- We define the additive process z, termed a random walk, as

$$\begin{cases} z(t_{k+1}) = z(t_k) + \varepsilon(t_k)\sqrt{\Delta t} \\ t_{k+1} = t_k + \Delta t \end{cases}, \tag{1}$$

for $k = 0, \ldots, N$. In these equations $\varepsilon(t_i)$ is a normal variable with mean 0 and variance 1. These random variables are mutually uncorrelated; that is, $\mathbb{E}\left[\varepsilon(t_j)\varepsilon(t_k)\right] = 0$ for $j \neq k$.

The difference random variables $z(t_k) - z(t_j)$ for $j < k$ can be written as

$$z(t_k) - z(t_j) = \sum_{i=j}^{k-1} \varepsilon(t_i)\sqrt{\Delta t},$$

and using the independence of $\varepsilon(t_j)$'s we find

$$\mathrm{Var}\left[z(t_k) - z(t_j)\right] = \mathbb{E}\left[\sum_{i=j}^{k-1} \varepsilon(t_i)\sqrt{\Delta t}\right]^2 = \mathbb{E}\left[\sum_{i=j}^{k-1} \varepsilon(t_i)^2 \Delta t\right] = (k-j)\Delta t = t_k - t_j.$$

By taking the limit of the random walk process as $\Delta t \to 0$ we obtain the **Wiener process**, or alternatively, Brownian motion. In symbolic form, we write this as

$$\mathrm{d}z = \varepsilon(t)\sqrt{\mathrm{d}t},$$

which corresponds infinitesimal differences in (**??**). This process can be extended to **generalized Wiener process** by allowing for a deterministic shift $a\,\mathrm{d}t$, that is,

$$\mathrm{d}x(t) = a\,\mathrm{d}t + b\,\mathrm{d}z, \tag{2}$$

where z is a Wiener process (**??**) and a and b are constants.

- The generalized Wiener process can be further extended by allowing for shifts (both stochastic and deterministic) dependent on a function of x and t, other than a constant. An **Ito process** is defined as

$$\mathrm{d}x(t) = a(x,t)\,\mathrm{d}t + b(x,t)\,\mathrm{d}z, \tag{3}$$

where $a(x,t)$ and $b(x,t)$ denote some functions on x and t and z is again a Wiener process (**??**).

- Ito's lemma: Suppose that the random process x is defined by the Ito process (**??**) and that the process $y(t)$ is defined by $y(t) = F(x,t)$. Then $y(t)$ satisfies the Ito equation

$$\mathrm{d}y(t) = \left(\frac{\partial F}{\partial x}a + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}b^2\right)\mathrm{d}t + \frac{\partial F}{\partial x}b\,\mathrm{d}z, \tag{4}$$

where z is a Wiener process (**??**).

- The price of a security can be modelled with **geometric Brownian motion**, that is,

$$\mathrm{d}S = \mu S\,\mathrm{d}t + \sigma S\,\mathrm{d}z. \tag{5}$$

In this equation, $\mu = \nu + \sigma^2/2$, where ν is the growth rate of the logarithm of S (remember the multiplicative model for security price) and σ is the volatility of this growth rate. This is a special case of Ito process.

- Black-Scholes equation: Suppose that the price of a security is governed by (**??**) and the interest rate is r. A derivative of this security has a price $f(S,t)$, which satisfies the partial differential equation

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf.$$

The Black-Scholes equation can be proven by modelling an instantaneous change in the price a portfolio that replicates the derivative, and then setting this equal to the change in price implied by Ito's lemma (**??**).

- Usually it is impossible to find an analytic solution to the Black-Scholes equation. However, it is possible to find such a solution for a European call option. Let K be the strike price and T the expiration time of the option, and S be the price of the underlying asset that pays no dividends, and let r be the interest rate, compounded continuously. The Black-Scholes solution is $f(S,t) = C(S,t)$, defined by

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2),$$

where $N(x)$ denotes the standard cumulative normal probability distribution and d_1 and d_2 are parameters, defined as

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,\mathrm{d}y$$
$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}.$$

- The first order sensitivity of the value of a derivative security in the value of the underlying asset is described by the quantity delta (Δ); that is,

$$\Delta = \frac{\partial f(S,t)}{\partial S} \approx \frac{\Delta f(S,t)}{\Delta S} \Rightarrow \Delta f(S,t) \approx (\Delta S)\Delta.$$

For example, the sensitivity of a European call option is

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \Rightarrow \Delta = \frac{\partial C(S,t)}{\partial S} = N(d_1).$$

Other similar measures are the sensitivity with respect to change in time Θ and the second order sensitivity to the price change of the underlying asset Γ. These are defined as

$$\Theta = \frac{\partial f(S,t)}{\partial t} \qquad \Gamma = \frac{\partial^2 f(S,t)}{\partial S^2}.$$

11.1 (L13.3) (Sigma estimation) Traders in major financial institutions use the Black-Scholes formula in a backward fashion to infer other traders' estimates of σ from option prices. In fact, traders frequently quote sigmas to each other, rather than prices, to arrange trades. Suppose a call option on a stock that pays no dividend for 6 months has a strike price of 35 €, a premium of 2.15 €, and time to maturity of 7 weeks. The current risk-free rate is 7%, and the price of the underlying stock is 36.12 €. What is the implied volatility of the underlying security?

Solution:
We find the volatility σ such that the Black-Scholes call option formula gives the value $C = 2.15$. Inserting into Excel the formulas

$$
\begin{aligned}
C(S,t) &= SN(d_1) - Ke^{-r(T-t)}N(d_2), \text{ where} \\
N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\,\mathrm{d}y \\
d_1 &= \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t},
\end{aligned}
$$

and the parameter values

$$K = 35 \qquad T = 7/52 = 0.1346 \qquad t = 0 \text{ (present)} \qquad r = 7\% \qquad S = 36.12,$$

and finding the volatility with Goal Seek method of Excel (Data→What-If Analysis→Goal Seek) yields

$$\underline{\sigma = 25.10\%} \qquad d_1 = 0.4905 \qquad d_2 = 0.3984.$$

11.2 (L13.4) (Black-Scholes approximation) The first order Taylor approximation of $N(d)$ is $1/2 + d/\sqrt{2\pi}$. Use this to derive the value of a call option when the stock price is at the present value of the strike price; that is, $S = Ke^{-rT}$. Specifically, show that $C \approx 0.4S\sigma\sqrt{T}$. Also show that $\Delta \approx 1/2 + 0.2\sigma\sqrt{T}$.
Suppose the price of a stock is 62 € and the volatility is 20%. The risk-free rate is 10%. Use the above approximations to estimate the value of a 5-month European call option with a strike price $K = 60$ for the mentioned stock.

Solution:
The Black-Scholes call option formulas are

$$\begin{aligned} C(S,t) &= SN(d_1) - Ke^{-r(T-t)}N(d_2), \text{ where} \\ d_1 &= \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}, \end{aligned}$$

and $N(d)$ is the cumulative standard normal probability distribution. We approximate $N(d)$ as

$$N(d) \approx \frac{1}{2} + \frac{d}{\sqrt{2\pi}},$$

and set $t = 0$, yielding

$$C(S,t) = S\left(\frac{1}{2} + \frac{d_1}{\sqrt{2\pi}}\right) - Ke^{-rT}\left(\frac{1}{2} + \frac{d_2}{\sqrt{2\pi}}\right).$$

Because the stock price is at the present value of the strike price, we have

$$S = Ke^{-rT} \Leftrightarrow K = Se^{rT},$$

and substituting this into the value formula of the derivative security yields

$$C(S,t) = S\left(\frac{1}{2} + \frac{d_1}{\sqrt{2\pi}}\right) - S\left(\frac{1}{2} + \frac{d_2}{\sqrt{2\pi}}\right) = S\left(\frac{d_1 - d_2}{\sqrt{2\pi}}\right) = S\left(\frac{d_1 - d_1 + \sigma\sqrt{T}}{\sqrt{2\pi}}\right) = \frac{1}{\sqrt{2\pi}}S\sigma\sqrt{T},$$

and because $1/\sqrt{2\pi} \approx 0.4$, we have $\underline{C \approx 0.4S\sigma\sqrt{T}}$.

For the Black-Scholes call option formula we have

$$\Delta = N(d_1).$$

Approximating $N(d) \approx 1/2 + d/\sqrt{2\pi}$ and substituting $S = Ke^{-rT}$ yields

$$\Delta \approx \frac{1}{2} + \frac{d_1}{\sqrt{2\pi}} = \frac{1}{2} + \frac{\ln(Ke^{-rT}/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}\sqrt{2\pi}} = \frac{1}{2} + \frac{-rT + rT + \sigma^2 T/2}{\sigma\sqrt{T}\sqrt{2\pi}} = \frac{1}{2} + \frac{1}{2\sqrt{2\pi}}\sigma\sqrt{T} \approx \frac{1}{2} + 0.2\sigma\sqrt{T}.$$

The parameter values for the call of this exercise are

$$S = 62 \qquad \sigma = 20\% \qquad r = 10\% \qquad T = \frac{5}{12} = 0.4167 \qquad K = 60.$$

We cannot approximate the value of the call directly, because the the value of the stock is not at the present value of the strike price as seen by

$$S' = Ke^{-rT} = 60e^{-0.10 \cdot 0.4167} \approx 57.551 \neq 62 = S.$$

However, we can calculate the option value at the present value of the strike price and approximate the value of the option at the current stock price using the approximation for Δ. The option value at the present value of the strike price is

$$C(S') = 0.4S'\sigma\sqrt{T} = 2.972$$

The value of Δ at this base point is

$$\Delta \approx \frac{1}{2} + 0.2\sigma\sqrt{T} = 0.526.$$

The difference in S is $62 - 57.55 = 4.45$. Hence the final value is

$$C(S) \approx C(S') + \Delta(S - S') = 2.972 + 0.526 \cdot 4.45 = 5.31$$

11.3 (L13.6) (A special identity) Kalle Virtanen believes that for a derivative security with price $P(S)$, the values of Δ, Γ and Θ are related. Show that in fact

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rP.$$

Solution:

Black-Scholes equation:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf$$

The definitions of delta Δ, gamma Γ and theta Θ are:

$$\Delta = \frac{\partial f}{\partial S} \qquad \Gamma = \frac{\partial^2 f}{\partial S^2} \qquad \Theta = \frac{\partial f}{\partial t}$$

Substituting these into Black-Scholes equation and setting $f = p$ gives

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rP,$$

which completes the proof.

11.4 (L13.11) (Pay-later options) Pay-later options are options for which the buyer is not required to pay the premium up front (i.e., at the time that the contract is entered into). At expiration, the holder of a pay-later options *must* exercise the option if it is in the money, in which case he pays the premium at that time. Otherwise the option is left unexercised and no premium is paid.
The stock of the CCC Corporation is currently valued at 12 € and is assumed to possess all the properties of geometric Brownian motion. It has an expected annual return of 15%, an annual volatility of 20%, and the annual risk-free rate is 10%.
a) Using a binomial lattice, determine the price of a call option of CCC stock maturing in 10 months' time with a strike price of 14 €. (Let the distance between nodes on your tree be 1 month in length.)
b) Using a similar methodology, determine the premium for a pay-later call with all the same parameters as the call in part a).
c) Compare your answers to part a) and b). Do the answers differ; if so why, if not why not? Under what conditions would you prefer to hold which option?

Solution:
The given values are:

$$K = 14 \qquad S = 12 \qquad \mu = 15\% \qquad \sigma = 20\% \qquad r_y = 10\% \qquad \Delta t = 1/12 \text{ (period length)}$$

We construct the binomial price lattice using the parameters $u = e^{\sigma\sqrt{\Delta t}} = 1.0594$ and $d = e^{-\sigma\sqrt{\Delta t}} = 0.9439$ as follows:

0	1	2	3	4	5	6	7	8	9	10
12.00	12.71	13.47	14.27	15.12	16.02	16.97	17.98	19.04	20.18	21.38
	11.33	12.00	12.71	13.47	14.27	15.12	16.02	16.97	17.98	19.04
		10.69	11.33	12.00	12.71	13.47	14.27	15.12	16.02	16.97
			10.09	10.69	11.33	12.00	12.71	13.47	14.27	15.12
				9.53	10.09	10.69	11.33	12.00	12.71	13.47
					8.99	9.53	10.09	10.69	11.33	12.00
						8.49	8.99	9.53	10.09	10.69
							8.01	8.49	8.99	9.53
								7.56	8.01	8.49
									7.14	7.56
										6.74

a) The value of a European call option:
Next we calculate $R = 1 + \Delta t \cdot r_y = 1.0083$. This gives $q = (R - d)/(u - d) = 0.56$ as the risk-neutral probability for an up move. The value of the call option in the last column (at the expiration date) is

$$C_T = \max\{S_T - K, 0\},$$

and the value at the other periods are calculated using discounted risk-neutral valuation as

$$C = \frac{1}{R}\left(qC_u + (1 - q)C_d\right).$$

The standard lattice for the valuation of the European call option is presented below. The resulting price of the option is <u>0.53 €</u>

0	1	2	3	4	5	6	7	8	9	10
0.53	0.76	1.08	1.49	2.02	2.67	3.44	4.32	5.28	6.29	7.38
	0.25	0.38	0.57	0.85	1.23	1.75	2.40	3.20	4.09	5.04
		0.09	0.15	0.24	0.38	0.61	0.95	1.45	2.13	2.97
			0.02	0.03	0.06	0.10	0.19	0.34	0.62	1.12
				0.00	0.00	0.00	0.00	0.00	0.00	0.00
					0.00	0.00	0.00	0.00	0.00	0.00
						0.00	0.00	0.00	0.00	0.00
							0.00	0.00	0.00	0.00
								0.00	0.00	0.00
									0.00	0.00
										0.00

b) Pay-later option:
Pay-later option is an option for which premium is paid only if the option ends up in the money. Because no premium is paid at the time of writing the option, the value of the pay-later option must be zero when issued. If the premium p is greater than the difference $S - K > 0$, holding the option causes losses.

The pay-later option lattice is set up exactly the same way as that of the standard European call option except that the final values are

$$C_T = \begin{cases} S_T - K - p & , S_T - K > 0 \\ 0 & , S_T - K \leq 0 \end{cases}$$

The value p is unknown. We use Goal Seek method of Excel (Data→What-If Analysis→Goal Seek) to adjust the value of p so that the initial price is zero. In this case that value is 2.04 €. The corresponding lattice with that value is shown below.

0	1	2	3	4	5	6	7	8	9	10
0.00	0.04	0.13	0.29	0.55	0.95	1.54	2.33	3.27	4.27	5.33
	-0.05	-0.06	-0.06	-0.04	0.04	0.23	0.59	1.19	2.07	3.00
		-0.04	-0.07	-0.10	-0.14	-0.19	-0.22	-0.17	0.11	0.93
			-0.01	-0.03	-0.05	-0.09	-0.16	-0.28	-0.51	-0.92
				0.00	0.00	0.00	0.00	0.00	0.00	0.00
					0.00	0.00	0.00	0.00	0.00	0.00
						0.00	0.00	0.00	0.00	0.00
							0.00	0.00	0.00	0.00
								0.00	0.00	0.00
									0.00	0.00
										0.00

c) Obviously the pay-later option premium is higher than that of a standard option: the premium is not paid until later (meaning there is interest rate advantage), and more importantly, no premium is paid if the option does not end up in the money.