Math Camp - Final Exam

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The exam is 2 hours and has a total of 120 points. It is closed book, and calculators are prohibited. Please answer as many questions as you can. Answer shortly but justify your answers and explain accurately what you are doing. If you are confused about some question statement, please explain clearly what you assume when answering. Point totals reflect the difficulty of the problem and give a rough estimate for how long the question should take.

1. Short questions:

   (a) *(5 points)* True or false, every set that is not open is closed. If true, provide a proof, if false, a counterexample.

   Solution. *The set* $X = (0, 1]$ *is not open or closed.* 0 *is a limit point of this set not contained in this set, and for every* $\varepsilon > 0$, $B_\varepsilon(1)$ *is not contained in* X.

   (b) *(5 points)* Show that any strictly concave function being maximized over a convex set has at most one maximizer.

   Solution. *Suppose not. Then there would be at least two maximizers* x, x'. *But then* $f(\lambda x + (1-\lambda)x') > \lambda f(x) + (1-\lambda) f(x') = f(x)$, *for any* $\lambda \in (0, 1)$, *which contradicts* x *being a maximizer.*

   (c) *(5 points)* What is the expected value of a random variable with pdf $f(x) = \frac{4}{\pi}\frac{1}{1+x^2}$ on $[0, 1]$.

   Solution.

$$\begin{aligned}\int_0^1 \frac{4}{\pi}\frac{x}{1+x^2}\,dx &= \int_1^2 \frac{2}{\pi}\frac{1}{u}\,du \\ &= \frac{2}{\pi}\ln 2\end{aligned}$$

   (d) *(5 points)* Suppose X and Y are independent and both have pdfs e^{-x} on $[0, \infty)$. What is the joint distribution of $X + Y$ and Y.

   Solution. *Let* $U = X + Y$ *and* $V = Y$. *Then* $X = U - V$ *and* $Y = V$ *so the Jacobian is*

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

   The pdf is then $f_{u,v}(u, v) = e^{-u}$ *on* $0 < V < U < \infty$.

2. Consider the following constrained maximization problem:

$$\max_{x,y\in\mathbb{R}^2_+} \sqrt{x}+\sqrt{y}$$
$$\text{s.t. } x^2+y^2\leq 9$$
$$\frac{x^2}{4}+\frac{y^2}{9}\geq 1$$

(a) *(5 points)* For any non-zero x, y that satisfy the constraints, show that the Hessian matrix of the objective function is negative semi-definite.

Solution. *The Hessian is*

$$\begin{pmatrix} -\frac{1}{4}x^{-3/2} & 0 \\ 0 & -\frac{1}{4}y^{-3/2} \end{pmatrix}$$

so negative definite on $\mathbb{R}^2_+$.

(b) *(15 points)* What are the KKT conditions for this problem? (feel free to disregard non-negativity constraints)

Solution. *Ignoring non-negativity, we get*

$$\begin{aligned} \frac{1}{2\sqrt{x}} &= 2\lambda x - \frac{1}{2}\mu x \\ \frac{1}{2\sqrt{y}} &= 2\lambda y - \frac{2}{9}\mu y \\ \lambda(x^2+y^2-9) &= 0 \\ \mu(1-\frac{x^2}{4}-\frac{y^2}{9}) &= 0 \\ \lambda \geq 0,\ \mu &\geq 0 \end{aligned}$$

and the constraints.

(c) *(15 points)* Are there any points that satisfy the KKT conditions and have $x^2/4+y^2/9>1$ and $x, y>0$?

Solution. *Complementary slackness then implies that* $\mu=0$. *So the FOCs sim-*

plifies to

$$\frac{1}{2\sqrt{x}} = 2\lambda x$$
$$\frac{1}{2\sqrt{y}} = 2\lambda y$$

and thus $x = y$. *The mutliplier* λ *must be positive, so* $x^2 + y^2 = 9$. *So* $x = y = 3/\sqrt{2}$ *solves this, with multiplier* $\lambda = 1/(4(3/\sqrt{2})^{3/2})$, *and all other multipliers are* 0.

(d) *(15 points)* Consider the alternative problem

$$\max_{x,y \in \mathbb{R}^2_+} \sqrt{x} + \sqrt{y}$$
$$\text{s.t. } x^2 + y^2 \leq 9.$$

Are the KKT conditions necessary and sufficient here? What does this tell you about the point you found in c?

Solution. *The objective is strictly concave, the constraints are convex (and thus quasiconvex). There is a point that lies strictly on the interior of the feasible set (e.g.* $(1, 1)$*) and the objective has non-zero gradient everywhere. Therefore the KKT conditions are necessary and sufficient. The point we found in c also satisfies the KKT conditions for this problem, and thus is a maximum not only of this problem but also of the original problem (which had a non-convex constraint set).*

3. Consider the consumer problem

$$\max_{x,y \in \mathbb{R}^2_+} u(x, y)$$
$$\text{s.t. } p_1 x + p_2 y \leq m$$

$u(x, y)$ is twice continuously differentiable, strictly concave, and strictly increasing (i.e. if $(x, y) \geq (x', y')$ then $u(x, y) \geq u(x', y')$, strictly so if $(x, y) \neq (x', y')$).

(a) *(15 points)* What are the KKT conditions for this problem.

Solution.

$$\frac{\partial u}{\partial x} = \lambda p_1 - \mu_1$$
$$\frac{\partial u}{\partial y} = \lambda p_2 - \mu_2$$
$$\lambda(p_1 x + p_2 y - m) = 0$$
$$\mu_1 x = 0, \mu_2 y = 0$$
$$\lambda, \mu_1, \mu_2 \geq 0$$

and the constraints.

(b) *(5 points)* Argue that $p_1 x + p_2 y \leq m$ holds with equality at any maximum.

Solution. *$u(x, y)$ is strictly increasing, so if the constraint was slack, we could always increase either x or y and increase the objective. Alternatively, KKT are nec + suff, so if this constraint was slack, then $\lambda = 0$, which would imply the multipliers on the non-negativity constraint are negative, violating the KKT conditions (this argument is not quite complete, as a strictly increasing function can still have 0 derivative at points, but can be made precise).*

(c) *(10 points)* Let $v(p, m)$ be the value function for this problem (the utility achieved at the maximum) and $x(p, m)$ and $y(p, m)$ be the demands (the arg maxes) and $\lambda(p, m)$ be the corresponding multiplier. If non-negativity constraints do not bind at the maximum and $x(p, m)$ is differentiable, show that $\frac{\partial v}{\partial p_1} = -\lambda(p, m)x(p, m)$.

Solution. *By the envelope theorem*

$$\frac{\partial v}{\partial p_1} = \frac{\partial}{\partial p_1}[u(x, y) - \lambda(p_1 x + p_2 y - m)]$$

which immediately gives the desired equation.

(d) *(20 points)* Assume $(x^*, y^*, \lambda^*) = (1, 1, 1)$ satisfies the Lagrange multiplier conditions, is the maximum at $(p_1, p_2, m) = (1, 1, 2)$ and

$$D^2 u(1, 1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Solve for $\partial x/\partial m, \partial y/\partial m$ and $\partial \lambda/\partial m$ at $(p_1, p_2, m) = (1, 1, 2)$ (hint: what three equations must the maximum satisfy).

Solution. *We have three conditions we know any maximum must satisfy:*

$$\frac{\partial u}{\partial x} - \lambda p_1 = 0$$
$$\frac{\partial u}{\partial y} - \lambda p_2 = 0$$
$$p_1 x + p_2 y - m = 0$$

Call these $f(x, y, \lambda, p_1, p_2, m)$. *We can apply the implicit function theorem if* $D_{x,y,\lambda} f$ *is invertible. This is given by*

$$D_{x,y,\lambda} f = \begin{pmatrix} u_{xx} & u_{xy} & -p_1 \\ u_{xy} & u_{yy} & -p_2 \\ p_1 & p_2 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

which is invertible. Then

$$D_m(x, y, \lambda) = -(D_{x,y,\lambda} f)^{-1} D_m f$$

which gives us the system of equations

$$-\frac{\partial x}{\partial m} - \frac{\partial \lambda}{\partial m} = 0$$
$$-\frac{\partial y}{\partial m} - \frac{\partial \lambda}{\partial m} = 0$$
$$\frac{\partial x}{\partial m} + \frac{\partial y}{\partial m} = 1$$

So $\partial x / \partial m = \partial y / \partial m = 1/2$ *and* $\partial \lambda / \partial m = -1/2$.