentiable mappings, see [7] pages 494–503. Formally it can be seen as the substition $y = \Phi(x)$. This means that we replace $f(y)$ by $f(\Phi(x))$, $\Phi(U)$ by U and dy by $|\det D\Phi(x)|\,dx$. In particular, if $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, $\Phi(x) = tx + y$, for a fixed $t > 0$ and $y \in \mathbb{R}^n$, then $D\Phi(x) = tI$, and $|\det D\Phi(x)| = |tI| = t^n$. Here I is the $n \times n$ identity matirix.

(39) Let us consider spherical coordinates in $\mathbb{R}^n$. Let

$$U = (0,\infty) \times (0,\pi)^{n-2} \times (0,2\pi) \subset \mathbb{R}^n, \quad n \geqslant 2.$$

Denote the coordinates of a point in U by $r,\theta_1,\dots,\theta_{n-2},\theta_{n-1}$, respectively. We define $\Phi : U \to \mathbb{R}^n$ by the spherical coordinate formulas as follows. If $x = \Phi(r,\theta)$, then

$$x_i = r\sin\theta_1\cdots\sin\theta_{i-1}\cos\theta_i, \quad i = 1,\dots,n,$$

where $\theta_n = 0$ so that $x_n = r\sin\theta_1 \cdots \sin\theta_{n-1}$. Then ϕ is a bijection from U onto the open set $\mathbb{R}^n \setminus (\mathbb{R}^{n-1} \times [0,\infty) \times \{0\})$. The change of variables formula implies that

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_0^r \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} f(\Phi(r,\theta)) r^{n-1} (\sin\theta_1)^{n-2} \ldots$$
$$\ldots (\sin\theta_{n-3})^2 \sin\theta_{n-2}\, d\theta_{n-1} \ldots d\theta_1\, dr.$$

It can be shown that

$$\beta(n) = \int_0^\pi (\sin\theta_1)^{n-2}\, d\theta_{n-1} \cdots \int_0^\pi \sin\theta_{n-2}\, d\theta_{n-2} \int_0^{2\pi} d\theta_{n-1},$$

where $\beta(n) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ is the $(n-1)$-dimensional volume of the unit sphere $\partial B(0,1)$.

(40) Assume that $f : \mathbb{R}^n \to [0,\infty]$ is radial. Thus f depends only on $|x|$ and it can be expressed as $f(|x|)$, where f is a function defined on $[0,\infty)$. Then

$$\int_{\mathbb{R}^n} f(|x|)\,dx = \beta(n) \int_0^\infty f(r) r^{n-1}\,dr = \int_0^\infty \int_{\partial B(0,r)} f(r)\,dS(y)\,dr$$

see [7] pages 503–504.

(41) The convolution $f * g : \mathbb{R}^n \to \mathbb{C}$ is

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y)\,dy,$$

whenever this integral exists.

(42) The space-time cylinder is $\Omega_T = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is open and bounded and $0 < T < \infty$.

(43) The parabolic boundary of Ω_T is $\Gamma_T = (\partial\Omega \times [0,T]) \cup (\Omega \times \{t = 0\})$. Observe that the parabolic boundary consists of the base and the lateral sides of the cylinder Ω_T.

Bibliography

[1] N.H. Asmar, Partial Differential Equations with Fourier Series and Boundary Value Problems, Second Edition, Pearson Prentice Hall, 2005.

[2] W.E. Boyce and R.C. DiPrima, Elementary Differential Equations and Boundary Value Problems, 9th Edition, John Wiley & Sons, 2010.

[3] L.C. Evans, Partial Differential Equations, American Mathematical Society, Second Edition, 2010.

[4] E. DiBenedetto, Partial Differential Equations, Birkhäuser 1995.

[5] G.B. Folland, Introduction to Partial Differential Equations, Princeton University Press, Second Edition, 1995.

[6] Q. Han, A Basic Course in Partial Differential Equations, American Mathematical Society, 2011.

[7] F. Jones, Lebesgue Integration on Euclidean Space (revised edition), Jones and Bartlett Publishers, 2001.

[8] R. McOwen, Partial Differential Equations. Methods and Applications. Prentice-Hall, 1996.

[9] E.M. Stein and R. Sakarchi, Fourier Analysis. An Introduction. Princeton University Press, 2003.

[10] W.A. Strauss, Partial Differentianl Equations. An Introduction. Second edition. John Wiley & Sons, 2008

[11] A. Vasy, Partial Differential Equations. An Accessible Route through Theory and Applications. American Mathematical Society, 2015.