Macaulay duration is defined as the duration in which the present values are calculated using the yield of the bond (yield to maturity). Specifically, suppose a financial instrument makes m payments in a year, with payment k being c_k (both coupon payment and possibly the face value), and there are n periods remaining. Then the payment times are $t_k = k/m$ and the Macaulay duration can be calculated as

$$D = \frac{\sum_{k=1}^{n} \frac{k}{m} \frac{c_k}{(1+\frac{\lambda}{m})^k}}{PV}, \text{ where } PV = \sum_{k=1}^{n} \frac{c_k}{(1+\frac{\lambda}{m})^k}.$$

If the coupon payments are identical ($c_k = C/m \,\forall\, k < n$ and $c_n = C/m + F$, where F is the face value of the bond and C the annual coupon payment), noting the coupon rate as $c = C/(mF)$, the explicit formula for the Macaulay duration is

$$D = \frac{1+y}{my} - \frac{1+y+n(c-y)}{mc[(1+y)^n - 1] + my}, \text{ where } y = \frac{\lambda}{m}.$$

This formula can be derived as follows. Assume coupon rate $c = C/(mF)$, when the periodical coupon payment is cF, and yield $y = \lambda/m$ per period.

The present value of the bond is $P = \sum_{k=1}^{n} \frac{c_k}{(1+\frac{\lambda}{m})^k} = \frac{cF}{(1+y)} + \frac{cF}{(1+y)^2} + \cdots + \frac{cF}{(1+y)^n} + \frac{F}{(1+y)^n}$. Differentiating the present value yields

$$\begin{aligned} \frac{dP}{dy} &= -\frac{1}{1+y}\left[\frac{cF}{(1+y)} + \frac{2cF}{(1+y)^2} + \cdots + \frac{ncF}{(1+y)^n} + \frac{nF}{(1+y)^n}\right] \\ &= -\frac{Pm}{1+y}\frac{1}{P}\left[\frac{1}{m}\frac{cF}{(1+y)} + \frac{2}{m}\frac{cF}{(1+y)^2} + \cdots + \frac{n}{m}\frac{(cF+F)}{(1+y)^n}\right] \\ &= -\frac{Pm}{1+y} D \left(= -mD_M P, \text{ as it should be}\right). \end{aligned} \tag{1}$$

The present value can also be written with the annuity formula as $P = \frac{cF}{y}\left[1 - \frac{1}{(1+y)^n}\right] + \frac{F}{(1+y)^n}$. Differentiating this yields

$$\frac{dP}{dy} = -\frac{cF}{y^2}\left[1 - \frac{1}{(1+y)^n}\right] + \frac{cF}{y}\frac{n}{(1+y)^{n+1}} - F\frac{n}{(1+y)^{n+1}} = -\frac{cF}{y^2}\left[1 - \frac{1}{(1+y)^n}\right] - \frac{(1-c/y)nF}{(1+y)^{n+1}} \tag{2}$$

Setting the two formulas (**??**) and (**??**) for dP/dy equal then gives

$$-\frac{Pm}{1+y} D = -\frac{cF}{y^2}\left[1 - \frac{1}{(1+y)^n}\right] - \frac{(1-c/y)nF}{(1+y)^{n+1}} \tag{3}$$

Then, we multiply both sides of (**??**) with the denominators y^2 and $(1+y)^{n+1}$ to get

$$y^2(1+y)^n PmD = \left[c(1+y)^{n+1} + ny^2 - c(1+y+ny)\right] F. \tag{4}$$

We see that the annuity formula form of the value P of the bond can be modified into

$$P = \frac{cF}{y}\left[1 - \frac{1}{(1+y)^n}\right] + \frac{F}{(1+y)^n} = \frac{c\left[(1+y)^n - 1\right] + y}{y(1+y)^n}F, \tag{5}$$

and substituting P from (**??**) into (**??**) yields

$$y\left[c\left[(1+y)^n - 1\right] + y\right] FmD = \left[c(1+y)^{n+1} + ny^2 - c(1+y+ny)\right] F \tag{6}$$

Now, we eliminate F and divide the factor of mD into the right side of the above equation to get

$$mD = \frac{c(1+y)^{n+1} + ny^2 - c(1+y+ny)}{y\left\{c\left[(1+y)^n - 1\right] + y\right\}} \tag{7}$$

Last step of the proof is to take the partial fraction decomposition of the right side of (**??**). We solve $A(y)$ and $B(y)$ from

$$\frac{c(1+y)^{n+1} + ny^2 - c(1+y+ny)}{y\left\{c\left[(1+y)^n - 1\right] + y\right\}} = \frac{A(y)}{y} + \frac{B(y)}{c\left[(1+y)^n - 1\right] + y}, \tag{8}$$

and get (detailed steps of solving these skipped here) $A(y) = 1+y$ and $B(y) = -\left[1 + y + n(c-y)\right]$. Substituting (**??**) with the previous formulas for $A(y)$ and $B(y)$ into (**??**) and dividing m into the right side yields

$$D = \frac{1+y}{my} - \frac{1+y+n(c-y)}{mc[(1+y)^n - 1] + my}, \tag{9}$$

which completes the proof.