- As a general rule, "long" bonds (bonds with very distant maturity dates) tend to offer higher yields than "short" bonds of the same quality. The spot rate s_t is the rate of interest, expressed in yearly terms, charged for money held from the present time ($t = 0$) until time t. Typically, spot rates are expressed in yearly basis. If the interest is compounded yearly, and an amount A is deposited for a year in a bank, the bank pays back $(1 + s_1)A$ after a year. Moreover, if your bank promises to pay a rate of s_2 for a 2-year deposit of an amount A compounded yearly, it will repay $(1 + s_2)^2 A$ at the end at 2 years.
  More generally, under a convention of compounding m periods per year, if an amount A is deposited for t years, the capital has grown to $(1 + s_t/m)^{mt} A$ after t years.

- Under a continuous compounding convention, $\lim_{m\to\infty}(1+s_t/m)^{mt}A = e^{s_t t}A$, which applies to all values of t.

- Spot rates can be determined either from the prices of a series of zero-coupon bonds with various maturity dates, or from the prices of coupon-bearing bonds by beginning with short maturities and working toward longer maturities.

- Constructing spot rates from coupon bearing bonds of increasing maturities works as follows. Consider a 1-year bond that has price P_1, makes a coupon payment of amount C_1 at the end of year 1, and has a face value F_1. First, define s_1 from a 1-year bond by solving it from the equality

$$P_1 = \frac{C_1 + F_1}{1 + s_1}.$$

  Then, consider a 2-year bond with price P_2, coupon C_2, and face value F_2. Using the s_1 solved in the previous phase, we can then solve s_2 from the identity

$$P_2 = \frac{C_2}{1 + s_1} + \frac{C_2 + F_2}{(1 + s_2)^2}$$

  Working forward this way, by next considering 3-year bonds, then 4-year bonds, and so forth, we can determine the spot rate curve $s_3, s_4, \ldots,$ step by step. When working forward in this manner, the general price formula for a bond with maturity n is

$$P_n = \frac{F_n}{(1 + s_n)^n} + \sum_{k=1}^{n} \frac{C_n}{(1 + s_k)^k}.$$

- A zero-coupon bond is a bond that pays no coupons, with price P that is smaller than the face value F of the bond. The price P and the face value F of a zero-coupon bond are related by $P = F/(1 + r)^n$, where r is the yield of the bond and n the maturity.

- The set of available zero-coupon bonds is typically rather sparse. Nevertheless, two bonds of different coupon rates but identical maturity dates can be used to construct the equivalent of a zero-coupon bond with a replicating portfolio. Let bonds A and B have maturity n, face value $F = 100$, prices P_A and P_B, and coupon rates c_A and c_B, such that $c_A < c_B$. A zero-coupon bond can be constructed by purchasing an amount N_A of bond A and selling (or shorting) an amount $N_B(< 0)$ of bond B such that

$$N_A c_A + N_B c_B = 0.$$

The price of the replicating portfolio can be found by first scaling the par value of the bond to 100 through

$$N_A \cdot 100 + N_B \cdot 100 = 100,$$

and then solving N_A and N_B from the two equations to calculate $P = N_A P_A + N_B P_B$. Using the price of the replicating portfolio, the spot rate s_n can be solved from equation

$$P = \frac{F}{(1 + s_n)^n}.$$

- Forward rates are interest rates for money to be borrowed between two dates in the future, *but under terms agreed upon today.* Suppose a 2-year situation, and that s_1 and s_2 are known. If we leave 1€ in a 2-year account it will grow to $(1+s_2)^2$ €. Alternatively, we might place the 1€ in a 1-year account and simultaneously make arrangements that the proceeds $(1+s_1)$ € will be lent for 1 year starting a year from now. That loan will accrue interest at a prearranged rate (agreed upon now) of f. The rate f is the forward rate for money to be lent in this way. The final amount of money we receive at the end of 2 years under this compound plan is $(1+s_1)(1+f)$. Moreover, through no arbitrage theorem, it can be shown that $(1+s_2)^2 = (1+s_1)(1+f)$, and hence $f = (1+s_2)^2/(1+s_1) - 1$. More generally, under yearly compounding, the (implied) forward rate $f_{i,j}$ between times i and $j > i$ satisfies the following equation:

$$(1+s_j)^j = (1+s_i)^i(1+f_{i,j})^{j-i} \Leftrightarrow f_{i,j} = \left[\frac{(1+s_j)^j}{(1+s_i)^i}\right]^{\frac{1}{j-i}} - 1.$$

- Consider a parallel shift in the spot rates, that is, $s_1, s_2, \ldots, s_n \rightarrow s_1+\lambda, s_2+\lambda, \ldots, s_n+\lambda$, and then consider the changes in a value of a bond. The present value of the bond is

$$PV(\lambda) = \sum_{k=0}^{n} \frac{x_k}{(1+\frac{s_k+\lambda}{m})^k}.$$

First order change in the spot rates in the present value depends on the derivative at $\lambda = 0$:

$$\frac{dP(\lambda)}{d\lambda}\Big|_{\lambda=0} = -\sum_{k=0}^{n} \frac{k}{m} \frac{x_k}{(1+\frac{s_k}{m})^{k+1}}.$$

Dividing the derivative with $-PV(0)$ gives the *quasi-modified duration* as

$$-\frac{1}{PV(0)}\frac{dP(\lambda)}{d\lambda}\Big|_{\lambda=0} = \frac{1}{PV(0)}\sum_{k=0}^{n} \frac{k}{m} \frac{x_k}{(1+\frac{s_k}{m})^{k+1}} = D_Q$$

- Suppose an obligation stream and a portfolio that is used pay these obligation as soon as they rise. By matching durations as well as present values of the portfolio and the obligation stream, the cash value of the portfolio and the present value of the obligation stream will respond identically (to first order) to a change in interest rates. This procedure is called *immunization*, which means protecting a (bond) portfolio against interest rate risk.

- When the present value of a cash flow stream is calculated in the term structure framework, one multiplies each cash flow by the discount factor associated with the period of the flow and then sums these discounted values. That is, $PV(k=0) = x_0 + d_1x_1 + d_2x_2 + \cdots + d_nx_n$. A simple modification of this formula gives

$$PV(0) = x_0 + d_1[x_1 + (d_2/d_1)x_2 \cdots + (d_n/d_1)x_n = x_0 + d_1[x_1 + d_{1,2}x_2 \cdots + d_{1,n}x_n] = PV(0) = x_0 + d_1PV(1),$$

where the discount factors $d_{1,i} = d_i/d_1$ are the discount factors *1 year from now.* More generally, it can be shown that the *running present values* satisfy the recursion

$$PV(k) = x_k + d_{k,k+1}PV(k+1).$$

To carry out the computation of $PV(0)$ in a recursive manner, the process is initiated by starting at the final time n. One first calculates $PV(n)$ as $PV(n) = x_n$, then $PV(n-1) = x_{n-1} + d_{n-1,n}PV(n)$, and so forth, until $PV(0)$ is found.

3.1 (Construction of a zero-coupon bond and forward rates) Consider two 4-year and 5-year bonds as presented in table below.
a) Find the prices of 4- and 5-year zero-coupon bond.
b) Find the short rate at year 4.

Bond	Coupon rate	Maturity (years)	Price
A	8%	4	98.30
B	7%	4	95.00
C	9%	5	101.00
D	7%	5	93.20

Solution:

Bond\Year	1	2	3	4	5	P
A	8	8	8	108		98.30
B	7	7	7	107		95.00
C	9	9	9	9	109	101.00
D	7	7	7	7	107	93.20
$n_A A + n_B B$	0	0	0	100		?
$n_C C + n_D D$	0	0	0	0	100	?

a) We find the prices of zero-coupon bonds that have the par value scaled to $F = 100$. For this par value, the price is denoted as percentage of the par value. We solve n_A,n_B,n_C and n_D from:

$$
\begin{aligned}
8n_A + 7n_B &= 0 \\
100n_A + 100n_B &= 100 \\
9n_C + 7n_D &= 0 \\
100n_C + 100n_D &= 100
\end{aligned}
$$

We find a solution $n_A = -7, n_B = 8, n_C = -3.5, n_D = 4.5$. Thus, bonds A and C will be sold and bonds B and D will be purchased. The price of such zero-coupon bonds (the replicating portfolio) are

$$P_4 = n_A P_A + n_B P_B = -7 \cdot 98.30 + 8 \cdot 95.00 \approx \underline{71.90} \text{ scores.}$$
$$P_5 = n_C P_C + n_D P_D = -3.5 \cdot 101.00 + 4.5 \cdot 93.20 \approx \underline{65.90} \text{ scores.}$$

b) First, we calculate the spot rates using the prices of zero-coupon bond replicating portfolios as

$$P = \frac{F}{(1+s_n)^n} \Rightarrow s_n = \left(\frac{F}{P}\right)^{1/n} - 1,$$

and substituting the values P_4 and P_5 yields

$$s_4 = 0.0860, \quad s_5 = 0.0870.$$

The short rate at year 4 is the forward rate from year 4 to 5. Thus

$$r_4 = f_{4,5} = f_{i,j} = \left[\frac{(1+s_j)^j}{(1+s_i)^i}\right]^{\frac{1}{j-i}} - 1 = \frac{(1+s_5)^5}{(1+s_4)^4} - 1 = 0.910.$$

3.2 (L4.13) (Stream immunization) A company faces a stream of obligations over the next 8 years as shown in the table below: where the numbers denote thousands of dollars. The spot rate curve is also presented in this table. The company has decided to invest in two bonds. Bond 1 has a maturity of 12 years, 6% coupon and price $P_1 = 65.95$ and bond 2 has maturity of 5 years, 10% coupon and price $P_2 = 101.66$. Find a portfolio, consisting of the two bonds, that has the same present value as the obligation stream and is immunized against an additive shift in the spot rate curve.

Year	1	2	3	4	5	6	7	8	9	10	11	12
Payment	500	900	600	500	100	100	100	50	0	0	0	0
Spot rate	7.67	8.27	8.81	9.31	9.75	10.16	10.52	10.85	11.15	11.42	11.67	11.89

Solution:
To construct and immunized portfolio we first calculate the present value and the quasi-modified duration of the obligation stream.

The present value, after a parallel shift λ in the spot rates, is:

$$P(\lambda) = \sum_{k=1}^{n} x_k \left(1 + \frac{s_k + \lambda}{m}\right)^{-k}$$

Quasi-modified duration is the relative (first order) change in the present value of a bond under a parallel shift λ in the spot rates. It is formally defined as:

$$D_Q \equiv -\frac{1}{P(0)} \frac{dP(\lambda)}{d\lambda}\Big|_{\lambda=0} = \frac{1}{P(0)} \sum_{k=1}^{n} \left(\frac{k}{m}\right) x_k \left(1 + \frac{s_k}{m}\right)^{-(k+1)}$$

In the tables below, duration factor denotes the summed terms in the quasi-modified duration formula, without the payments x_k. Duration term denotes same terms, but including the payments x_k.

Year	1	2	3	4	5	6	7	8	9	10	11	12
Spot rate	7.67	8.27	8.81	9.31	9.75	10.16	10.52	10.85	11.15	11.42	11.67	11.89
Discount factor	0.93	0.85	0.78	0.70	0.63	0.56	0.50	0.44	0.39	0.34	0.30	0.26
Duration factor	0.86	1.58	2.14	2.56	2.86	3.05	3.14	3.17	3.13	3.04	2.93	2.79

Year	1	2	3	4	5	6	7	8	9	10	11	12
Obligation	500	900	600	500	100	100	100	50	0	0	0	0
PV term	464	768	466	350	63	56	50	22	0	0	0	0
Duration term	431	1418	1284	1282	286	305	314	158	0	0	0	0

Present value of the obligation stream is P_O = sum of present value terms = 2238.44. Quasi-modified duration of the obligation stream is D_O = sum of duration terms / present value = 2.45. We then construct similar tables for bond 1 and 2.

Table 1: Duration calculation of bond 1.

Year	1	2	3	4	5	6	7	8	9	10	11	12
Bond 1	6	6	6	6	6	6	6	6	6	6	6	106
Duration term	5.18	9.45	12.84	15.38	17.17	18.29	18.87	18.99	18.76	18.26	17.55	295.26

QM-duration is D_1 = sum of duration terms / price of the bond =7.07.

Table 2: Duration calculation of bond 2.

Year	1	2	3	4	5	6	7	8	9	10	11	12
Bond 2	10	10	10	10	110	0	0	0	0	0	0	0
Duration term	8.63	15.76	21.40	25.63	314.73	0	0	0	0	0	0	0

QM-duration is D_2 = sum of duration terms / price of the bond =3.80.

We solve equation system

$x_1 P_1 + x_2 P_2 = P_O$
$x_1 P_1 D_1 + x_2 P_2 D_2 = P_O D_O$

This can be solved (using, e.g., Excel) to find the amounts of the two bonds $\underline{x_1 = -14.0 \text{ and } x_2 = 31.1}$.

3.3 (L4.5) (Instantaneous rates) Let $s(t)$, $0 \leq t \leq \infty$ denote a spot rate curve; that is, the present value of a dollar to be received at time t is $e^{-s(t)t}$. For $t_1 < t_2$, let $f(t_1, t_2)$ be the forward rate between t_1 and t_2 implied by the given spot rate curve.
a) Find an expression for $f(t_1, t_2)$ (using $s(t)$).
b) Let $r(t) = \lim_{t_2 \to t} f(t, t_2)$. $r(t)$ is the instantaneous interest rate at time t. Show that $r(t) = s(t) + s'(t)t$.
c) If an amount x_0 is invested at $t = 0$ in a bank which pays the instantaneous rate of interest $r(t)$ at all t (compounded), the bank balance $x(t)$ will satisfy $\mathrm{d}x(t)/\,\mathrm{d}t = r(t)x(t)$. Find an expression for $x(t)$.

Solution:
a) By the definition of the forward rate:

$$e^{s(t_2)t_2} = e^{s(t_1)t_1} e^{f(t_1,t_2)(t_2-t_1)} = e^{s(t_1)t_1 + f(t_1,t_2)(t_2-t_1)}$$

$$\Leftrightarrow s(t_2)t_2 = s(t_1)t_1 + f(t_1, t_2)(t_2 - t_1) \Leftrightarrow \underline{f(t_1, t_2) = \frac{s(t_2)t_2 - s(t_1)t_1}{t_2 - t_1}}$$

b) By the definition of derivative and using the product rule (of derivatives):

$$r(t) = \lim_{t_2 \to t} f(t, t_2) = \lim_{t_2 \to t} \frac{s(t_2)t_2 - s(t)t}{t_2 - t} = \frac{\mathrm{d}}{\mathrm{d}t}[s(t)t] = s(t) \cdot 1 + s'(t) \cdot t = s(t) + s'(t)t$$

Note that for conventional term structure, $s'(t) > 0 \Rightarrow r(t) > s(t)$.

c) Part b) gives $r(t) = \mathrm{d}[s(t)t]/\,\mathrm{d}t$, and using this, the differential equation can be written as

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}[s(t)t]x(t) \Leftrightarrow \frac{1}{x(t)} \frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}[s(t)t].$$

Note that using the chain rule and derivative rule for natural logarithm, the derivative of $\ln x(t)$ is

$$\frac{\mathrm{d}\ln x(t)}{\mathrm{d}t} = \frac{x'(t)}{x(t)} = \frac{1}{x(t)} \frac{\mathrm{d}x(t)}{\mathrm{d}t}.$$

Hence, we can write the differential equation as

$$\frac{\mathrm{d}\ln(x(t))}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}[s(t)t].$$

This can be solved by integrating both sides to get

$$\ln x(t) = s(t)t + C \Leftrightarrow e^{\ln x(t)} = e^{s(t)t+C} = e^{s(t)t}e^C = De^{s(t)t} \Leftrightarrow x(t) = De^{s(t)t},$$

where C and D are constants. Using the initial condition $x(0) = x_0$ we get $D = x_0$, and hence $\underline{x(t) = x_0 e^{s(t)t}}$. Note that the solution is equal to the balance of an account that according to spot rates according to term t.

3.4 (L4.7) (Bond taxes) An investor is considering the purchase of 10-year US Treasury bonds and plans to hold them to maturity. Federal taxes on coupons must be paid during the year they are received, and tax must also he paid on the capital gain realized at maturity (defined as the difference between face value and original price). Federal bonds are exempt from state taxes. This investor's federal tax bracket rate is $t = 30\%$, as it is for most individuals. There are two bonds that meet the investor's requirements. Bond 1 is a 10-year, 10% bond with a price (in decimal form) of $P_1 = 92.21$. Bond 2 is a 10-year, 7% bond with a price of $P_2 = 75.84$. Based on the price information contained in those two bonds, the investor would like to compute the theoretical price of a hypothetical 10-year zero-coupon bond that has no coupon payments and requires tax payment only at maturity equal in amount to 30% of the realized capital gain (the face value minus the original price). This theoretical price should be such that the price of this bond and those of bonds 1 and 2 are mutually consistent on an after-tax basis. Find this theoretical price, and show that it does not depend on the tax rate t. (Assume all cash flows occur at the end of each year.)

Solution:
Let t be the tax rate,
x_i be the number of bond i bought,
c_i be the coupon of bond i, and
p_i be the price of of i.
The after tax coupon payments are $(1-t)c_i$ and the after tax final cash flows (which is only taxed by the difference between face value and the original price) are $100 - (100 - p_i)t$ (for face value 100).
To create a zero-coupon bond, we require first that the after tax coupons match. Hence

$$x_1(1-t)c_1 + x_2(1-t)c_2 = 0,$$

which reduces to

$$x_1c_1 + x_2c_2 = 0.$$

We scale the face value of the replicating portfolio to 100 through

$$100x_1 + 100x_2 = 100.$$

Next, we require that the after tax final cash flow matches. Hence

$$\begin{aligned}
x_1[100 - (100 - p_1)t] + x_2[100 - (100 - p_2)t] &= [100 - (100 - p_0)t] \\
\Leftrightarrow 100x_1 + 100x_2 - (100 - p_1)x_1t - (100 - p_2)x_2t &= 100 - (100 - p_0)t \\
\Leftrightarrow (100 - p_1)x_1t + (100 - p_2)x_2t &= (100 - p_0)t \\
\Leftrightarrow 100x_1 + 100x_2 - p_1x_1 - p_2x_2 &= 100 - p_0 \\
\Leftrightarrow p_0 &= x_1p_1 + x_2p_2,
\end{aligned}$$

where we used the identity $100x_1 + 100x_2 = 100$ twice and eliminated t. We note that the price of the replicating portfolio is a weighted sum of the prices of bonds 1 and 2, where the numbers of bonds 1 and 2 can be solved from equation system

$$x_1c_1 + x_2c_2 = 0 \text{ and } x_1 + x_2 = 1$$

to be $x_1 = -7/3 = -2.33$ and $x_2 = 10/3 = 3.33$. Hence the price of the hypothetical 10-year zero-coupon bond is $p_0 = x_1p_1 + x_2p_2 = -2.33 \cdot 92.21 + 3.33 \cdot 75.84 \approx \underline{37.6}$.

3.5 (L4.15) (Short rate sensitivity) The quasi-modified duration measures the sensitivity of a price of an asset to a parallel shift in the spot rate curve. A measure for the sensitivity of an asset's price to a parallel shift in the short rates (that is, $r_k \to r_k + \lambda$.) can also be useful. This can be solved using the running present value method. Specifically, letting P_k be the present value as seen at time k and $S_k = \mathrm{d}P_k/\,\mathrm{d}\lambda|_{\lambda=0}$, the S_k's can be found recursively by an equation of the form $S_{k-1} = -a_k P_k(\lambda = 0) + b_k S_k$, while P_k's are found by the running method. Find a_k and b_k.

Solution:
Let the present value of bond at time k be P_k and parallel in short rates be λ. Under a parallel shift in short rates, the running present value formula becomes
$P_{k-1}(\lambda) = x_{k-1} + \dfrac{P_k(\lambda)}{1 + r_{k-1} + \lambda}$
Differentiation at $\lambda = 0$ leads to

$$\frac{\mathrm{d}P_{k-1}(\lambda)}{\mathrm{d}\lambda} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(x_{k-1} + \frac{P_k(\lambda)}{1 + r_{k-1} + \lambda}\right) = 0 + \frac{1}{1 + r_{k-1} + \lambda}\frac{\mathrm{d}P_k(\lambda)}{\mathrm{d}\lambda} - \frac{1}{(1 + r_{k-1} + \lambda)^2}P_k(\lambda)$$

$$\Leftrightarrow \frac{\mathrm{d}P_{k-1}(\lambda)}{\mathrm{d}\lambda}\Big|_{\lambda=0} = \frac{1}{1 + r_{k-1}}\frac{\mathrm{d}P_k(\lambda)}{\mathrm{d}\lambda}\Big|_{\lambda=0} - \frac{1}{(1 + r_{k-1})^2}P_k(0)$$

$$\Leftrightarrow S_{k-1} = -\frac{1}{(1 + r_{k-1})^2}P_k(0) + \frac{1}{1 + r_{k-1}}S_k$$

$$\Leftrightarrow \underline{a_k = \frac{1}{(1 + r_{k-1})^2} \text{ and } b_k = \frac{1}{1 + r_{k-1}}}$$