- Securities with payoffs that depend on interest rates are called **interest rate derivatives**. Examples of these derivatives are bonds, mortgages and swaps. These financial instruments entail exposure to interest rate risk.

- To analyse the interest rate risk systematically, it is best to model interest rate fluctuations. An arbitrage-free framework for constructing interest rate models is a **binomial short rate lattice**. The nodes of this lattice define the one-period rates to the next period. For the purposes of our analysis, it suffices to simply assign the multiplicative factors u and d and the risk-neutral probabilities q and $(1-q)$ of the lattice. We then study pricing of interest derivatives according to the interest rate fluctuations implied by this model.

- We use the following indexing convention for the binomial short rate lattice. We index the columns of the lattice with t that corresponds the period number of each node. Each node is also assigned an index i that denotes how many ups it has taken to reach the node. Hence, we use indexing (t,i) as illustrated in the below table:

		$(2,2)$
	$(1,1)$	$(2,1)$
$(0,0)$	$(1,0)$	$(2,0)$

- For sake of demonstration, we assign an up factor of $u = 1.2$ and a down factor of $d = 0.9$, and the current short rate is $r = 0.06$. We then construct a simple short rate lattice up to year $T = 3$ using these parameter values:

0	1	2
		0.0864
	0.072	0.0648
0.06	0.054	0.0486

  For example, the short rate at node $(2,1)$ is calculated as $r_{21} = 0.06ud = 0.06du = 0.0648$.

- This lattice can be used for pricing interest rate securities by using risk-neutral pricing. We calculate the value V_{ti} of a security in each node (t,i) by noting that the value of the security is related to the value of the security at the next two possible successor nodes according to the risk-neutral pricing formula

$$V_{ti} = \frac{1}{1+r_{ti}}\left(qV_{t+1,i+1} + (1-q)V_{t+1,i}\right) + D_{ti}, \tag{1}$$

  where D_{ti} is the dividend paid at node (t,i). Using this formula we calculate the value of a 6% bond that matures in $T = 3$ years (the coupon is first paid in period 1). The values in last column are simply the sum of the coupon and principal payments. The other values are calculated using the recursive formula (**??**), where risk-neutral probabilities were assigned as $q = 0.5$ and $1-q = 0.5$.

0	1	2	3
			106
		103.57	106
	103.54	105.55	106
105.25	106.87	107.09	106

12.1 (L14.1) (A callable bond) Construct a short rate lattice for periods (years) 0 through 9 with an initial rate of 6% and with successive rates determined by a multiplicative factor of either $u = 1.2$ or $d = 0.9$. Assign the risk-neutral probabilities to be 0.5.
a) Using this lattice, find the value of a 10-year 6% bond.
b) Suppose this bond can be called by the issuing party at any time after 5 years. (When the bond is called, the face value plus the currently due coupon are paid at that time and the bond is cancelled.) What is the fair value of this bond?

Solution

We first construct the short rate lattice using parameters $u = 1.2$ and $d = 0.9$.

0	1	2	3	4	5	6	7	8	9
									31.0 %
								25.8 %	23.2 %
							21.5 %	19.3 %	17.4 %
						17.9 %	16.1 %	14.5 %	13.1 %
					14.9 %	13.4 %	12.1 %	10.9 %	9.8 %
				12.4 %	11.2 %	10.1 %	9.1 %	8.2 %	7.3 %
			10.4 %	9.3 %	8.4 %	7.6 %	6.8 %	6.1 %	5.5 %
		8.6 %	7.8 %	7.0 %	6.3 %	5.7 %	5.1 %	4.6 %	4.1 %
	7.2 %	6.5 %	5.8 %	5.2 %	4.7 %	4.3 %	3.8 %	3.4 %	3.1 %
6.0 %	5.4 %	4.9 %	4.4 %	3.9 %	3.5 %	3.2 %	2.9 %	2.6 %	2.3 %

a) We calculate the values of the bond in the last period as the sum of the coupon and principal payments, that is, $V_{iT} = 6 + 100 = 106$. The rest of the values are calculated with the recursive formula

$$V_{ti} = \frac{1}{1 + r_{ti}} \left(qV_{t+1,i+1} + (1 - q)V_{t+1,i}\right) + D_{ti},$$

working backwards the lattice until the initial time. Risk-neutral probabilities $q = 0.5$ and $1 - q = 0.5$ were assigned for the pricing and $D_{ti} = 6\ \forall t, i$. The values in each node of the lattice are presented below.

0	1	2	3	4	5	6	7	8	9	10
										106
									86.9	106
								77.1	92.0	106
							72.7	84.9	96.3	106
						71.6	82.0	91.6	99.8	106
					72.8	81.9	90.2	97.2	102.5	106
				75.6	83.8	91.2	97.2	101.8	104.7	106
			79.8	87.2	93.8	99.1	103.1	105.5	106.5	106
		84.9	91.8	97.7	102.4	105.8	107.8	108.4	107.8	106
	91.0	97.2	102.5	106.7	109.6	111.2	111.5	110.7	108.8	106
91.7	103.5	108.2	111.8	114.3	115.5	115.5	114.5	112.5	109.6	106

b) We now calculate the value of the bond when the issuer of the bond has an option to call the bond after 5 years. The issuer will obviously exercise the call option if the value of the bond will become greater than the call price, which is now $F + D_{ti} = 100 + 6 = 106$. If the interest rates in the market have gone down by the time of the call date, the issuer will be able to refinance the debt at a cheaper rate.

After the call date (that is, starting from year 5), the recursive formula for calculating the value of the bond becomes

$$V_{ti} = \min\left\{F + D_{ti}, \frac{1}{1+r_{ti}}\left(qV_{t+1,i+1} + (1-q)V_{t+1,i}\right) + D_{ti}\right\}$$

Hence, we construct the following value lattice for the callable bond. **Boldfaced values** correspond times when call option can be exercised and underlined values correspond nodes when the issuer will exercise the call option.

0	1	2	3	4	5	6	7	8	9	10
										106
									86.9	**106**
								77.1	**92.0**	**106**
							72.7	**84.9**	**96.3**	**106**
						71.6	**82.0**	**91.6**	**99.8**	**106**
					72.8	**81.9**	**90.2**	**97.2**	**102.5**	**106**
				75.6	**83.8**	**91.2**	**97.2**	**101.8**	**104.7**	**106**
			79.8	87.2	**93.7**	**99.1**	**103.0**	**105.3**	**106.0**	**106**
		84.9	91.7	97.4	**101.9**	**104.9**	**106.0**	**106.0**	**106.0**	**106**
	90.7	96.7	101.5	104.8	**106.0**	**106.0**	**106.0**	**106.0**	**106.0**	**106**
91.0	102.1	105.9	107.9	108.0	**106.0**	**106.0**	**106.0**	**106.0**	**106.0**	**106**

The value of the callable bond is lower than the standard bond because the buyer of the bond experiences losses of $S_{ti} - 106$ if the issuer exercises the call option, where S_{ti} is the value of the bond without the call option.

12.2 (L14.8) (Swaps) Consider a plain vanilla interest rate swap where party A agrees to make six yearly payments to party B of a fixed rate of interest on a notional principal of 10 M€ and in exchange party B will make six yearly payments to party A at the floating short rate on the same notional principal. Assume that the short rate process is described by the lattice of Exercise 1.
a) Set up a lattice that gives the value of the floating rate cash flow stream at every short rate node, and thereby determines the initial value of this stream.
b) What rate of interest would equalize the present values of the cash flows on both sides of the swap?

Solution:

The difference of parts a) and b) are the paid interest: in part a) the paid interest follows the fluctuations of the short rate and in part b) the paid interest is constant. However, the cash flow streams of the both parts are discounted using the fluctuating short rate. The short rate lattice of Exercise 1 up to year 6 is:

0	1	2	3	4	5	6
						17.9 %
					14.9 %	13.4 %
				12.4 %	11.2 %	10.1 %
			10.4 %	9.3 %	8.4 %	7.6 %
		8.6 %	7.8 %	7.0 %	6.3 %	5.7 %
	7.2 %	6.5 %	5.8 %	5.2 %	4.7 %	4.3 %
6.0 %	5.4 %	4.9 %	4.4 %	3.9 %	3.5 %	3.2 %

a) The value of the floating rate cash flow can be calculated with backward recursion. Specifically, the value of each node is equal to the amount of interest to be paid at the end of the year plus the risk-neutral sum of the values in the next two nodes - all of this discounted by the risk-free rate r_{ti}. That is,

$$V_{ti} = \frac{1}{1 + r_{ti}} \left(10 \times r_{ti} + 0.5 V_{t+1,i+1} + 0.5 V_{t+1,i}\right),$$

where $10 \times r_{ti}$ is the paid interest and risk-neutral probabilities $q = 1 - q = 0.5$ are used. Note that, unlike in formula (**??**), the paid interest is discounted because this interest is paid at the end of period t. (If the recursion would be written in the standard form (**??**), the interest rates of the earlier nodes would be required to determine the interest paid in period k, which would make the formula more complicated.)

Using the above recursion, the following lattice can be constructed for the value of the floating rate cash flow stream. The values in the last nodes are simply the discounted interest payments $D_{5i}/(1+r_{5i})$, because there is no future value for the cash flow stream, because the last payment is made at the start of period 6.

0	1	2	3	4	5
					1.299
				2.132	1.007
			2.661	1.668	0.775
		2.984	2.095	1.293	0.593
	3.164	2.359	1.632	0.994	0.451
3.240	2.505	1.842	1.260	0.760	0.342

For example, the value at the top of the last column is $10 \times 0.1493/1.1493 = 1.299$; and the value at the top of the second to last column is $(10 \times 0.1244 + 0.5 \times 1.299 + 0.5 \times 1.007)/1.1244 = 2.132$. The final value is 3.240 M€. (We used a greater precision than presented in the short rate lattice.)

b) The third lattice is constructed with a fixed interest rate r. The values are the present values of the fixed-rate payments. Hence, we use the recursion

$$V_{ti} = \frac{1}{1 + r_{ti}} \left(10r + 0.5V_{t+1,i+1} + 0.5V_{t+1,i}\right).$$

We use Goal Seek method of Excel to find the value of r such that the initial value of this lattice is equal to that of the one above. This value is $r = 0.0669$.

0	1	2	3	4	5
					0.582
				1.121	0.601
			1.643	1.169	0.617
		2.165	1.723	1.207	0.629
	2.695	2.276	1.787	1.238	0.639
3.240	2.836	2.366	1.838	1.261	0.646

12.3 (L14.9) (Swaption pricing) A swaption is an option to enter a swap arrangement in the future. Suppose that company B has debt of 10 M€ financed over 6 years at a fixed rate of 6.69%. Company A offers to sell company B a swaption to swap the fixed rate obligation for a floating rate obligation, with payments equal to the short rate, with the same principal and the same termination date. The swaption can be exercised at the beginning of year 2 (just after the payment for the previous year and when the short rate for the coming year is known). Assuming that the short rate process is that of Exercise 1, how much is this swaption worth?

Solution:

A swaption is an option on an interest rate swap. The swaption of this exercise can be used to swap the fixed rate stream of Exercise 2 to the floating rate stream. The swaption will be exercised at the beginning of year 2 only if it is beneficial for company B (that pays the obligations), that is, if the value of the floating rate stream is lower than the value of the fixed rate stream. (Conversely, if company A would hold the swaption, the option should be exercised if the value of the floating rate stream would be greater.)

To find the value of the swaption, we merely put the minimum of the two value lattices (from the previous exercise) in the third column. (The case where this minimum indicates exercise of the swaption is shown in bold in the below lattice.) We then value the lattice backward in the usual way. The value of the swaption is the difference between the value obtained and the value without the swaption, that is,

$$V_{swaption} = 3.240 - 3.123 = 0.117 \text{ M€}$$

0	1	2
		2.165
	2.695	2.276
3.123	2.588	**1.842**

12.4 (L14.2) (General adjustable formula) Let V_{ks} be the value of an adjustable-rate loan initiated at period k and state s with initial principal of 100. The loan is to be fully paid at period n. The interest rate charged each period is the short rate of that period plus a premium p. The loan payment for a period is the amount that would be required to amortize the loan at the charged interest rate equally over the remaining periods. Write an explicit backward recursion formula for V_{ks} as a function of k and s.

Solution:
The standard annuity formula is (see, e.g., Exercise round 2)

$$A = \frac{r(1+r)^n P}{(1+r)^n - 1}.$$

The interest rate in each period k and state s is $r_{ks} + p$, and the number of remaining periods is $n - k$, where n is the total number of periods and k the current period. Hence the annuity in node (k, s) is

$$A_{ks} = \frac{(r_{ks} + p)(1 + r_{ks} + p)^{n-k} P_{ks}}{(1 + r_{ks} + p)^{n-k} - 1}, \tag{2}$$

where P_{ks} is the remaining loan principal in node (k, s).

The value of the adjustable rate loan could be calculated as the value of the payment stream A_{ks} seen by the bank. As can be seen from above formula, each payment depends on two variables, the short rate r_{ks} and the remaining loan principal P_{ks}. The short rates can be modelled with a binomial lattice. However, the remaining principal cannot be directly modelled using a binomial lattice, because the remaining loan principal depends on the earlier payments and is thus **path dependent**.

Nevertheless, for path dependent problems with two variables - variable j that can be directly modelled with a lattice (here the short rate) and variable x that cannot (the remaining loan principal) - the lattice structure can be preserved by using **levelling**, if the price at a node is (linearly) proportional to the variable x. If this is the case, we can decide on a fixed level x_0 of x, and use this level at all nodes. Later, we can scale the results appropriately. (If the values at this fixed level are V_j, the values at any level x can be calculated by linearity on x as $(x/x_0)V_j$.)

Hence we construct a backward recursion that calculates the value of a fixed level of loan principal $x_0 = 100$ in each node of a binomial lattice. The payment in each node then becomes

$$A_{ks} = \frac{(r_{ks} + p)(1 + r_{ks} + p)^{n-k} 100}{(1 + r_{ks} + p)^{n-k} - 1}$$

We denote the remaining principal (from the fixed level $x_0 = 100$) at the nodes following the node (k, s) as L_{ks}. After each period the principal has grown at interest rate $r_{ks} + p$ and the payment A_{ks} will be subtracted from the principal. Hence the remaining principal will be

$$L_{ks} = 100(1 + r_{ks} + p) - A_{ks}.$$

In the next period the changing interest rate can be interpreted as refinancing the loan at a new interest rate. Effectively, the borrower will loan the amount L_{ks} at the current rate $r_{k+1,s}$ or $r_{k+1,s+1}$ and then use this loan to pay back the remaining principal L_{ks}. This process gives bank the following cash flows in the period after node (k, s):

- The bank receives the payment L_{ks} immediately.
- The bank will gain the profits of the newly issued loan, that is, it will receive a value of either $(x/x_0)V_{k+1,s} = (L_{ks}/100)V_{k+1,s}$ or $(L_{ks}/100)V_{k+1,s+1}$, which correspond the value of the loan in the nodes $(k+1, s)$ and $(k+1, s+1)$. The expected value of these two alternative values can be calculated using risk-neutral probabilities as

$$(L_{ks}/100)\left(qV_{k+1,s+1} + (1-q)V_{k+1,s}\right) = (L_{ks}/100)\left(0.5V_{k+1,s+1} + 0.5V_{k+1,s}\right),$$

  where we assigned $q = 0.5$.
- The bank also receives the periodical payment A_{ks} (that is, the agreed payment before "refinancing" the loan).

To calculate the value at node (k, s), all the above values will be discounted to period k using the prevailing short rate r_{ks} at the node. Also, the bank pays the principal of the loan 100 to the borrower, which causes a negative cash flow of 100. We can now write the recursion for the value of the adjustable-rate loan as

$$V_{ks} = \frac{L_{ks} + (L_{ks}/100)\left(0.5V_{k+1,s+1} + 0.5V_{k+1,s}\right) + A_{ks}}{1 + r_{ks}} - 100,$$

where A_{ks} and L_{ks} are calculated as

$$A_{ks} = \frac{(r_{ks} + p)(1 + r_{ks} + p)^{n-k}100}{(1 + r_{ks} + p)^{n-k} - 1} \text{ and } L_{ks} = 100(1 + r_{ks} + p) - A_{ks}.$$

Because the levelling was made using a fixed level that equals the initial principal of 100, this recursion calculates exactly the value of the adjustable-rate loan of this exercise. For this backward recursion, initial values $V_{ns} = 0\ \forall s$ should be used at the end nodes (because the value of a loan is zero if paid immediately).