MSE2114 - Investment Science
Lecturer Notes I

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Contents

Conventions

$\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.
$\mathbb{N}$ denotes the set $\{1, 2, 3, \ldots\}$ of natural numbers (excluding 0).
Inner products are taken to be linear in the first argument and conjugate linear in the second.
The Einstein summation convention is used for tensors unless otherwise specified.

1 Investment science

Let us start with some definitions:

Definition 1.1

Investment = **Commitment** of current **resources** with the aim of **receiving later benefits**

Some common examples can be found in daily life, such as preparing for a course or exam to successfully learn from it and earn the credits (as in this course, for example) or training every day for a marathon to beat your personal best. Other practical examples such as:

- Construct a new factory for profitable growth in the future
- Purchase shares in company X to receive dividends and to profit from possibly increasing share price

In the same context, we can define **investment science** as the scientific **principles**, **theory** and **methods** which assist investors in making **investment** decisions.

Behind investments, there are people or organizations that are responsible for making decisions related to it. Such personnel, known as **investors**, might act in their own behaviour or sometimes in someone's interests (as fund managers.)

In investments, another keyword is an asset, which refers to any **resource** owned or controlled by a **business** or a legal economic **entity**. A special case of assets is **financial asset**, which corresponds to a non-physical asset whose value is derived from a **contractual** claim. There is no physical sense of ownership but a legal bind contract between resources and owner. Some examples such as:

- E.g., bank deposits, bonds, companies' share capital
- Economic value of financial assets is measured in <u>monetary terms</u> (=money)
- Money is a generally accepted medium of exchange, a measure of value, or a means of payment

Finally, financial investments regard purchasing assets which can either generate further income or appreciate in value (so that it can be sold at a profit later on). Obviously, there is always some risk associated with it (more about that in the future).

Remark: This course is focused on financial investments

2 Investments types

There are several types of investments. Those can range from more straightforward and easily accessible bank procedures to becoming a shareholder in a big company. Some examples:

1. **Bank deposit**
   - Deposit 80 000 € for one year at the 0.25% interest rate
2. Saving for **future pension**
   - Pay an extra 200 € per month for 15 years
   - Get an extra 250 € in addition to your monthly pension starting at the age of 65
3. **Stock investment**
   - Buy shares of a stock at the current price 2000€
   - Sell once the stock has (hopefully) appreciated to 2400€
   - Make a profit of 400 € per share

In those examples, they can categorized into different categories regarding several factors. For instance, regarding how many assets are involved, there are single or multiple assets: buying a single stock or a portfolio of many stocks (more details about portfolio optimization in the future). Also, either actively manage mutual funds or passive index funds.

Another category is regarding the volatility of such investments, where they can have a fixed interest rate (more on that later) (deterministic value) or investments with uncertain future value (stochastic behaviour - random value).

Some investments regard exchange-traded assets, a common behaviour (an intrinsic part of the stock market - liquid market). Some unique scenarios can also occur, such as when rare works of art are involved (illiquid market).

The time frame for each investment is also important. It can be categorized in a single period (where the decision to invest is decided in the moment), multiperiod (investments can be made at the moment with the option of further investments in fixed time points) or continuous (investments are made continuously).

When it comes to markets, the comparison principle has an important role.

Definition 2.1

The **attractiveness** of an investment is determined by **comparing** it with other comparable investments

Arbitrage, which can be defined as:

Definition 2.2

the simultaneous buying and selling of securities, currency, or commodities in different markets or in derivative forms in order to take advantage of differing prices for the same asset.

is normally eliminated from competitive markets(e.g., M. Friedman: *There's No Such Thing as a Free Lunch*, Open Court Publ. Company, 1975). In this case, it prevents deliberate exploitation for someone's benefit.

The price of assets evolves dynamically according to a (stochastic) process, which leads to possible fluctuation in profit values and, therefore, creates an associated risk. Different investors react differently: private persons tend to be more risk-averse than institutional investors.

In the stock market, the leading trade is done via assets typically free of transaction costs and other taxes. As in other markets, demand and supply control the prices, especially the bid and ask price. The former refers to the highest price that a buyer is willing to pay for an asset, while the latter regards the lowest price for which a seller is willing to sell an asset. In a scenario of **no arbitrage** implies bid price $\leq$ ask price. As mentioned before, with the comparison principle, iff two assets have the same cash flows, then they must have the same price.

Some important investment problems that we might cover in this course are:

1. **Pricing**
   - What is the **right** price of a financial asset in light of all available information, based on the given assumptions?
2. **Hedging**
   - If the investor is exposed to financial risks, how should they invest to reduce these risks?
   - Insurance is one approach to hedging against risks
3. **Portfolio optimization**
   - Which portfolio of available (financial) investment opportunities best matches the investor's risk-return preferences?

3 Cash flow model of investments

Any investments yield a **cash flow stream**, which is the money that streams in and out of an investment, and it is a crucial indicator of overall financial health. It is important to remember that cash flow differs from profit.

Let $\mathbf{x} = (x_0, x_1, x_2, \ldots, x_n)$ be the cash flow stream:

- x_i = cash flow at time $t_i, t_i < t_j$ for all $i < j$
- Negative cash flows $x_i < 0$ are *outflows*
  - , E.g., expenses due to buying investment equipment
- Positive cash flows $x_i > 0$ are *inflows*
  - E.g., revenues from selling assets

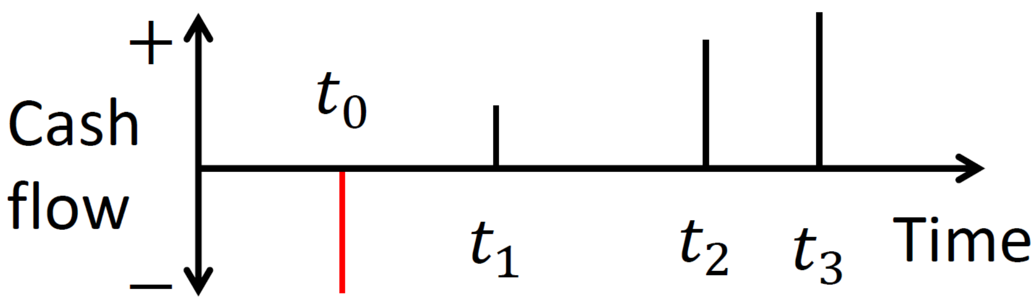

Figure 1: Cash flow through time

4 Time value of money

In many cases when it comes to **money**, the basic principle is:

Definition 4.1

The sooner cash is available, the better

However, there are certain conditions to allow such behaviour. For instance, if the interest rate is positive, you can invest your cash now and get it back, plus the accrued interest. Opportunities might come or go very abruptly, and an attractive, unexpected investment opportunity might present itself.

Let us consider the following alternatives:

A. Receive $A = 100$€ now

B. Receive $B = (100 + x)$€ after one year

The main challenge is:
The sum x for which your prefer these alternatives equally represents the time value of money

How can we guarantee that x will be in our favour? Volatility and uncertainty are vital factors to be considered here, especially in the form of inflation, marginal utility (preferences depend on one's current level of wealth - how much someone is willing to try/risk) and risk aversion. This makes it very challenging to measure the time value of money.

Further reading: Frederick, S., Loewenstein, G., & O'Donoghue, T. (2002). Time discounting and time preference: A critical review. *Journal of Economic Literature*, 351-401.

5 Interest

Interest can be defined as the monetary charge for the privilege of borrowing money. Interest expense or revenue is often expressed as a dollar amount, while the interest rate used to calculate interest is typically expressed as an annual percentage rate (APR). Interest is the amount of money a lender or financial institution receives for lending out money. Interest can also refer to the amount of ownership a stockholder has in a company, usually expressed as a percentage.

Corollary 5.1. Investor lends a **principal** A_0 (in €) for a single period, then at the end the investor receives $A_1 = A_0 + x$ (in €)

Interest x can compensate, for example, for:

- lost time value of money to the investor
- credit risk of the borrower
- administrative expenses of lending
- the profit margin required by the investor

The measure of interest is done by the interest rate, which is:

$$r = \frac{x}{A_0} \tag{1}$$

In finance, there are two types of interest calculation. Let A_n be the value of the loan in period $n = 0, 1, \dots$:

- **Simple interest** is paid only on the <u>principal</u> (**Linear** growth):
$$A_n = A_0 + rnA_0 = (1 + rn)A_0$$
- **Compound interest** pays <u>interest on interest</u> as well (**Geometric** growth):
$$A_n = (1 + r)A_{n-1} = (1 + r)^2 A_{n-2} = \cdots$$
$$\Rightarrow A_n = (1 + r)^n A_0$$
  - The term comes from the fact that interest is *compounded* (=added to the principal) at the end of <u>each</u> period
  - If the interest is added to the principal of the loan rather than paid out to the investor, the interest is said to be *paid in kind* (PIK)

Rule of thumb: The value of the investment doubles in

- about 10 periods when the interest rate is $r = 7\%$
- about 7 periods when the interest rate is $r = 10\%$

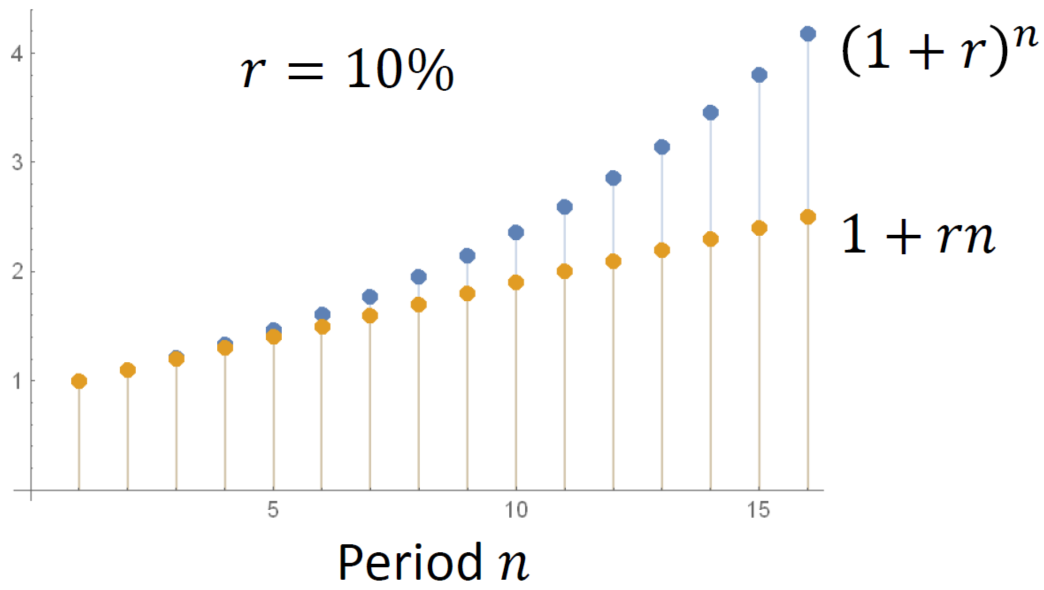

Figure 2: Linear vs Geometric growth of different interests

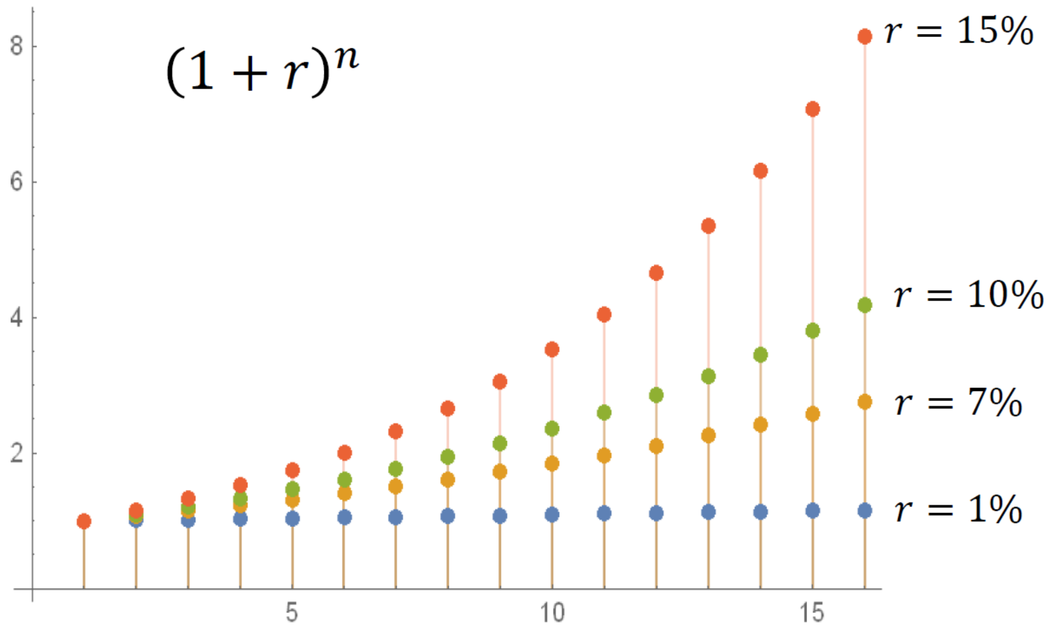

Figure 3: Comparison between different rates in geometric growth

Question: If a client are borrowing money from a bank, which type of interest is the most beneficial for the bank? And for the client?

In finance, interest rates are usually **annualized**. However, they can be expressed in different time measures such as r' = interest rate for a single period of length $t = k/m,\ m, k \in \mathbb{N}$.

- In k years, the interest rate is r' compounded $m = k/t$ times
- In k years, the one-year interest rate r is compounded k times

The annual interest rate r which yields the same return as r' after k years is thus:

$$(1 + r')^m = (1 + r)^k$$

The rate $r > 0$ which solves this equation is the **annualised rate** of r', computed as the principal k-th root

$$r = (1 + r')^{m/k} - 1$$

.

Considering $k = 1$, **Effective rate** gives the equivalent interest with once-per-year compounding:

$$1 + r_e = (1 + r/m)^m \Leftrightarrow r_e = (1 + r/m)^m - 1$$

When $m \to \infty$ in the previous expression, we have the continuous compounding with the following effective rate:

$$r_e = \lim_{m\to\infty} (1 + r/m)^m - 1$$
$$\Rightarrow r_e = e^r - 1$$

Similar to different interest rates, the growth of value also behaves differently with different interest types. For simple interest, the calculation is straightforward:

$$A_k = A_0 + (A_0 \frac{k}{m} k)$$

For compounding interest, the calculation is as follows:

$$A_k = \left(1 + \frac{r}{m}\right)^k A_0$$

after k periods, where r is the nominal rate and A_0 is the principal value. At time $t = k/m$, the value is $(1+r/m)^{mt} A_0$, hence:

$$A_t = \lim_{m\to\infty} \left(1 + \frac{r}{m}\right)^{mt} A_0 = e^{rt} A_0$$

6 Inflation

The interest rate can be misleading due to various factors. One of the most common elements is inflation, which is the reduction of purchasing power due to higher prices. Similarly to interest, it is calculated yearly as **inflation rate** f. Therefore, the actual interest rate (real interest rate r_0) must account for inflation.

The following equation calculates r_0:

$$1 + r_0 = \frac{1 + r}{1 + f}$$
$$\Rightarrow r_0 = \frac{r - f}{1 + f}\%$$

7 Present value and future value

Consider a scenario where an investor considers the investment and can deposit and borrow money at a risk-free interest rate. This scenario creates the concept of **ideal bank**. Although unrealistic, it is a convenient theoretical assumption that greatly simplifies the analysis. In the same scenario, we can define the future value **Future value** which is the amount of cash that the investment is worth to the investor at the chosen future time (normally the time of the last cash flow of the investment).

Starting with the future value, it can be calculated as:

$$FV = (1 + r)A_0$$

which denotes the value of the cash flow after accounting interest. At the same time, the current value of a future sum of money or stream of cash flows given a specified rate of return. It can be calculated:

$$PV = \frac{1}{1 + r} A_1 = dA_1$$

Consider the cash flow stream $\mathbf{x} = (x_1, \ldots, x_n)$ with cash flows $x_1, x_2, \ldots, x_n$ at periods $1, 2, \ldots, n$. The present value of a cash flow stream $\mathbf{x}$ is:

$$PV = x_0 + \frac{x_1}{1 + r} + \cdots + \frac{x_n}{(1 + r)^n}$$
$$= x_0 + dx_1 + \cdots + d^n x_n$$

where,

- r = interest rate of a single period = discount rate
- d = discount factor of a single period

Finally, the **net present value** (NPV) accounts for both positive and negative cash flows.

Similarly, the future value of a cash flow stream $\mathbf{x}$ at period n is:

$$FV = (1+r)^n x_0 + (1+r)^{n-1} x_1 + \cdots + x_n$$

With this information in mind, the following theorem:

Theorem 7.1. Main theorem on present value

The cash flow streams $\mathbf{x} = (x_0, x_1, \ldots, x_n)$ and $\mathbf{y} = (y_0, y_1, \ldots, y_n)$ are equivalent for an investor who has access to an ideal bank employing a constant interest rate r if and only if the present values of the two streams, evaluated at this interest rate, are equal.

Proof. The cash flow $\mathbf{x}$ is equivalent to $(PV_{\mathbf{x}}, 0, \ldots, 0)$ and $\mathbf{y}$ is equivalent to $(PV_{\mathbf{y}}, 0, \ldots, 0)$. These are equal only if $PV_{\mathbf{x}} = PV_{\mathbf{y}}$ (otherwise, there would be an arbitrage opportunity). 😎

8 Internal rate of return

In a special case, when the present value of x is zero, the interest rate r becomes the internal rate of return (IRR). To find r the following equation should be solved:

$$0 = x_0 + \frac{x_1}{1+r} + \frac{x_2}{(1+r)^2} + \cdots + \frac{x_n}{(1+r)^n}$$

Equivalently, IRR is a number r satisfying $1/(1+r) = c$, where c satisfies the polynomial equation:

$$0 = x_0 + x_1 c + x_2 c^2 + \cdots x_n c^n$$

Theorem 8.1. Main theorem of internal rate of return

ssume that the cash flow stream $(x_0, x_1, \ldots, x_n)$ is such that $x_0 < 0$ and $x_k \geq 0, k = 1, 2, \ldots, n$ with a strict inequality for some $k \geq 1$. Then the equation

$$0 = x_0 + x_1 c + x_2 c^2 + \cdots + x_n c^n$$

has a unique positive solution $c^* > 0$. If $\sum_{k=0}^{n} x_k > 0$, then the corresponding internal rate of return $r^* = (1/c^*) - 1$ is positive.

Proof: Let $f(c) = x_0 + x_1 c + \cdots + x_n c^n$. By assumption $f(0) = x_0 < 0$. Since $x_k \geq 0 \; \forall \; k \geq 1$ with a strict inequality for some $k \geq 1$, it follows that $f'(c) > 0, c > 0$ and $f(c) > 0$ for some large enough c. Thus, $f(c) = 0$ has a unique solution $c^* > 0$. If $\sum_{k=0}^{n} x_k > 0$, then $f(1) > 0$ and thus $c^* = 1/(1+r^*) < 1 \Rightarrow r^* > 0$ 😎

Comparison between NPV and IRR:

	Selection	Pros	Cons
NPV	Higher NPV	Easy to compute and aims to maximize value	Discount rate must be defined and insensitive to investment size
IRR	Higher IRR	Ranks by rate of return	Assumes that all cash flows can be reinvested at the same IRR. IRR is defined by non-linear function.