Imagine a general problem with f(x) as objective function and a set of inequality and equality constraints. Its general form could be written as:

$$\min f(x)$$
$$s.t.:$$
$$g_i(x) \geq 0, \forall i \in \{1, \dots, m\}$$
$$h_j(x) = 0, \forall j \in \{1, \dots, n\}$$
$$x \in \mathbb{R}^n$$

Now imagine a general, **linear problem** (key word here is **linear**):

$$\max 3\, x_1 + 4x_2$$
$$s.t.:$$
$$\frac{1}{2} x_1 + 2x_2 \leq 30$$
$$3x_1 + x_2 \leq 25$$
$$x_1, x_2 \geq 0$$

How can we come up with the dual formulation for this problem? Let us start with the following constraint:

$$\frac{1}{2} x_1 + 2x_2 \leq 30$$

By multiplying both sides by 6, we have:

$$3x_1 + 12x_2 \leq 180$$

The coefficient for x_1 in this constraint is the same as the coefficient for x_1 in the objective function in the primal (both are equal to 3). Also, the coefficient for x_2 in the constraints in larger than the coefficient for x_2 in the objective function (12 > 4). Also, considering that both x_1 and x_2 is larger than zero (last line of the primal problem), we can assume that:

$$3x_1 + 4x_2 \leq 3x_1 + 12x_2 \leq 180$$

We also can conclude that, for this constraint, the objective function of the primal will always be less or equal than 180.

Now for the second constraint, same analogy.

$$3x_1 + x_2 \leq 25$$

Take the constrain and multiply by 4 (as an example).

$$12x_1 + 4x_2 \leq 100$$

The coefficient for x_1 is the new constraint above are also larger than the coefficient for x_1 in the objective function (12 >3). For x_2, same case (4 >= 4).

Once again, we can assume that the objective function cannot exceed 100. Key point here is the objective function cannot exceed 100 given this particular constraint.

The primal problem is a maximization problem. Based on the two constraints that I looked into above, my objective function has to below 180 and 100. Which means, it has to be below 100 (an upper bound or upper limit)

If I keep repeating this, I will find lower and lower values. Which means that I would be basically **minimizing** an alternative model.

So let us approach the problem differently, because we find the minimum value allowed would correspond to finding the maximum value in my primal problem.

Next step, let us try a linear combination of the constraint. We gonna multiply the first constraint by 2 and add to the second constraint:

$$2\left(\frac{1}{2}x_1 + 2x_2\right) + 1(3x_1 + x_2) \leq 2 \cdot 30 + 25$$

Resulting in:

$$4x_1 + 5x_2 \leq 85$$

Once again, the coefficient of x_1 in this constraint above is larger than the coefficient of x_1 in the objective function of the primal (4 >3) and also for the x_2 (5 > 4). Then, the following is also true:

$$3x_1 + 4x_2 \leq 4x_1 + 5x_2 \leq 85$$

Also, we found a better upper bound for our objective function in the primal, which is 85.

By this point, the question should be: what are the values that I should multiply my constraints in a way that I found the best upper bound possible. In other words, what are the values that I should multiply the constraints in order that the combined coefficients of those new constraints is greater or equal than the coefficients in my primal objective function.

Let’s try this. Let y1 and y2 the some values that I’m multiplying both constraint such that:

$$y_1\left(\frac{1}{2}x_1 + 2x_2\right) + y_2(3x_1 + x_2) \leq 30y_1 + 25y_2 \quad \text{(condition I)}$$

Because I want the coefficients for x_1 to be greater than 3 and the coefficients for x_2 to be equal to x_2, I can rewrite the condition above as:

$$\frac{1}{2}y_1 + 3y_2 \geq 3 \text{ For x\_1}$$

$$2y_1 + y_2 \geq 4 \text{ For x\_2}$$

We also want to minimize the right-side of the condition $30y_1 + 25y_2$

In this case, we just build “accidently” a new problem:

$$\begin{gathered} \min \quad 30y_1 + 25y_2 \\ s.t.\ \frac{1}{2}y_1 + 3y_2 \geq 3 \\ 2y_1 + y_2 \geq 4 \\ y_1, y_2 \geq 0 \end{gathered}$$

Pay close attention to the coefficients of this new problem. The values that were in the coefficients of the objective function (in the primal) are now bound for the new constraint. If we write the coefficients for the constraint in the primal in a matrix form: A = [½ 2; 3 1]. Now in the new problem above, those values are transposed. Finally, the bounds for the constraints in the primal are now part of the objective function in the new problem.

This new problem, we call it the dual problem for a linear problem.

So, generalizing this we have, that for a primal problem:

$$\begin{gathered} \min \quad c^T x \\ s.t.\ Ax = b \\ x \geq 0 \end{gathered}$$

Dual problem:

$$\begin{gathered} \max \quad b^T y \\ s.t.\ A^T y = c \\ y \geq 0 \end{gathered}$$

If we consider the as part of this analysis (note that in the previous case, on the non-domain constraints were involved), the same conclusion can be observed and the following representation of dual and primal can be certified.

$$(P):\ \min \quad c^T x$$

$s.t.\ Ax = b :\ v$ -> one type of dual variable for each type of constraint

$x \geq 0 :\ u$ -> one type of dual variable for each type of constraint

Results in:

$$(D):\ \max \quad b^T v$$
$$s.t.\ A^T v + u = c$$
$$u \geq 0,\ v \in \mathbb{R}^m$$

<u>**This is only valid for linear problems.**</u>

Another conclusion is that we can get the KKT conditions for those in a very simple way:

Feasibility on the primal:

$$Ax = b,\ x \geq 0$$

Feasibility on the dual:

$$A^T v + u = c,\ u \geq 0,\ v \in \mathbb{R}^m$$

Complementary slackness (only inequality constraints):

$$u^T x = 0$$

Please be aware that this is KKT for linear problem (do not confuse that with KKT for nonlinear problems).

Moving to the questions regarding the Problem 4.2 in the final homework assignment. There are two problems P and D, primal and dual. Assuming that you understood how to get one from the other without the extra element on the objective. Meaning without:

$\frac{1}{2}x^TQx$ -> is a quadratic term

We could repeat the same procedure done for linear problems and get that particular dual, but the simplest way to do it is. From the primal:

$$
\begin{aligned}
(P):\ \min \quad & c^Tx + \frac{1}{2}x^TQx \\
s.t.\ Ax &= b \\
x &\geq 0
\end{aligned}
$$

Write Lagragian dual function for this:

$$L(x, u, v) = c^Tx + \frac{1}{2}x^TQx + v^T(Ax - b) + ux$$

It can be rewritten as:

$$L(x, v, u) = b^Tv - \frac{1}{2}x^TQx + x^T(Qx + c - A^Tv - u) \qquad (*)$$

Now, we can minimize L(x,v,u) with respect to x by setting the gradient of L with respect to x equal to 0, which yields:

$$Qx + c - A^Tv - u = 0$$

Which already corresponds to the constraint in the dual. But we keep going. Find the value of x for the equation above:

$$x = Q^{-1}(u - c + A^T v)$$

We can replace that term into my original Lagrangian function and simplify it a little bit after some heavy algebra. This would reveal function based on:

$$b^T v - \frac{1}{2} x^T Q x$$

However, the easier way to see is that in (*), the term in parentheses corresponds to what would have happened to an equality constraint if it was incorporated into a Lagrangian function:

$$h(x) = Qx + c - A^T v - u = 0 \longrightarrow A^T v + u - Qx = c$$

Therefore, the remaining part would be an objective function:

$$b^T v - \frac{1}{2} x^T Q x$$

Finally, a third way to prove that this is correct would be to generate the lagrangian from the dual and from the primal. They are required to be exact. And they're, given some simplification.

There is a bit of confusion regarding this formulation because x appears as a variable in both primal and dual. This is due to some solvers exploit of having x as a variable in the dual. We don't explore much of that in this course, but it helps solving quadratic programming problems.

Keep in mind that everything mentioned in this file regards linear problems (first half) and quadratic problems (second half).

For the Lagrangian of the QP problem, you write by combining objective function and constraints (both equality and inequality). So the first part should be:

$$L(x,u,v) = c^T x + \frac{1}{2} x^T Q x$$

Now it depends on how you are expressing the equality and inequality constraints. For equality, it could be:

$$Ax = b \rightarrow b - Ax = 0 \; or \; Ax - b = 0$$

Also, the inequality could be:

$$-ux = 0 \; or \; ux = 0$$

For <u>inequality</u>, let use the following: $-ux = 0$

For equality, let's try both. First:

$$Ax - b = 0$$

Then, Lagrangian equal to:

$$L(x,u,v) = c^T x + \frac{1}{2} x^T Q x + v^T (Ax - b) - ux = 0 \rightarrow$$
$$L(x,u,v) = -bv^T - \frac{1}{2} x^T Q x + x^T (Qx + c + A^T v - u)$$

Now, with:

$$b - Ax = 0$$

Leading to:

$$L(x,u,v) = c^T x + \frac{1}{2} x^T Q x + v^T (b - Ax) - ux = 0 \rightarrow$$
$$L(x,u,v) = bv^T - \frac{1}{2} x^T Q x + x^T (Qx + c - A^T v - u)$$

Both are correct, as long they're consistent. What I think it's confusing it that I go between those two interchangeably in the pdf. Perhaps, that wasn't clear, but both are valid.