- A simple and versatile model of asset price dynamics is the binomial lattice. In this model, the asset price is multiplied by either factor u (up) or d (down) in each period, according to probabilities p and $1-p$, respectively. That is, the binomial lattice model is discrete model both in time and possible asset prices.

  Another class of asset models are those where the asset price may take values on a continuum of possibilities. In these models, the asset value is changed in each period according to a continuous random variable. These are, for example, additive and multiplicative models.

  Models that are continuous in both time and asset price comprise the third class of asset price models. By letting the period length of the multiplicative model tend to zero, it becomes the Ito process.

- Let $S(k)$ be the asset price at time k. The additive model is

$$S(k+1) = aS(k) + u(k),\ k = 0, \ldots, N\ \Rightarrow S(k) = a^k S(0) + a^{k-1}u(0) + a^{k-2}u(1) + \cdots + u(k-1).$$

  The random fluctuations in price $u(k)$ are mutually independent random variables. If they are normally distributed with mean 0, the prices $S(k)$ are normally distributed and their mean is $\mathbb{E}[S(k)] = a^k S(0)$. One drawback of the additive model is that because normal random variables $u(k)$ can take negative values, the prices in this model might be negative as well, which lacks realism.

- Again, let $S(k)$ be the asset price at time k. The multiplicative model is

$$S(k+1) = u(k)S(k),\ k = 0, \ldots, N-1\ \Rightarrow S(k) = u(k-1)u(k-2)\cdots u(0)S(0).$$

  $u(k)$'s are again mutually independent random variables, and they represent the relative change in the asset price between k and $k+1$. The multiplicative model takes an additive form if we take the natural logarithm of both sides of the equation. This yields

$$\ln S(k+1) = \ln S(k) + \ln u(k)$$

  for $k = 0, 1, \ldots, N-1$. We specify that the random disturbances $w(k) = \ln u(k)$ are normally distributed, and consequently $u(k)$'s are lognormal random variables, because their logarithms are normally distributed. We can write

$$\ln S(k) = \ln S(0) + \sum_{i=0}^{k-1} w(i),$$

  which is a sum of a constant $\ln S(0)$ and a sum of normally distributed random variables $w(i)$ and is hence normally distributed. Moreover, denoting $\mathbb{E}[w(k)] = \nu$ and $\text{Var}[w(k)] = \sigma^2$, we write

$$\mathbb{E}[\ln S(k)] = \ln S(0) + \nu k \qquad \text{Var}[\ln S(k)] = k\sigma^2.$$

- The multiplicative model and the binomial lattice model are analogous, because at each step the price is multiplied by a random variable. Let ν be the expected yearly change in the logarithm of the price of the asset and σ the corresponding volatility in a multiplicative model. That is,

$$\nu = \mathbb{E}\left[\ln S(T) - \ln S(0)\right] = \mathbb{E}\left[\ln \frac{S(T)}{S(0)}\right] \qquad \sigma^2 = \mathrm{Var}\left[\ln \frac{S(T)}{S(0)}\right],$$

where T denotes the number of periods in a year.

- We define the parameters of the binomial lattice model u, d, and p so that they match the multiplicative model as closely as possible. We scale $S(0) = 1$ and find by direct calculation that

$$\mathbb{E}[\ln S(1)] = \mathbb{E}[\ln S(0) + w(0)] = p \ln u + (1-p)\ln d, \text{ and}$$
$$\mathrm{Var}[\ln S(1)] = \mathbb{E}[\ln^2 S(1)] - \mathbb{E}[\ln S(1)]^2 = p\ln^2 u + (1-p)\ln^2 d - [p\ln u + (1-p)\ln d]^2 = p(1-p)(\ln u - \ln d)^2.$$

For a period of length Δt, the per period expected change and the corresponding variance are $\nu\Delta t$ and $\sigma^2\Delta t$, where Δt is expressed as a fraction of year. Therefore the appropriate parameter matching equations are

$$\begin{aligned} pU + (1-p)D &= \nu\Delta t \\ p(1-p)(U-D)^2 &= \sigma^2\Delta t, \end{aligned}$$

where $U = \ln u$ and $D = \ln d$. There are three parameters U, D and p but only two equations. We use the resulting one degree of freedom to set $D = -U$ (when also $d = 1/u$). In this case the above equations reduce to

$$\begin{aligned} (2p-1)U &= \nu\Delta t \\ 4p(1-p)U^2 &= \sigma^2\Delta t. \end{aligned}$$

By solving these we obtain

$$\begin{aligned} p &= \frac{1}{2}\left(1 + \frac{1}{\sqrt{\sigma^2/(\nu^2\Delta t) + 1}}\right) \\ U &= \sqrt{\sigma^2\Delta t + (\nu\Delta t)^2}. \end{aligned}$$

When Δt is small, the denominator of the second term of p approaches $\sqrt{\sigma/(\nu^2\Delta t)}$, and for U the second order term in the square root is small. Hence for small Δt we can approximate

$$\begin{aligned} p &= \frac{1}{2}\left(1 + \frac{\nu}{\sigma}\sqrt{\Delta t}\right) \\ U &= \sigma\sqrt{\Delta t} \qquad \Rightarrow u = e^{\sigma\sqrt{\Delta t}} \\ D &= -\sigma\sqrt{\Delta t} \qquad \Rightarrow d = e^{-\sigma\sqrt{\Delta t}} \end{aligned} \tag{1}$$

- For the lattice, the probability of attaining the various end nodes of the lattice is given by the binomial distribution. Specifically, the probability of reaching the value $S = u^k d^{n-k} S(0)$ is $\binom{n}{k} p^k (1-p)^{n-k}$, where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ is the binomial coefficient.

- Different options pricing theories can be applied when different asset price dynamics are assumed for the underlying stocks of the options.
- The simplest of these theories is based on the binomial model of stock price fluctuations. In this model, the value of the option in each cell of the binomial lattice is found by noting that it must be equal to the value of its replicating portfolio (because the cash flows of the option and the replicating portfolio are identical).

  The replicating portfolio is constructed by purchasing x dollars worth of stock and b dollars worth of the risk free asset. Denoting the value of the option by C_u when the stock price goes up and by C_d when the stock goes down, to match the option outcomes with the value of the replicating portfolio we require

  $$ux + Rb = C_u$$
  $$dx + Rb = C_d,$$

  where R is the risk-free rate. These equations can be solved for x and b as

  $$x = \frac{C_u - C_d}{u - d} \qquad b = \frac{uC_d - dC_u}{R(u-d)}.$$

  These can be combined to find the value of the portfolio. Using the no-arbitrage principle, we know that the value of the option must equal the value of the portfolio $x + b$. Hence the value of the option is

  $$C = x + b = \frac{1}{R}\left[\frac{R-d}{u-d}C_u + \left(1 - \frac{R-d}{u-d}\right)C_d\right].$$

  This formula can be further simplified by noting

  $$q = \frac{R-d}{u-d}$$

  to get

  $$C = \frac{1}{R}\left[qC_u + (1-q)\,C_d\right]. \tag{2}$$

  Because this formula reminds calculation of the discounted expected value of a random variable C_i, the factor q is termed a risk neutral probability. These are a real probability measure in that, e.g., they sum up to one, and therefore we can treat them as probabilities. For example, we can define an expectation with respect to the risk neutral probabilities as $\hat{\mathbb{E}}[X] = \sum_i q_i X_i$. The risk-neutral probabilities are not the actual probabilities that define the likelihoods of events, but they are those probabilities that should be used when pricing an asset based on an expected value (because otherwise you would lose money by enabling an arbitrage).
- The value of a European option can be calculated simply by calculating the values at the expiration nodes of the binomial lattice and then working out backwards the formula (2) until the current date is reached.
- When calculating the value of an American put option, the value at each node is the maximum of the value of the option and the payoff of exercising the option at the current node, that is,

  $$P = \max\left\{\frac{1}{R}\left(qP_u + (1-q)P_d\right), K - S\right\}.$$

  For American call options, early exercise is never optimal and hence it can be valued similarly as a European call option.

10.1 (L11.1) (Stock lattice) A stock with current value $S(0) = 100$ has an expected growth rate of its logarithm of $\nu = 12\%$ and a volatility of that growth rate of $\sigma = 20\%$. Find suitable parameters of a binomial lattice representing this stock with a basic elementary period of 3 months. Draw the lattice and enter the node values of 1 year. What are the probabilities of attaining the various final nodes?

Solution:
If we consider that $\Delta t = 3/12 = 0.25$ is small, then by the formulas (1), we set

$$p = \frac{1}{2} + \frac{1}{2}\left(\frac{\nu}{\sigma}\right)\sqrt{\Delta t} = 0.65$$
$$u = e^{\sigma\sqrt{\Delta t}} = 1.1052$$
$$d = e^{\sigma\sqrt{-\Delta t}} = 0.9048$$

We construct the binomial lattice with parameters u and d in a tabular form in Table 1. The periods k have the length of three months. The column # u/d indicates the steps taken in the lattice when arriving to different final values $S(4)$. The probabilities of attaining the values in the final nodes p' are calculated as the binomial probabilities $p' = \binom{n}{k}p^k(1-p)^{n-k}$, where $n = 4$ and k is the number of upward movements required to arrive at the specific final node.

Table 1: The binomial lattice.

$S(0)$	$S(1)$	$S(2)$	$S(3)$	$S(4)$	# u/d	Arrival probability p'	$p' \times S(4)$	$\ln\left(\frac{S(T)}{S(0)}\right)$	$p' \times \ln\left(\frac{S(T)}{S(0)}\right)$
100.00	110.52	122.14	134.99	149.18	uuuu	17.9 %	26.63	0.4	0.0714
	90.48	100.00	110.52	122.14	uuud	38.4 %	46.96	0.2	0.0768
		81.87	90.48	100.00	uudd	31.1 %	31.05	0.0	0.0000
			74.08	81.87	uddd	11.1 %	9.13	-0.2	-0.0223
				67.03	dddd	1.5 %	1.01	-0.4	-0.0060
$\sum$							114.78		0.12

The expected rate of return after 1 year is 14.78%. Note that 12% is the expected growth rate of the logarithm of the price.

10.2 (American put option) Consider the stock of Exercise 10.1. An American put option has been written on the stock with a strike price of 90 € and an expiration date after 1 year. Using a binomial lattice with a basic elementary period of 3 months, find the probability that it will be beneficial to exercise this option. The yearly risk-free rate is 10%.

Solution:

To find out the probability of getting to exercise the option, we first have to calculate the value lattice of the option. The calculation will be made with backward recursion, and hence the policy cannot be calculated directly.

The value lattice requires the risk-neutral probabilities q and $1-q$ of moving up and down in the lattice. The per-period risk-free rate is $R = 1 + 0.1/4 = 1.025$. We calculate q as

$$q = \frac{R-d}{u-d} = \frac{1.025 - 1.1052}{1.1052 - 0.9048} = 0.600,$$

where u and d were calculated in the previous exercise. Then, we calculate the value lattice of the American put option by checking in every node of the lattice, whether exercising or holding the option yields better returns by risk-neutral expectation, that is, by setting

$$P = \max\left\{\frac{1}{R}(qP_u + (1-q)P_d), K - S\right\},$$

where P_u and P_d are the values in the following period when the value of the stock has gone up or down, respectively. We get the value of the option in each node of the lattice as

Table 2: Value of the option.

$0m$	$3m$	$6m$	$9m$	$12m$
1.80	0.48	0.00	0.00	0.00
	3.90	1.24	0.00	0.00
		8.13	3.17	0.00
			15.92	8.13
				22.97

To find out the optimal exercising policy in each node of Table 2, we see for each node, whether the value of the node is $(qP_u + (1-q)P_d)/R$ or $K - S$, and set a binary decision variable 0 or 1, respectively. This indicates whether it is beneficial to hold or exercise the option in each node, respectively. We get

Table 3: Option exercising policy.

$0m$	$3m$	$6m$	$9m$	$12m$
0	0	0	0	0
	0	0	0	0
		1	0	0
			1	**1**
				1

It is clear that the nodes ddd and dddd will never be reached, because in order to arrive at these nodes, the node dd should be arrived first, and it is beneficial to exercise the option already at node dd. Hence the nodes ddd and dddd will never be reached while still holding the option, because it is optimal to exercise the option before them.

To calculate the probability that the option will yield profits, we calculate the probabilities of either (i) attaining node dd, or (ii) attaining node uddd, given that the node dd was not reached. The first probability equals having 0 successes in 2 Bernoulli trials with $p = 0.65$ (from Exercise 10.1.), and hence it is the binomial probability

$$\mathbb{P}(\mathrm{dd}) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{2}{0} p^0 (1-p)^2 = 0.1225.$$

The only way to arrive at uddd without visiting node dd is to arrive there via node ud. Hence the second probability a joint probability of two binomial probabilities, with first one corresponding one success and second no successes, both from a trial of $n = 2$. Hence we have

$$\mathbb{P}(\mathrm{uddd} \mid \overline{\mathrm{dd}}) = \mathbb{P}(\mathrm{ud}) \cdot \mathbb{P}(\mathrm{uddd} \mid \mathrm{ud}) = \binom{2}{1} p^1 (1-p)^1 \cdot \binom{2}{0} p^0 (1-p)^2 = 0.0557.$$

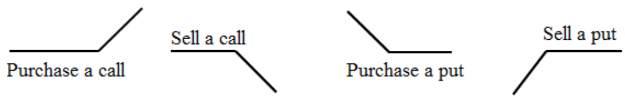
Thus the total probability that the option will be profitable and will be exercised is

$$\mathbb{P}(\text{Option will be exercised}) = \mathbb{P}(\mathrm{dd}) + \mathbb{P}(\mathrm{uddd} \mid \overline{\mathrm{dd}}) = 0.1225 + 0.0557 = 0.1782.$$

10.3 (L12.1) (Bull spread) An investor who is bullish about a stock (believing that it will rise) may wish to construct a *bull spread* for that stock. One way to construct such a spread is to buy a call with strike price K_1 and sell a call with the same expiration date but with a strike price of $K_2 > K_1$. Draw the payoff curve for such a spread. Is the initial cost of the spread positive or negative?

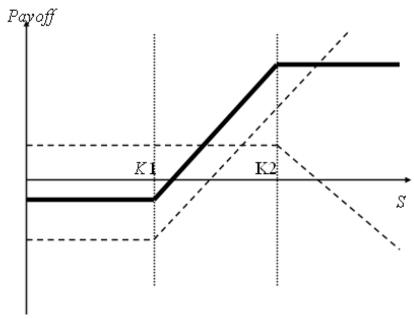
Solution:

There are two kinds of basic options. An option that gives the right to purchase something is called a call option, whereas an option that gives the right to sell something is called a put. Both of these can be both bought and sold. The graphs below present the values of different options at their expiration.



The payoff curves fold at the strike price K and the horizontal parts of the graphs correspond zero value of the options. The curve of selling an option is the opposite of the curve of purchasing the same option. If the price of the option is taken into account, the curve of selling an option is shifted upwards and the curve of purchasing an option will be shifted downwards.

The below figure shows the payoffs of buying a call with strike price K_1 and selling a call with strike price $K_2 > K_1$, shown with dashed lines. The sum of the two payoffs is shown with the thick solid line.



The price of a call option $C(K)$ increases when the strike price K decreases. (The opposite stands for put options.) The initial cost of the spread is positive since $C(K_1) > C(K_2)$ for $K_1 < K_2$.

10.4 (L12.5) (Fixed dividend) Suppose that a stock will pay a dividend of amount D at time τ. We wish to determine the price of a European call option on this stock using the lattice method. Accordingly, the time interval $[0, T]$ covering the life of the option is divided into N intervals, and hence $N + 1$ time periods are assigned. Assume that the dividend date τ occurs somewhere between period k and $k + 1$. One approach to the problem would be to establish a lattice of stock prices in the usual way, but subtract D from the nodes at period k. This produces a tree with nodes that do not recombine, as shown in Figure 1.

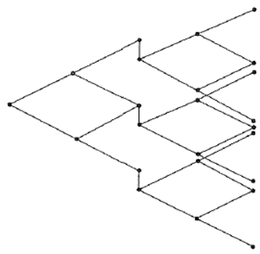


Figure 1: Non-recombining dividend tree.

The problem can be solved this way, but there is another representation that does recombine. Since the dividend amount is known, we regard it as a non-random component of the stock price. At any time before the dividend we regard the price as having two components: a random component S^* and a deterministic component equal to the present value of the future dividend. The random component S^* is described by a lattice with initial value $S(0) - De^{-r\tau}$ and with u and d determined by the volatility σ of the stock. The option is evaluated on this lattice. The only modification that must be made in the computation is that when valuing the option at a node, the stock price used in the valuation formula is not just S^* at that node, but rather $S = S^* + De^{-r(\tau - t)}$ for $t < \tau$. Use this technique to find the value of a 6-month call option with $S(0) = 50, K = 50, \sigma = 20\%, r = 10\%$, and $D = 3$ to be paid in $3\frac{1}{2}$ months.

Solution:
We have:
Initial value of the stock $S(0) = 50$
Strike price $K = 50$
Volatility $\sigma = 20\%$
Yearly rate of return $r_y = 10\%$
Monthly rate of return $r_m = 0.83\%$
Total rate of return (monthly) $R = 1.0083$
Dividend $D = 3$
Period length $\Delta t = 1/12$

We construct two binomial lattices; one that defines the present value of the dividend (deterministic component) and one that defines the random part of the value of the stock (stochastic component). The present value of the dividend in presented by the lattice below.

0	1	2	3	4	5	6
2.914	2.938	2.963	2.988	0.000	0.000	0.000
	2.938	2.963	2.988	0.000	0.000	0.000
		2.963	2.988	0.000	0.000	0.000
			2.988	0.000	0.000	0.000
				0.000	0.000	0.000
					0.000	0.000
						0.000

The stochastic part of the value of the stock is calculated by subtracting the present value of the deterministic component from the initial value of the stock and then working out the values in the lattice in the conventional way. The parameters of the lattice are

$$u = e^{\sigma\sqrt{\Delta t}} = 1.059 \qquad d = \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}} = 0.944.$$

Hence the stochastic component of the stock value is:

0	1	2	3	4	5	6
47.09	49.88	52.85	55.99	59.32	62.84	66.58
	44.44	47.09	49.88	52.85	55.99	59.32
		41.95	44.44	47.09	49.88	52.85
			39.60	41.95	44.44	47.09
				37.38	39.60	41.95
					35.28	37.38
						33.30

We then find the lattice of the stock value S by summing the two lattices. The initial value of the stock is 50 as it should be.

0	1	2	3	4	5	6
50.00	52.82	55.81	58.98	59.32	62.84	66.58
	47.38	50.05	52.87	52.85	55.99	59.32
		44.91	47.43	47.09	49.88	52.85
			42.59	41.95	44.44	47.09
				37.38	39.60	41.95
					35.28	37.38
						33.30

We then calculate the values of the European and American call options on this stock. First we calculate the risk neutral probability as

$$q = \frac{R - d}{u - d} = \frac{1.0083 - 0.944}{1.059 - 0.944} = 0.558.$$

Then, for the European option we set the values at the last column of the lattice as $\max(S-K,0)$, and work the value of the option backwards the lattice using

$$C = \frac{1}{R}\left(qC_u + (1-q)C_d\right).$$

These procedures yield the lattice of the European call option as presented in the table below.

0	1	2	3	4	5	6
2.514	3.698	5.324	7.466	10.142	13.257	16.579
	1.067	1.718	2.724	4.233	6.404	9.318
		0.267	0.482	0.872	1.576	2.850
			0.000	0.000	0.000	0.000
				0.000	0.000	0.000
					0.000	0.000
						0.000

The values of the American call option at the expiration date are calculated the same way. However, in the other cells of the lattice, early exercise is possible, and hence the values in these cells are calculated as

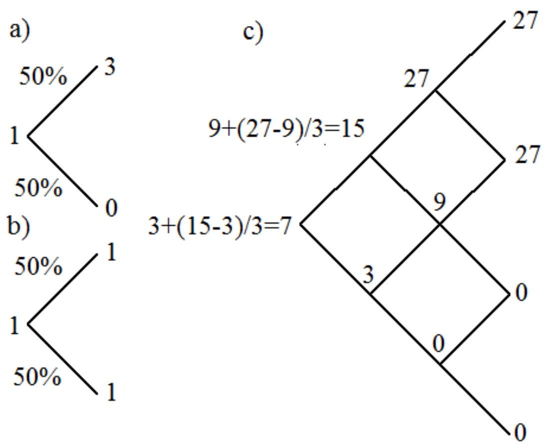
$$C = \max\left\{\frac{1}{R}(qC_u + (1-q)C_d), S-K\right\}.$$

The lattice of the American call option is shown below.

0	1	2	3	4	5	6
2.829	4.233	6.226	<u>8.978</u>	10.142	13.257	16.579
	1.113	1.800	<u>2.872</u>	4.233	6.404	9.318
		0.267	0.482	0.872	1.576	2.850
			0.000	0.000	0.000	0.000
				0.000	0.000	0.000
					0.000	0.000
						0.000

10.5 (L12.9) (My coin) There are two propositions:
a) I flip a coin. If it is heads, you are paid 3 €; if it is tail, you are paid 0 €. It cost you 1 € to participate in this proposition. You may do so at any level, or repeatedly, and the payoffs scale accordingly.
b) You may keep your money in your pocket (earning no interest).
Here is a third proposition:
c) I flip the coin three times. If at least two of the flips are heads, you are paid 27 €; otherwise zero. How much is this proposition worth?

Solution:
We evaluate how much proposition c) is worth based on propositions a) and b) by breaking it into parts. The figure below shows the three propositions. Upward movement in the trees represent flipping heads and a downward movement represents tails.



We work out the values at each node backwards the tree, starting with the final nodes.

- The value of proposition c) at the final nodes are clear.
- The values at the second to last nodes are worked out as follows. The uppermost of these has flipped heads twice; hence it has already earned the 27 € and it is the value of this node. The mid node corresponds to the first proposition scaled to play at bet 9 €, and the lowest node can no longer win anything, giving a value 0 €.
- Roll back one more stage to the second nodes. The upper node corresponds to participating in proposition a) for 6 € to win 18 € or 0 € AND in proposition b) for 9 € so that an additional 9 € is kept in both cases. Hence the total value of this node is 9+6=15. The value of the bottom node correspond participating a) for 3 €.
- The first node corresponds to participating a) for 4 € to win 12 € or 0 € AND participating b) for 3 € to win either 12+3=15 or 0+3=3. Hence the total value of the proposition c) is 7.