MS-E2122 - Nonlinear Optimization
Lecture I

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Outline of this lecture

Context

- Mathematical programming and optimisation
- Types of mathematical optimisation models

Example of Applications

- Resource allocation
- The pooling problem: refinery operations planning
- Robust optimisation
- Combinatorial optimisation

Outline of this lecture

Definition

NonLinear (or Non-Linear):

- *adjective*
- not arranged in a straight line.
- not sequential or straightforward.

Optimization (or Optimisation):

- *noun*
- the action of making the best or **most effective** use of a **situation** or **resource**.

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Field of applied mathematics. The goal is to search values for variables in a given domain that maximise/minimise function values.

Can be achieved by:

- Analysing/Visualizing properties of functions / extreme points or

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Optimisation has important applications in fields such as

- **operations research (OR)**;
- economics;
- statistics;
- bioinformatics;
- machine learning and artificial intelligence.

What is optimisation?

In this course, optimisation is viewed as the core element of mathematical programming.

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However, math. programming has many applications in fields other than OR, which causes some confusion;

We will study math. programming in its most general form: both constraints and objectives are nonlinear functions.

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- $h_i(x) = 0$ for $i = 1, \ldots, l$ - equality constraints.

Types of programming

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- **Mixed-integer nonlinear programming (MINLP):** MIP+NLP.

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Resource allocation and portfolio optimisation

Problem statement. Plan production that maximises return. Let

- $I = \{1, \ldots, i, \ldots, M\}$ resources;
- $J = \{1, \ldots, j, \ldots, N\}$ products;
- c_j - return per unit of product $j \in J$;
- a_{ij} - resource $i \in I$ requirement for making product $j \in J$;
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- notice that max. $f(x) =$ min. $-f(x)$;

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Remark:

- notice that max. $f(x) =$ min. $-f(x)$;
- the base of most practical optimisation problems; exploits mature LP technology.

Portfolio optimization

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Remarks:

- The term $x^\top \Sigma x$ measures exposition to risk. It is credited to Harry Markowitz (1952).
- Another important class: quadratic programming (nonlinear).

Refinery Operations Planning Problem

Oil refinery operational planning

- Goal is to maximize profit;
- Several possible configurations;
- Product property specifications must be met;

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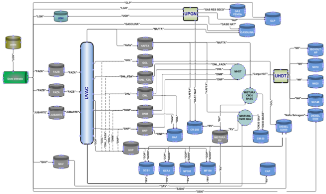
Model characteristics:

- Bilinear (nonconvex) and mixed-integer;
- Large number of flows;
- Several nonlinear constraints.

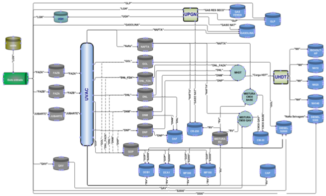


Refinery Operations Planning Problem

Objective: maximize profit

Variables:

- Stream Flows (crude, intermediate and final products);
- Storage;
- Stream properties.



Refinery Operations Planning Problem

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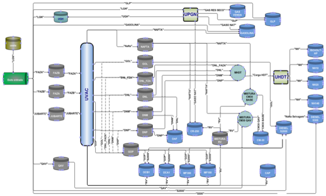
Constraints

- Mass balance;
- Market features (supply and demand);
- Unit capacities;
- Stream property limits;
- Calculation of mix properties (nonlinear).

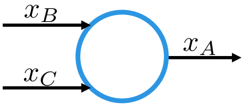


Refinery Operations Planning Problem

The challenging aspect is how to model the calculation of product properties in a mix. Let:

- x_p be the volume of product $p \in P$ and
- q_p the value of a given chemical property (sulphur content, octane content, viscosity...).

In a given mix, mass and property balances are calculated as:



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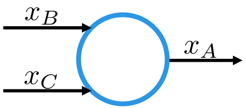

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Remarks:

- More complex mixes (such as nonlinear balances) might need to be considered.
- These are bilinear programming problems (nonlinear).

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Is a subarea of mathematical programming concerned with uncertainty in the input data.

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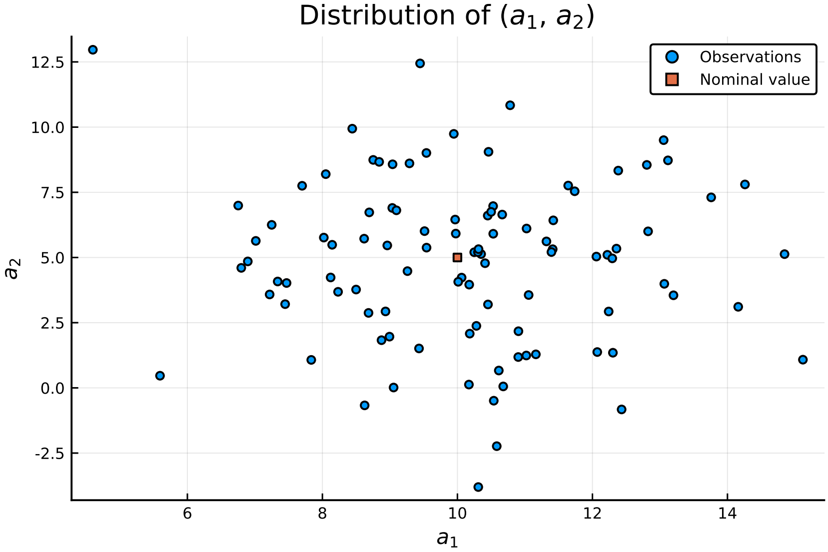
Consider the resource allocation problem under uncertainty:

$$\begin{aligned} \text{max.} \quad & c^\top x \\ \text{subject to:} \quad & \tilde{a}_i^\top x \leq b_i, \forall i \in I \\ & x_j \geq 0, \forall j \in J, \end{aligned}$$

where $\tilde{a}_i$ is a random variable.

Robust optimisation



Robust optimisation

Assume that, for any $i \in I$, $\tilde{a}_i \in \epsilon_i = \{\overline{a}_i + P_i u : ||u||_2 \leq \Gamma_i\}$, where

- $\overline{a}_i$ is the nominal (average) value;
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Then, the robust counterpart can be stated as

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Notice that

$$\max_{a_i \in \epsilon_i} \left\{ a_i^\top x \right\} = \overline{a}_i^\top x + \max_{u} \left\{ u^\top P_i x : ||u||_2 \leq \Gamma_i \right\} = \overline{a}_i^\top x + \Gamma_i ||P_i x||_2$$

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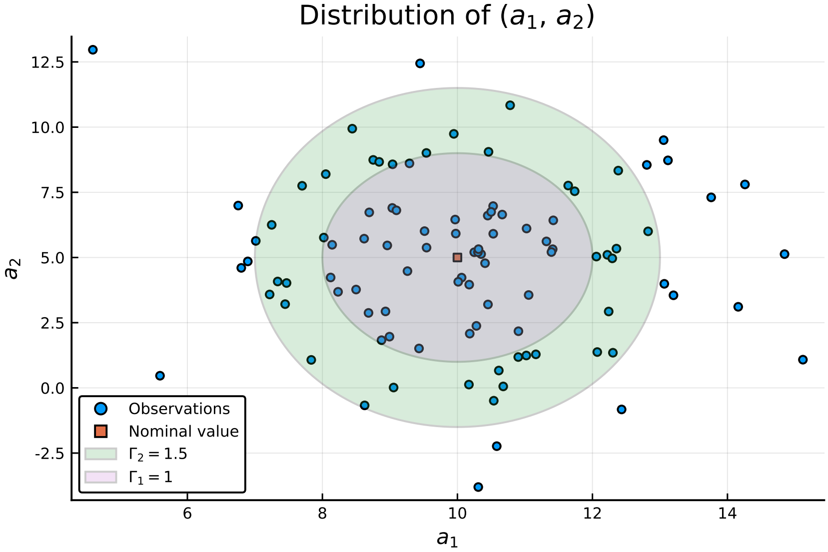
- In case data is available, P_i can be obtained from the empirical covariance matrix;
- Values of Γ_i can be drawn, for example, from a Chi-squared distribution. Γ_i is sometimes called the budget of uncertainty.

Robust optimisation



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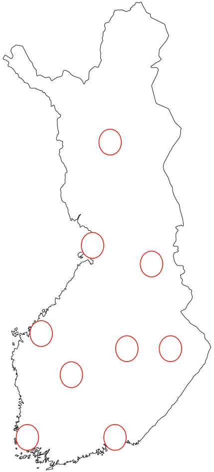
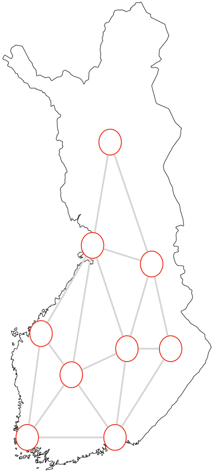
Combinatorial optimisation

One of most classic problems is Travelling Salesman Problem (TSP)





Combinatorial optimisation

Assume a number of cities labeled with numbers $1, \cdots, n$, and the distance (or any cost) between two cities i and j is defined as c_{ij}.

The main variable is x_{ij} such that:

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x_{ij} is binary variable $\leftarrow$ integer programming

Combinatorial optimisation

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The objective function is:

$$\min \sum_{i=1}^{n} \sum_{j \neq i, j=1}^{n} c_{ij} x_{ij} \tag{1}$$

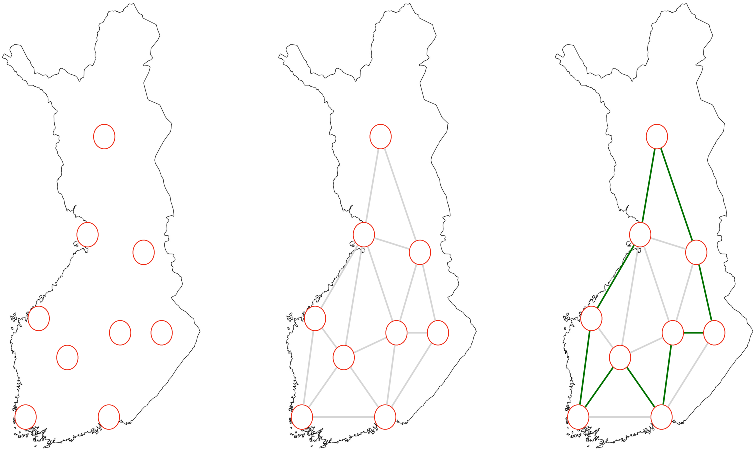
Combinatorial optimisation

The constraints can be written as:

$$\sum_{i \neq j, i=1}^{n} x_{ij} = 1 \qquad \forall j = 1, \cdots, n \qquad (2)$$

$$\sum_{j \neq i, j=1}^{n} x_{ij} = 1 \qquad \forall j = 1, \cdots, n \qquad (3)$$

Combinatorial optimisation




THANK YOU

Reference

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