- Suppose you purchase an asset at price X_0 and sell it 1 year later with price X_1. The total return on your investment is $R = X_1/X_0$, and the rate of return is $r = (X_1 - X_0)/X_0$. It is clear that the two notions are related by $R = 1 + r$.
- Generally speaking, being short on an asset means arranging a financial position that creates profit when the value of the asset declines. Specifically, short selling (or shorting) means selling an asset that you do not own. To do this, you borrow an asset from someone who owns it. You then sell the borrowed asset to someone else, receiving an amount X_0. Later, you repay your loan by purchasing the asset for, say, X_1 and return the asset to your lender. If $X_1 < X_0$, you will have made a profit of $X_0 - X_1$. If $X_1 > X_0$, the loss is $X_1 - X_0$. Short selling is considered quite risky, because the potential loss for shorting is unlimited.
- Let x, y and z be random variables and a and b be constants. The mean, variance and covariance of x and y have following properties:
  - $\mathbb{E}[ax + by] = a\mathbb{E}[x] + b\mathbb{E}[y]$
  - $\sigma_x^2 = \mathrm{Var}[x] = \mathbb{E}[(x - \mathbb{E}[x])^2] = \mathbb{E}[x^2] - 2\mathbb{E}[x]\mathbb{E}[x] + \mathbb{E}[x]^2 = \mathbb{E}[x^2] - \mathbb{E}[x]^2$
  - $\sigma_{xy} = \mathrm{Cov}[x, y] = \mathbb{E}[(x - \mathbb{E}[x])(y - \mathbb{E}[y])] = \mathbb{E}[(y - \mathbb{E}[y])(x - \mathbb{E}[x])] = \mathrm{Cov}[y, x] = \sigma_{yx}$
  - $\mathrm{Var}[ax + by] = a^2\mathrm{Var}[x] + b^2\mathrm{Var}[y] + 2ab\mathrm{Cov}[x, y]$
  - $\mathrm{Cov}[ax + by, z] = a\mathrm{Cov}[x, z] + b\mathrm{Cov}[y, z]$
  - $\rho = \dfrac{\sigma_{xy}}{\sigma_x \sigma_y}$ (correlation)
- Suppose that n different assets are available. We can form a portfolio of these assets so that the fraction of an asset i in the portfolio is w_i (that is, the weight of the asset i in the portfolio). For the weights w_i, we have $\sum_{i=1}^{n} w_i = 1$.
- Let R_i and r_i denote the total return and rate of return of asset i, respectively. The overall return of the portfolio is $R = \sum_{i=1}^{n} w_i R_i$ and the overall rate of return is $r = \sum_{i=1}^{n} w_i r_i$.

- The Markowitz model. Suppose we want to find the portfolio that has a rate of return $\bar{r}$ and has minimum variance $\text{Var}[r] = \text{Var}[\sum_{i=1}^{n} w_i r_i] = \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$, where σ_{ij} are the variances ($i = j$) and covariances ($i \neq j$) of the assets in the portfolio. We denote the mean rate of return of the asset i as $\bar{r}_i$ and formulate the optimization problem as

$$\begin{aligned}\min \frac{1}{2} \sum_{i,j=1}^{n} w_i w_j \sigma_{ij} \\ \text{s.t.} \sum_{i=1}^{n} w_i \bar{r}_i = \bar{r} \\ \sum_{i=1}^{n} w_i = 1.\end{aligned}$$

- The above optimization problem can be solved using Lagrange multipliers λ and μ. The Lagrangian is

$$L = \frac{1}{2} \sum_{i,j=1}^{n} w_i w_j \sigma_{ij} - \lambda \left(\sum_{i=1}^{n} w_i \bar{r}_i - \bar{r} \right) - \mu \left(\sum_{i=1}^{n} w_i - 1 \right).$$

Setting the derivatives of the Lagrangian with respect to $w_i, i = 1, \ldots, n$, λ and μ to zero gives the equations for the efficient set as

$$\begin{aligned}\frac{\partial}{\partial w_i} L = 0, \quad \forall i = 1, 2, \ldots, n \\ \frac{\partial}{\partial \lambda} L = 0 \\ \frac{\partial}{\partial \mu} L = 0 \\ \Rightarrow \sum_{j=1}^{n} \sigma_{ij} w_j - \lambda \bar{r}_i - \mu = 0, \quad \forall i = 1, 2, \ldots, n \\ \sum_{i=1}^{n} w_i \bar{r}_i = \bar{r} \\ \sum_{i=1}^{n} w_i = 1.\end{aligned}$$

There are $n + 2$ equations with $n + 2$ variables (w_i's, λ and μ). The equation system can be solved to find the weights w_i that minimize the variance of the portfolio that has overall rate of return of $\bar{r}$

5.1 (L6.3) (Two correlated assets) The correlation ρ between assets A and B is 0.1, and other data are given in Table 1 (*Note* $\rho = \sigma_{AB}/(\sigma_A \sigma_B)$).

Table 1: Two Correlated Cases

Asset	$\bar{r}$	σ
A	10%	15%
B	18%	30%

a) Find the proportions α of A and $(1-\alpha)$ of B that define the portfolio of A and B which has the minimum standard deviation.
b) What is the value of this minimum standard deviation?
c) What is the expected return of this portfolio?

Solution:

We have
$\bar{r}_A = 0.10$, $\bar{r}_B = 0.18$, $\sigma_A = 0.15$ and $\sigma_B = 0.30$.
The correlation is $\rho = 0.1$ and the covariance of the assets is
$\rho = \dfrac{\sigma_{AB}}{\sigma_A \sigma_B} \Rightarrow \sigma_{AB} = \rho\sigma_A\sigma_B.$

We construct a portfolio that has proportions α of A and $(1-\alpha)$ of B. The rate of return of this portfolio is a random variable $r = \alpha r_A + (1-\alpha) r_B$, of which mean $\bar{r}$ and variance σ^2 can be calculated as

$$\bar{r} = \alpha\bar{r}_A + (1-\alpha)\bar{r}_B$$
$$\sigma^2 = \alpha^2\sigma_A^2 + (1-\alpha)^2\sigma_B^2 + 2\alpha(1-\alpha)\sigma_{AB} = \alpha^2\sigma_A^2 + (1-\alpha)^2\sigma_B^2 + 2\alpha(1-\alpha)\rho\sigma_A\sigma_B.$$

a) We find $\arg\min_\alpha \sigma^2$ by setting the derivative of σ^2 with respect to α to zero:

$$\frac{\mathrm{d}\sigma^2}{\mathrm{d}\alpha} = 2\alpha(\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B) - 2\sigma_B^2 + 2\rho\sigma_A\sigma_B = 0 \Rightarrow \alpha = \frac{\sigma_B^2 - \rho\sigma_A\sigma_B}{\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B}$$

Substituting the values of σ_A, σ_B, and ρ yields $\alpha = 0.826$. Figure 1 presents the minimum variance set.

b) Substituting values of α, ρ, σ_A and σ_B gives
$\sigma^2 = \alpha^2\sigma_A^2 + (1-\alpha)^2\sigma_B^2 + 2\alpha(1-\alpha)\rho\sigma_A\sigma_B = 0.0194 \Rightarrow \sigma = 13.92\%$
c) Substituting values of $\alpha, \bar{r}_A$ and $\bar{r}_B$ gives
$\bar{r} = \alpha\bar{r}_A + (1-\alpha)\bar{r}_B = 11.39\%$

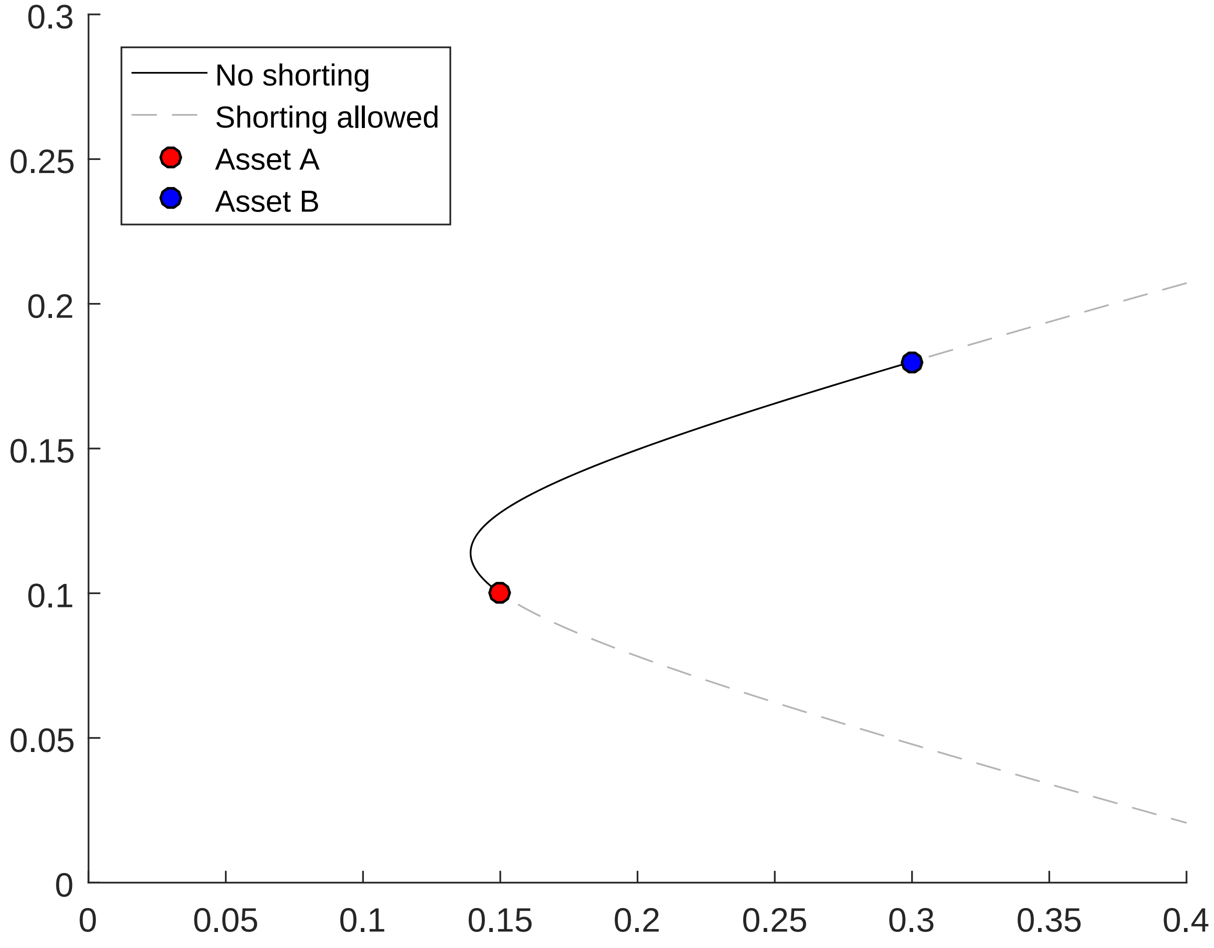

Figure 1: Mean-standard deviation curve

5.2 (L6.7) (Markowitz fun) There are just three assets with rates of return r_1, r_2 and r_3, respectively. The covariance matrix and the expected rates of return are

$$\mathbf{V} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad \bar{\mathbf{r}} = \begin{bmatrix} 0.4 \\ 0.8 \\ 0.8 \end{bmatrix}$$

a) Find the minimum-variance portfolio.
b) If the risk-free rate is $r_f = 0.2$, find the efficient portfolio of risky assets.

Solution:

a) The equations for the efficient set are

$$\sum_{j=1}^{3} \sigma_{ij} w_j - \lambda \bar{r}_i - \mu = 0, \quad \forall i = 1, 2, 3$$
$$\sum_{i=1}^{3} w_i \bar{r}_i = \bar{r} \tag{1}$$
$$\sum_{i=1}^{3} w_i = 1,$$

where $\sigma_{ij} = V_{ij}$. Minimum variance portfolio can be found by setting $\lambda = 0$, and because $\bar{r}$ is free, equation (1) can be dropped as redundant. Hence we solve

$$\sum_{j=1}^{n} \sigma_{ij} w_j - \mu = 0, \quad \forall i = 1, 2, 3$$
$$\sum_{i=1}^{n} w_i = 1$$

$$\begin{aligned} \Rightarrow 2w_1 + \ w_2 \qquad\quad -\mu &= 0 && (2) \\ w_1 + 2w_2 + \ w_3 - \mu &= 0 && (3) \\ w_2 + 2w_3 - \mu &= 0 && (4) \\ w_1 + \ w_2 + \ w_3 - 1 &= 0. && (5) \end{aligned}$$

Subtracting (5) from (3) yields $\mu = w_2 - 1$, and substituting this into equations (2)-(4) yields

$$\begin{aligned} 2w_1 \qquad\qquad +1 &= 0 \\ w_1 + w_2 + \ w_3 + 1 &= 0 \\ 2w_3 + 1 &= 0, \end{aligned}$$

and hence $w_1 = 0.5, w_2 = 0, w_3 = 0.5$.

b) The equations to solve the one fund F have been derived in the lecture. With three risky assets they are

$$\sum_{j=1}^{3} \sigma_{ij} v_j = \bar{r}_i - r_f, \quad \forall i = 1, 2, 3,$$

where v_i are unnormalized weights $v_i = \lambda w_i$. We solve this set of equations for the given covariance matrix $\mathbf{V}$ and expected rates of return $\bar{\mathbf{r}}, r_f$.

$$\begin{aligned} 2v_1 + v_2 \quad &= 0.4 - 0.2 = 0.2 && (6)\\ v_1 + 2v_2 + v_3 &= 0.8 - 0.2 = 0.6 && (7)\\ v_2 + 2v_3 &= 0.8 - 0.2 = 0.6. && (8) \end{aligned}$$

Solving v_2 and v_3 from (6) and (8) and substituting these into (7) yields

$$v_1 + 2(0.2 - 2v_1) + 0.3 - (0.2 - v_1)/2 = 0.6 \Rightarrow v_1 = 1.11 \cdot 10^{-17},$$

and substituting this into (6) and (8) yields $v_2 = 0.2, v_3 = 0.2$. We note that v_1 is practically 0, and we normalize v_2 and v_3 to find the normalized asset weights in the one fund as $\underline{w_1 = 0, w_2 = 0.5, w_3 = 0.5}$.

5.3 (L6.1) (Shorting with margin) Suppose that to short a stock you are required to deposit an amount equal to $1.5X_0$, where X_0 is the initial price of the stock. At the end of first year the stock price is X_1 and you liquidate your position. If R is the total return of the stock, what is the total return on your short?

Solution:
We seek to find the total return on the short, that is,

$$\text{total return} = \frac{\text{amount received}}{\text{amount invested}}.$$

Of the required deposit of $1.5X_0$, an amount X_0 can be covered from the proceeds of selling the stock when the contract is made. However, an amount $0.5X_0$ must be invested from other sources than selling the shorted stock, and hence

$$\text{amount invested} = 0.5X_0.$$

In a typical short-selling process, the deposit $1.5X_0$ will be made into a *margin account*. The balance of this account is then either increased (if the price of the stock declines) or decreased (if the price of the stock rises). If the position is liquidated at the end of the first year, when the stock price is X_1, then the balance of the margin account will be $1.5X_0 - X_1$, as the cash to purchase the stocks back from the market is reduced from the account. Hence the amount received is

$$\text{amount received} = 1.5X_0 - X_1.$$

Then the total return will be

$$\text{total return} = \frac{1.5X_0 - X_1}{0.5X_0} = 3 - 2\frac{X_1}{X_0} = 3 - 2R.$$

To see when shorting is more profitable purchasing the stock, we consider

$$3 - 2R > R \Leftrightarrow R < 1,$$

and hence we see that if $R < 1$, going short is is more profitable than going long (i.e., purchasing the stock).

5.4 (L6.5) (Rain insurance) Kalle Virtanen's friend is planning to invest 1 M€ in a rock concert to be held 1 year from now. The friend figures that he will obtain 3 M€ revenue from his 1 M€ investment - unless, my goodness, it rains. If it rains, he will lose his entire investment. There is a 50% chance that it will rain the day of the concert. Kalle suggests that he buys rain insurance. He can buy one unit of insurance for 0.50 €, and this unit pays 1 € if it rains and nothing if it does not. He may purchase as many units as he wishes, up to 3 M€.
a) What is the expected rate of return on his investment if he buys u units of insurance? (The cost of insurance is in addition to his 1 M€ investment.)
b) What number of units will minimize the variance of his return? What is this minimum value? And what is the corresponding expected rate of return? (*Hint*: Before calculating a general expression for variance, think about a simple answer.)

Solution:
Initial investment is $1\ 000\ 000 + 0.5u$
With a probability of 50% it rains, and the revenue is $0 + u$
With a probability of 50% it does not rain, and the revenue is $3\ 000\ 000 + 0$

a) Total return is R = revenue/investment. Hence the expected total return is

$$\bar{R} = 0.50\frac{u}{1\ 000\ 000 + 0.5u} + 0.50\frac{3\ 000\ 000}{1\ 000\ 000 + 0.5u} = \frac{0.50u + 1\ 500\ 000}{1\ 000\ 000 + 0.5u},$$

and the expected rate of return is

$$\bar{r} = \bar{R} - 1 = \frac{500\ 000}{1\ 000\ 000 + 0.5u}.$$

b) By inspection, it can be seen that buying $u = 3\ 000\ 000$ units of insurance eliminates all uncertainty regarding the return. So, $u = 3\ 000\ 000$ units of insurance results in a variance of 0 and a corresponding expected rate of return equal to

$$\bar{r} = \frac{3\ 000\ 000}{1\ 000\ 000 + 1\ 500\ 000} - 1 = 0.2.$$

5.5 (L6.6) Suppose there are n assets which are uncorrelated. You may invest in any one, or in any combination of them. The mean rate of return $\bar{r}$ is the same for each asset, but the variances are different. The return of an asset i has a variance of σ_i^2 $(i = 1, 2, \ldots, n)$.
a) Show the situation on an $\bar{r} - \sigma$ diagram. Describe the efficient set.
b) Find the minimum-variance point. Express your result in terms of

$$\bar{\sigma}^2 = \left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2} \right)^{-1}.$$

Solution:
a) The three assets are on a single horizontal line. The efficient set is a single point on the same line (shaded black in Figure 2), but to the left of the left-most of the three original points.

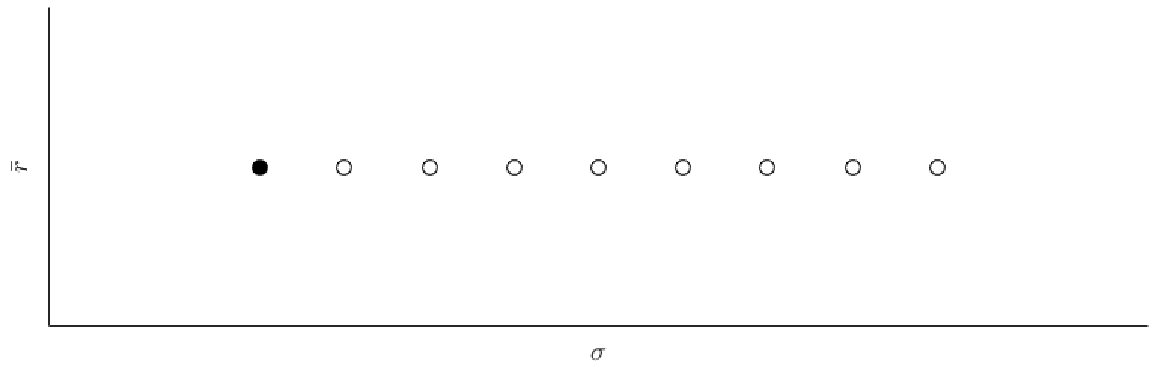

Figure 2: $\bar{r} - \sigma$ diagram of Exercise 4.

b) Let w_i be the percentage of the total investment invested in asset i. Then, because the assets are uncorrelated, we have

$$\text{Var(total investment)} = \sum_{i=1}^{n} w_i^2 \sigma_i^2,$$

where $\sum_{i=1}^{n} w_i = 1$. We set up the the Lagrangian and set its derivatives w.r.t. w_i's to zero as follows:

$$L = \sum_{i=1}^{n} w_i^2 \sigma_i^2 - \lambda \left(\sum_{i=1}^{n} w_i - 1 \right) \Rightarrow \frac{\partial L}{\partial w_i} = 2 w_i \sigma_i^2 - \lambda = 0 \Rightarrow w_i = \frac{\lambda}{2\sigma_i^2} \; \forall i = 1, \ldots, n.$$

Substituting these to the constraint $\sum_{i=1}^{n} w_i = 1$ yields

$$\sum_{i=1}^{n} \frac{\lambda}{2\sigma_i^2} = 1 \Leftrightarrow \sum_{i=1}^{n} \frac{1}{\sigma_i^2} = \frac{2}{\lambda} \Leftrightarrow \lambda = 2 \left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2} \right)^{-1}$$

Substituting λ into $w_i = \lambda/(2\sigma_i^2)$ and denoting $\bar{\sigma}^2 = \left(\sum_{i=1}^{n} 1/\sigma_i^2 \right)^{-1}$ yields

$$w_i = \frac{\bar{\sigma}^2}{\sigma_i^2}$$

5.6 (L6.8) (Tracking) Suppose that it is impractical to use all the assets that are incorporated into a specified portfolio (such as a given efficient portfolio). One alternative is to find the portfolio, made up of a given set of n stocks, that tracks the specified portfolio most closely - in the sense of minimizing the variance of the difference in returns.
Specifically, suppose that the target portfolio has (random) rate of return r_M. Suppose that there are n assets with (random) rates of return $r_1, r_2, \ldots, r_n$. We wish to find the portfolio rate of return

$$r = \alpha_1 r_1 + \alpha_2 r_2 + \ldots \alpha_n r_n$$

(with $\sum_{i=1}^n \alpha_i = 1$) minimizing $\mathrm{Var}[r - r_M]$.
a) Find a set of equations for the α_i's.
b) Although this portfolio tracks the desired portfolio most closely in terms of variance, it may not have the desired the mean. Hence a logical approach is to minimize the variance of the tracking error subject to achieving a given mean return. As the mean is varied, this results in a family of portfolios that are efficient in a new sense - say, tracking efficient. Find the equation for the α_i's that are tracking efficient.

Solution:
a) Using formula for the variance of a sum $\mathrm{Var}[ax + by] = a^2\mathrm{Var}[x] + 2ab\mathrm{Cov}[x, y] + b^2\mathrm{Var}[y]$ we write

$$\mathrm{Var}[r - r_M] = \mathrm{Var}[r] - 2\mathrm{Cov}[r, r_M] + \mathrm{Var}[r_M] = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \sigma_{ij} - 2 \sum_{i=1}^n \alpha_i \sigma_{iM} + \sigma_M^2,$$

where σ_{ij} is the covariance of stocks i and j, σ_{iM} is the covariance of stock i and the tracked portfolio, and σ_M^2 is the variance of the return of the tracked portfolio. In the last equality we used the formula for the variance of the return of a portfolio $\mathrm{Var}[r] = \mathrm{Var}\left[\sum_{i=1}^n \alpha_i r_i\right] = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \mathrm{Cov}[r_i, r_j]$ and the linearity of covariance with respect to another of the variates $\mathrm{Cov}\left[\sum_{i=1}^n a_i x, y\right] = \sum_{i=1}^n a_i \mathrm{Cov}[x, y]$.

So, to minimize $\mathrm{Var}[r - r_M]$ subject $\sum_{i=1}^n \alpha_i = 1$ set up the Lagrangian

$$\begin{aligned} L &= \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \sigma_{ij} - 2 \sum_{i=1}^n \alpha_i \sigma_{iM} + \sigma_M^2 - \lambda \left(\sum_{i=1}^n \alpha_i - 1 \right) \\ &= \sum_{i=1}^n \alpha_i^2 \sigma_i^2 + \sum_{i=1}^n \sum_{j=1, j\neq i}^n \alpha_i \alpha_j \sigma_{ij} - 2 \sum_{i=1}^n \alpha_i \sigma_{iM} + \sigma_M^2 - \lambda \left(\sum_{i=1}^n \alpha_i - 1 \right) \end{aligned}$$

Differentiation with respect to α_is and λ and setting the derivatives to zero yields

$$\frac{\partial L}{\partial \alpha_i} = 2\alpha_i \sigma_i^2 + 2 \sum_{j=1, j\neq i}^n \alpha_j \sigma_{ij} - 2\sigma_{iM} - \lambda = 0, \ \ \forall i = 1, \ldots, n$$

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^n \alpha_i - 1 = 0,$$

and we have $n + 1$ equations and $n + 1$ variables from which the α_i's can be solved.

b) Similar to a) with the added constraint $\sum_{i=1}^{n} \alpha_i \bar{r}_i = \bar{r}_M$. The Lagrangian is now

$$L = \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \alpha_i \alpha_j \sigma_{ij} - 2 \sum_{i=1}^{n} \alpha_i \sigma_{iM} + \sigma_M^2 - \lambda \left(\sum_{i=1}^{n} \alpha_i - 1 \right) - \mu \left(\sum_{i=1}^{n} \alpha_i \bar{r}_i - \bar{r}_M \right)$$

Again, we differentiate with respect to α's, λ and μ and set the derivatives to zero and get

$$\frac{\partial L}{\partial \alpha_i} = 2\alpha_i \sigma_i + 2 \sum_{j=1, j \neq i}^{n} \alpha_j \sigma_{ij} - 2\sigma_{iM} - \lambda - \mu \bar{r}_i = 0 \ \forall i = 1, \ldots, n,$$

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^{n} \alpha_i - 1 = 0,$$

$$\frac{\partial L}{\partial \mu} = \sum_{i=1}^{n} \alpha_i \bar{r}_i - \bar{r}_M = 0.$$

These $n + 2$ equations can be solved to find the tracking efficient α_i's.