- The market portfolio is the summation of all assets. An asset's weight in the market portfolio, termed capitalization weights, is equal to the proportion of that asset's total capital value to the total market capital value.

- Capital market line emanates from point $(0, r_f)$ through $(\sigma_M, \bar{r}_M)$, and shows the relation between the expected rate of return and the risk of return for any efficient portfolio. The capital market line is

$$\bar{r} = r_f + \frac{\bar{r}_M - r_f}{\sigma_M}\sigma,$$

where the slope $K = (\bar{r}_M - r_f)/\sigma_M$ is termed the price of risk.

- The capital asset pricing model (CAPM): If the market portfolio M is efficient, the expected return $\bar{r}_i$ of any asset i satisfies

$$\bar{r}_i - r_f = \beta_i(\bar{r}_M - r_f), \qquad \beta_i = \frac{\sigma_{iM}}{\sigma_M^2},$$

where σ_{iM} is the covariance of an asset i and the market portfolio.

- The relation $\bar{r}_i = r_f + \beta_i(\bar{r}_M - r_f)$ is the security market line, which shows the relation between the expected rate of return and the risk of an asset. This highlights how CAPM emphasizes that the risk of an asset is a function of its covariance with the market or, equivalently, its beta β_i.

- CAPM is a pricing model. However, to see how the above expressions are related to pricing, we must go back to the definition of return. Suppose that an asset is purchased at price P and later sold at price Q. The rate of return is then $r = (Q - P)/P$, where P is known and Q is random. Substituting this into the CAPM formula yields

$$\frac{\bar{Q} - P}{P} = r_f + \beta(\bar{r}_M - r_f)$$
$$\Leftrightarrow P = \frac{\bar{Q}}{1 + r_f + \beta(\bar{r}_M - r_f)}. \tag{1}$$

Equation (1) is termed the pricing form of the CAPM. It uses risk-adjusted discount rate $r_f + \beta(\bar{r}_M - r_f)$ when discounting the expected value of a fund $\bar{Q}$ to get the present value.

- The pricing form of CAPM is linear, that is

$$P_1 + P_2 = \frac{\bar{Q}_1 + \bar{Q}_1}{1 + r_f + \beta_{1+2}(\bar{r}_M - r_f)},$$

where β_{1+2} is the beta of a new asset, which is the sum of assets 1 and 2.

- The form of the CAPM that clearly displays linearity is called certainty equivalent form, which can be derived as follows. Suppose that we have an asset with price P and final value Q. We can then write $r = Q/P - 1$, from which follows

$$\beta = \frac{\mathrm{Cov}[r, r_M]}{\sigma_M^2} = \frac{\mathrm{Cov}[Q/P - 1, r_M]}{\sigma_M^2} = \frac{\mathrm{Cov}[Q, r_M]}{P\sigma_M^2}.$$

Substituting this into (1) yields

$$P = \frac{\bar{Q}}{1 + r_f + \frac{\mathrm{Cov}[Q, r_M]}{P\sigma_M^2}(\bar{r}_M - r_f)} \Leftrightarrow P(1 + r_f) + \frac{\mathrm{Cov}[Q, r_M]}{\sigma_M^2}(\bar{r}_M - r_f) = \bar{Q}$$

$$\Leftrightarrow P = \frac{1}{1 + r_f}\left[\bar{Q} - \frac{\mathrm{Cov}[Q, r_M]}{\sigma_M^2}(\bar{r}_M - r_f)\right] \tag{2}$$

Equation (2) is the certainty equivalent pricing formula. The term in brackets is called the certainty equivalent of Q, because it is treated as a certain amount and then the risk-free discount factor $1/(1 + r_f)$ is applied to obtain the present value P (it can be thought as a market-risk adjusted expected value). This term is linear with respect to Q, because expected value and covariance are linear. For example, we can calculate a price P_{1+2} for a combination of assets 1 and 2 as follows.

$$\begin{aligned}
P_{1+2} &= \frac{1}{1 + r_f}\left[\mathbb{E}[Q_1 + Q_2] - \frac{\mathrm{Cov}[Q_1 + Q_2, r_M]}{\sigma_M^2}(\bar{r}_M - r_f)\right] \\
&= \frac{1}{1 + r_f}\left[\mathbb{E}[Q_1] + \mathbb{E}[Q_2] - \frac{\mathrm{Cov}[Q_1, r_M] + \mathrm{Cov}[Q_2, r_M]}{\sigma_M^2}(\bar{r}_M - r_f)\right] \\
\Leftrightarrow P_{1+2} &= \frac{1}{1 + r_f}\left[\bar{Q}_1 - \frac{\mathrm{Cov}[Q_1, r_M]}{\sigma_M^2}(\bar{r}_M - r_f)\right] + \frac{1}{1 + r_f}\left[\bar{Q}_2 - \frac{\mathrm{Cov}[Q_2, r_M]}{\sigma_M^2}(\bar{r}_M - r_f)\right] = P_1 + P_2
\end{aligned}$$

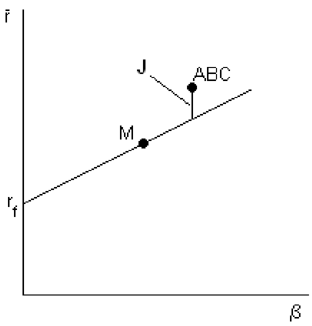
- The CAPM theory can be used to evaluate the performance of investment portfolios in terms of historical data. The expected rates of returns and covariances with the market can be estimated using standard formulas.

- The Jensen index J measures how much the performance of an individual asset has deviated from the security market line, that is,

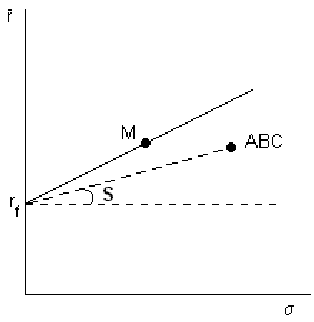
$$\hat{\bar{r}} - r_f = J + \beta(\hat{\bar{r}}_M - r_f).$$

- The Sharpe ratio S is the slope of the line drawn between the risk-free point $(0, r_f)$ and the point $(\hat{\sigma}, \hat{\bar{r}})$ on the $\bar{r} - \sigma$ diagram. Hence it measures the efficiency of a portfolio by examining where it falls relative to the capital market line. It is defined from the formula

$$\hat{\bar{r}} - r_f = S\hat{\sigma}.$$



(a) Jensen index.



(b) Sharpe ratio.

6.1 (L7.1) (Capital market line) Assume that thet expected rate of return on the market portfolio is 23% and the rate of return on T-bills (the risk-free rate) is 7%. The standard deviation of the market is 32%. Assume that the market portfolio is efficient.
a) What is the equation of the capital market line?
b) (i) If an expected return of 39% is desired, what is the standard deviation of this position? (ii) If you have 1 000 € to invest, how should you allocate it to achieve the above position?
c) If you invest 300 € in the risk-free asset and 700 € in the market portfolio, how much money should you expect to have at the end of the year?

Solution:

$$\bar{r}_M = 23\% \qquad \sigma_M = 32\% \qquad r_f = 7\% \qquad \bar{r} = 39\%$$

a) Equation of the capital market line:

$$\bar{r} = r_f + \frac{\bar{r}_M - r_f}{\sigma_M}\sigma = 0.07 + 0.5\sigma$$

b) We solve the standard deviation from the equation of the capital market line:

$$\sigma = \sigma_M \frac{\bar{r} - r_f}{\bar{r}_M - r_f} = 0.32\frac{0.39 - 0.07}{0.23 - 0.07} \approx 0.64 = 64\%$$

The expected rate of return of a portfolio is $\bar{r} = \sum_{i=1}^{n} w_i \bar{r}_i$, where $\sum_{i=1}^{n} w_i = 1$. We have

$$\bar{r} = w r_f + (1-w)\bar{r}_M = w \cdot 0.07 + (1-w) \cdot 0.23 = 0.39 \Rightarrow w = -1$$

Hence, borrow 1000 € at the risk-free rate; invest 2000 € in the market.

c)

$$\bar{r} = 0.3 \cdot 0.07 + 0.7 \cdot 0.23 = 0.182 \Rightarrow 1.182 \cdot 1000 = 1182$$

The expected amount of money at the end of the year is 1182 €.

6.2 (L7.5) (Uncorrelated assets) Suppose there are n mutually uncorrelated assets. The return on asset i has variance σ_i^2. The expected rates of return are unspecified at this point. The total amount of asset i in the market is X_i. We let $T = \sum_{i=1}^n X_i$ and then set $x_i = X_i/T$, for $i = 1, 2, \ldots, n$. Hence the market portfolio in normalized form is $\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n)$. Assume there is a risk-free asset with rate of return r_f. Find an expression for $\beta_i = \mathrm{Cov}[r_i, r_M]/\mathrm{Var}[r_M]$ in terms of the x_i's and σ_i's.

Solution:
The weights of the uncorrelated assets in the market portfolio are x_i. Hence the market portfolio's return and expected return are

$$r_M = \sum_{i=1}^n x_i r_i \qquad \bar{r}_M = \sum_{i=1}^n x_i \bar{r}_i.$$

The covariances of the uncorrelated assets and the market portfolio are

$$\mathrm{Cov}[r_i, r_M] = \mathrm{Cov}[r_i, \sum_{j=1}^n x_j r_j] = \sum_{j=1}^n x_j \mathrm{Cov}[r_i, r_j],$$

where the last equality follows from the linearity of the covariance with respect to one variate. We then note that because the assets are uncorrelated, we have

$$\mathrm{Cov}[r_i, r_j] = 0, \quad i \neq j,$$

and also note that

$$\mathrm{Cov}[r_i, r_i] = \mathrm{Var}[r_i] = \sigma_i^2.$$

Hence we have

$$\mathrm{Cov}[r_i, r_M] = \sum_{j=1}^n x_j \mathrm{Cov}[r_i, r_j] = x_i \mathrm{Cov}[r_i, r_i] = x_i \sigma_i^2.$$

Similarly, we can calculate the variance of the market portfolios return as follows.

$$\mathrm{Var}[r_M] = \mathrm{Var}\left[\sum_{i=1}^n x_i r_i\right] = \sum_{i=1}^n \sum_{j=1}^n x_i x_j \mathrm{Cov}[r_i, r_j] = \sum_{i=1}^n \sum_{\substack{i=1 \\ j \neq i}}^n x_i x_j \mathrm{Cov}[r_i, r_j] + \sum_{i=1}^n x_i^2 \mathrm{Cov}[r_i, r_i] = \sum_{i=1}^n x_i^2 \sigma_i^2,$$

where in the second equality we used the general formula for the variance of a sum. Substituting these into the formula of β_i yields

$$\beta_i = \frac{\mathrm{Cov}[r_i, r_M]}{\mathrm{Var}[r_M]} = \frac{x_i \sigma_i^2}{\sum_{j=1}^n x_j^2 \sigma_j^2}.$$

6.3 (L7.7) (Zero-beta assets) Let $\mathbf{w_0}$ be the portfolio (weights) of risky assets corresponding the minimum-variance point in the feasible region. Let $\mathbf{w_1}$ be any other portfolio on the efficient frontier. Define r_0, r_1, σ_0^2 and σ_1^2 to be the corresponding returns and variances of the returns.
a) There is a formula of the form $\sigma_{01} = A\sigma_0^2$. Find A. (*Hint*: Consider portfolios $\mathbf{p} = (1-\alpha)\mathbf{w_0} + \alpha\mathbf{w_1}$, and consider small variations of the variance of such portofolios near $\alpha = 0$. Note that $\mathrm{dVar}[r_\mathbf{p}]/\,\mathrm{d}\alpha\,|_{\alpha=0} = 0$, because $\mathbf{w_0}$ is the minimum variance point.)
b) Corresponding to the portfolio $\mathbf{w_1}$ there is a portfolio $\mathbf{w_z}$ on the minimum-variance set that has zero beta with respect to $\mathbf{w_1}$; that is, $\sigma_{1z} = 0$. This portfolio can be expressed as $\mathbf{w}_z = (1-\alpha)\mathbf{w_0} + \alpha\mathbf{w_1}$. Find the proper value of α.
c) Show the relation of the three portfolios on a diagram that includes the feasible region.
d) If there is no risk-free asset, it can be shown that other assets can be priced according to the formula

$$\bar{r}_i - \bar{r}_z = \beta_{iM}(\bar{r}_M - \bar{r}_z),$$

where the subscript M denotes the market portfolio and $\bar{r}_z$ is the expected rate of return of the portfolio that has zero beta with the market portfolio. Suppose that the expected returns on the market and the zero-beta portfolio are 15% and 9%, respectively. Suppose that stock i has a correlation with the market of 0.5. Assume also that the standard deviation of the returns of the market and stock i are 15% and 5%, respectively. Find the expected return of stock i.

Solution:

a) Now $\mathbf{p} = (1-\alpha)\mathbf{w_0} + \alpha\mathbf{w_1}$ and, using the formula for the variance of a sum $\mathrm{Var}[\sum_i a_i x_i] = \sum_i \sum_j a_i a_j \sigma_{ij}$,

$$\begin{aligned}\mathrm{Var}[r_\mathbf{p}] &= \mathrm{Var}[(1-\alpha)r_0 + \alpha r_1] \\ &= (1-\alpha)^2\sigma_0^2 + \alpha^2\sigma_1^2 + 2\alpha(1-\alpha)\sigma_{01} \\ \Rightarrow \frac{\mathrm{dVar}[r_\mathbf{p}]}{\mathrm{d}\alpha} &= -2(1-\alpha)\sigma_0^2 + 2\alpha\sigma_1^2 + 2(1-\alpha)\sigma_{01} - 2\alpha\sigma_{01}\end{aligned}$$

Because w_0 is the minimum-variance point,

$$\left.\frac{\mathrm{dVar}[r_\mathbf{p}]}{\mathrm{d}\alpha}\right|_{\alpha=0} = 0 \Leftrightarrow -2\sigma_0^2 + 2\sigma_{01} = 0 \Leftrightarrow \sigma_{01} = A\sigma_0^2, \text{ where } A = 1.$$

Note that we did not use the fact that $\mathbf{w}_1$ was assumed efficient, and hence the equation $\sigma_{01} = \sigma_0^2$ is valid for any feasible portfolio. Substituting this into the variance formula of $\mathbf{p}$ yields

$$\begin{aligned}\mathrm{Var}[r_\mathbf{p}] &= (1-\alpha)^2\sigma_0^2 + \alpha^2\sigma_1^2 + 2\alpha(1-\alpha)\sigma_0^2 = (1-\alpha)(1-\alpha+2\alpha)\sigma_0^2 + \alpha^2\sigma_1^2 \\ &= (1-\alpha)(1+\alpha)\sigma_0^2 + \alpha^2\sigma_1^2 = (1-\alpha^2)\sigma_0^2 + \alpha^2\sigma_1^2 = \sigma_0^2 + \alpha^2(\sigma_1^2 - \sigma_0^2).\end{aligned}$$

Because $\sigma_1^2 > \sigma_0^2$, the last term is strictly positive, and as a result the variance of this portfolio is strictly larger than the variance of the minimum variance portfolio. Hence no (risky) asset can be added to the minimum variance portfolio without increasing its variance.

b) The beta of portfolio $\mathbf{w_z}$ with respect to $\mathbf{w_1}$ is zero, and consequently the covariance σ_{z1} is zero because

$$\beta = \frac{\sigma_{z1}}{\sigma_1^2} = 0 \Rightarrow \sigma_{z1} = 0.$$

On the other hand the covariance of $\mathbf{w_z} = (1-\alpha)\mathbf{w_0} + \alpha\mathbf{w_1}$ with $\mathbf{w}_1$ (using $\text{Cov}[\mathbf{w_0}, \mathbf{w_1}] = \sigma_{01} = \sigma_0^2$) is

$$\text{Cov}[\mathbf{w_z}, \mathbf{w_1}] = \text{Cov}[(1-\alpha)\mathbf{w_0} + \alpha\mathbf{w_1}, \mathbf{w_1}] = \text{Cov}[(1-\alpha)\mathbf{w_0}, \mathbf{w_1}] + \text{Cov}[\alpha\mathbf{w_1}, \mathbf{w_1}] = (1-\alpha)\sigma_0^2 + \alpha\sigma_1^2$$

Because this is zero, we have

$$(1-\alpha)\sigma_0^2 + \alpha\sigma_1^2 = 0 \Leftrightarrow \alpha(\sigma_1^2 - \sigma_0^2) = -\sigma_0^2 \Leftrightarrow \alpha = \frac{\sigma_0^2}{\sigma_0^2 - \sigma_1^2} < 0,$$

because $\mathbf{w_0}$ is the minimum-variance point, that is, $\sigma_0^2 - \sigma_1^2 < 0$.

c) $\mathbf{w_0}$ is the minimum-variance point and $\mathbf{w_1}$ is in the efficient frontier, located above $\mathbf{w_0}$ in the minimum-variance set. Because $\mathbf{w}_0$ and $\mathbf{w}_1$ are efficient, the Two-fund theorem guarantees that $\mathbf{w}_z$ is in the minimum-variance set, and because $\alpha < 0$, $\mathbf{w_z}$ is below $\mathbf{w_0}$ in the minimum variance set.

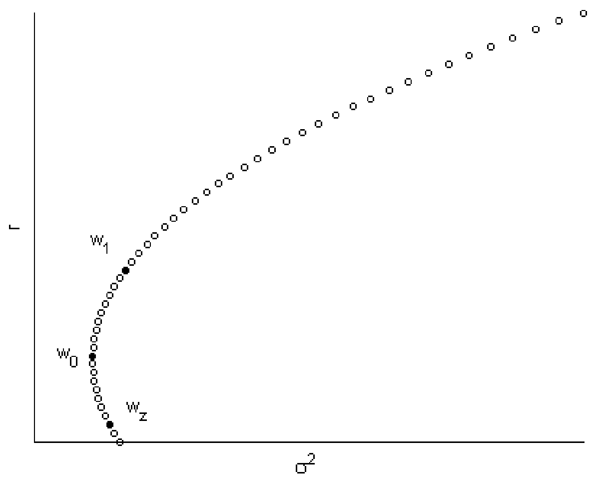


Figure 2: $\bar{r} - \sigma$ curve of Exercise 3c.

d) We have

$$\bar{r}_M = 0.15 \qquad \sigma_M = 0.15 \qquad \bar{r}_z = 0.09 \qquad \sigma_i = 0.05 \qquad \rho_{iM} = 0.5$$

Correlation ρ can be written using standard deviations and covariance as

$$\rho_{iM} = \frac{\sigma_{iM}}{\sigma_i \sigma_M} \Rightarrow \sigma_{iM} = \rho_{iM}\sigma_i\sigma_M.$$

Using the pricing formula $\bar{r}_i - \bar{r}_z = \beta_{iM}(\bar{r}_M - \bar{r}_z)$ we have

$$\bar{r}_i = \bar{r}_z + \beta_{iM}(\bar{r}_M - \bar{r}_z) = \bar{r}_z + \frac{\sigma_{iM}}{\sigma_M^2}(\bar{r}_M - \bar{r}_z) = \bar{r}_z + \frac{\rho_{iM}\sigma_i}{\sigma_M}(\bar{r}_M - \bar{r}_z)$$

Substituting numeral values yields the expected return of asset i as $\bar{r}_i = 0.10 = 10\%$.

6.4 (L7.9) Show that for a fund with return $r = (1-\alpha)r_f + \alpha r_M$, both CAPM pricing formulas (pricing form of the CAPM and certainty equivalent pricing formula) give the price of 100 € worth of fund assets as 100 €.

Solution:
Let P be the value of fund assets at the time of the purchase (known) and Q be their value after a year (a random variable). Using equation $r = (1-\alpha)r_f + \alpha r_M$ for the return of the fund we can write

$$Q = P(1+r) = P(1 + (1-\alpha)r_f + \alpha r_M)$$

$$\bar{Q} = P(1+\bar{r}) = P(1 + (1-\alpha)r_f + \alpha \bar{r}_M)$$

The beta of the fund is

$$\beta = \frac{\text{Cov}[r, r_M]}{\text{Var}[r_M]} = \frac{\text{Cov}[(1-\alpha)r_f + \alpha r_M, r_M]}{\text{Var}[r_M]} = \frac{(1-\alpha)\text{Cov}[r_f, r_M] + \alpha \text{Cov}[r_M, r_M]}{\text{Var}[r_M]} = \frac{0 + \alpha\sigma_M^2}{\sigma_M^2} = \alpha$$

Using the pricing form of the CAPM, the price of the fund is

$$P = \frac{\bar{Q}}{1 + r_f + \beta(\bar{r}_M - r_f)} = \frac{\bar{Q}}{1 + r_f + \alpha(\bar{r}_M - r_f)} = \frac{P(1 + (1-\alpha)r_f + \alpha\bar{r}_M)}{1 + (1-\alpha)r_f + \alpha\bar{r}_M} = P$$

That is, the price of the fund is the discounted expected value of the fund, as it should be.
Certainty equivalent pricing formula is

$$P = \frac{1}{1+r_f}\left[\bar{Q} - \frac{\text{Cov}[Q, r_M](\bar{r}_M - r_f)}{\sigma_M^2}\right],$$

where the covariance of Q and the market portfolio is

$$\text{Cov}[Q, r_M] = \text{Cov}[P(1+(1-\alpha)r_f+\alpha r_M), r_M] = P\text{Cov}[1, r_M] + (1-\alpha)P\text{Cov}[r_f, r_M] + \alpha P\text{Cov}[r_M, r_M] = P\alpha\sigma_M^2.$$

We have

$$P = \frac{1}{1+r_f}\left[\bar{Q} - \frac{\text{Cov}[Q, r_M](\bar{r}_M - r_f)}{\sigma_M^2}\right] = \frac{P}{1+r_f}\left[1 + (1-\alpha)r_f + \alpha\bar{r}_M - \alpha(\bar{r}_M - r_f)\right] = \frac{P}{1+r_f}\left[1 + r_f\right] = P.$$

The above equation states that the market risk adjusted expected value of Q is $(1+r_f)P$, and discounting this with the risk-free discount rate is the same as the price of the fund. Thus, the price of the fund is again equal to the discounted future value, as it should be.

Note that the value of the fund is P regardless of the relative amounts of capital allocated in the risk-free and risky assets.