Math Camp - Final Exam

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The exam is 4 hours and has a total of 120 points. Please answer as many questions as you can. Answer shortly but justify your answers and explain accurately what you are doing. If you are confused about some question statement, please explain clearly what you assume when answering. Point totals reflect the difficulty of the problem and give a rough estimate for how long the question should take.

1. Consider the following maximization problem

$$\max_{x \in \mathbb{R}^6_+} \sum_{i=1}^{6} -x_i \log(6x_i)$$
$$\text{s.t. } x_1 \geq 2x_2$$
$$\sum_{i=1}^{6} x_i \geq 1$$

   (a) *(10 points)* Are the KKT conditions necessary for a maximum? Sufficient? Why?

   (b) *(10 points)* Write down the KKT conditions, ignoring the non-negativity constraints (i.e. feel free to assume that all non-negativity constraints have 0 multipliers).

   (c) *(10 points)* Show that the KKT conditions imply that for all $i, j \notin \{1, 2\}$, $x_i = x_j$.

   (d) *(15 points)* Argue using the KKT conditions that $x_1 = 2x_2$ at a maximum.

Solution. *(a) The constraints are linear, so they are convex. The objective function $f(x) = \sum -x_i \log 6x_i$ has second derivatives $\frac{\partial^2 f}{\partial x_i^2} = -\frac{1}{x_i}$, and $\frac{\partial^2 f}{\partial x_j \partial x_i} = 0$ for all $i \neq j$, so it is strictly concave.*

For necessity, in addition to the convex constraints and concave objective, we need there to be a feasible point where all the constraints are slack, for instance $(100, 2, 2, 2, 2, 2)$.

For sufficiency, in addition to the (quasi)convex constraints and (quasi)concave objective, we need $\nabla f(x) \neq 0$ at every feasible point. $\frac{\partial f}{\partial x_i} = -\log 6x_i - 1$, so $\nabla f(x) = 0$ only at the vector that is $(6e)^{-1}$ in each component, which is not feasible. So the KKT conditions are sufficient.

(b) *In addition to the constraints, a maximum must satisfy*

$$\begin{aligned}
&-\log 6x_1 - 1 = -\lambda - \mu \\
&-\log 6x_2 - 1 = 2\lambda - \mu \\
&-\log 6x_i - 1 = -\mu \,\forall i > 2 \\
&\lambda(2x_2 - x_1) = 0 \\
&\mu\left(\sum_{i=1}^{6} x_i - 1\right) = 0 \\
&\lambda \geq 0, \mu \geq 0
\end{aligned}$$

(c) *This follows immediately from the KKT conditions.*

(d) *Combining KKT conditions, we get*

$$\log 6x_1 - \log 6x_2 = 3\lambda$$

Since $x_1 \geq 2x_2$*, the left hand side of this is always strictly positive, so by complementary slackness* $2x_2 = x_1$*. (Note that this implies that* $3\lambda = \log \frac{x_1}{x_2} = \log 2$*, from here solving this problem is pretty straightforward.)*

2. A firm is selling two goods, (x, y), to a consumer. To entice the consumer to purchase more, the firm offers a discount on the price of each good based on the amount of the other good the consumer purchases. The consumer solves the following problem

$$\begin{aligned}
&\max_{x,y \geq 0} \sqrt{x} + \sqrt{y} \\
&\text{s.t. } p(y)x + q(x)y \leq m
\end{aligned}$$

where $p : \mathbb{R}_+ \to \mathbb{R}_+$ and $q : \mathbb{R}_+ \to \mathbb{R}_+$ are decreasing, twice continuously differentiable functions.

(a) *(10 points)* Show that if $p(y), q(x) \geq K$ for some $K > 0$ for all $x, y \geq 0$, then a maximum exists.

(b) *(15 points)* What condition must the first and second derivatives of $p(y)$ and $q(x)$ satisfy to make $p(y)x + q(x)y - m$ a convex function? If this condition holds, how do you know the maximum is unique?

(c) *(15 points)* Suppose that $p(x) = q(x) = 1 + e^{-\alpha x}$ for some $\alpha > 0$, so the problem becomes

$$V(\alpha, m) = \max_{x,y \geq 0} \sqrt{x} + \sqrt{y}$$
$$\text{s.t. } x\left(1 + e^{-\alpha y}\right) + y\left(1 + e^{-\alpha x}\right) \leq m$$

and let $x(\alpha, m)$ and $y(\alpha, m)$ be the corresponding arg maxes. Using the envelope theorem, express $\frac{\partial V}{\partial \alpha} / \frac{\partial V}{\partial m}$ in terms of $x(\alpha, m)$ and $y(\alpha, m)$.

Solution. *(a) Note that* $p(y)x + q(x)y \geq K(x+y)$*, so* x *and* y *are each bounded above by* m/K*. Thus, since they are also assumed to be bounded below by* 0*, the feasible set is bounded.*

(b) To check convexity, we can just look at the Hessian, which is

$$\begin{pmatrix} yq''(x) & p'(y) + q'(x) \\ p'(y) + q'(x) & xp''(y) \end{pmatrix}$$

This is positive semidefinite if $q''(x)$ *and* $p''(y)$ *are both positive, and* $xyp''(y)q''(x) - (p'(y) + q'(x))^2 \geq 0$*. If this holds then the feasible set is convex and the objective is strictly concave, so the maximum is unique. Really all we need is quasiconvexity of the constraints for this.*

(c) Assuming we can apply the envelope theorem,

$$\frac{\partial V}{\partial \alpha} = \lambda y x e^{-\alpha y} + \lambda x y e^{-\alpha x}$$

and

$$\frac{\partial V}{\partial m} = \lambda$$

so

$$\frac{\frac{\partial V}{\partial \alpha}}{\frac{\partial V}{\partial m}} = xy(e^{-\alpha x} + e^{-\alpha y}).$$

3. Suppose X and Y are distributed

$$f_{x,y} = \begin{cases} \frac{4x^2}{y^2}\left(1 - x\left(\frac{1-y}{y}\right)\right) & \text{if } x \in (0,1), y \in (x/(x+1), 1) \\ 0 \text{ o.w.} \end{cases}$$

Consider the random variables U, V, with $U = X$, $V = X\left(\frac{1-Y}{Y}\right)$.

(a) *(10 points)* Show that (U, V) has support $(0,1)^2$.

(b) *(10 points)* What is the joint pdf of U, V.

(c) *(15 points)* Are U and V independent?

Solution. *(a) Note that* $\frac{x(1-y)}{y}$ *is continuous and monotone in* y*, and plugging in the endpoints of* y *we get* $\frac{x(1-\frac{x}{x+1})}{\frac{x}{x+1}} = 1$ *and* $\frac{x(0)}{1} = 0$*, so this ranges from* 0 *to* 1 *for any* x*.*

(b) Note that $X = U$ *and* $Y = \frac{U}{U+V}$*, so the derivative matrix of our transformation is*

$$\begin{pmatrix} 1 & 0 \\ \frac{V}{(U+V)^2} & \frac{-U}{(U+V)^2} \end{pmatrix}$$

So the transformed density is

$$\frac{4U^2}{\frac{U^2}{(U+V)^2}}(1-V)\frac{U}{(U+V)^2} = 4U(1-V)$$

(c) The marginals are

$$f_U = 2U$$

and

$$f_V = 2(1-V)$$

so $f_{U,V} = f_U f_V$*, these are independent.*