MSE2114 - Investment Science Lecturer Notes X

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Abstract

In this lecture, we will keep covering options price, extending binomial lattice for pricing options.

Contents

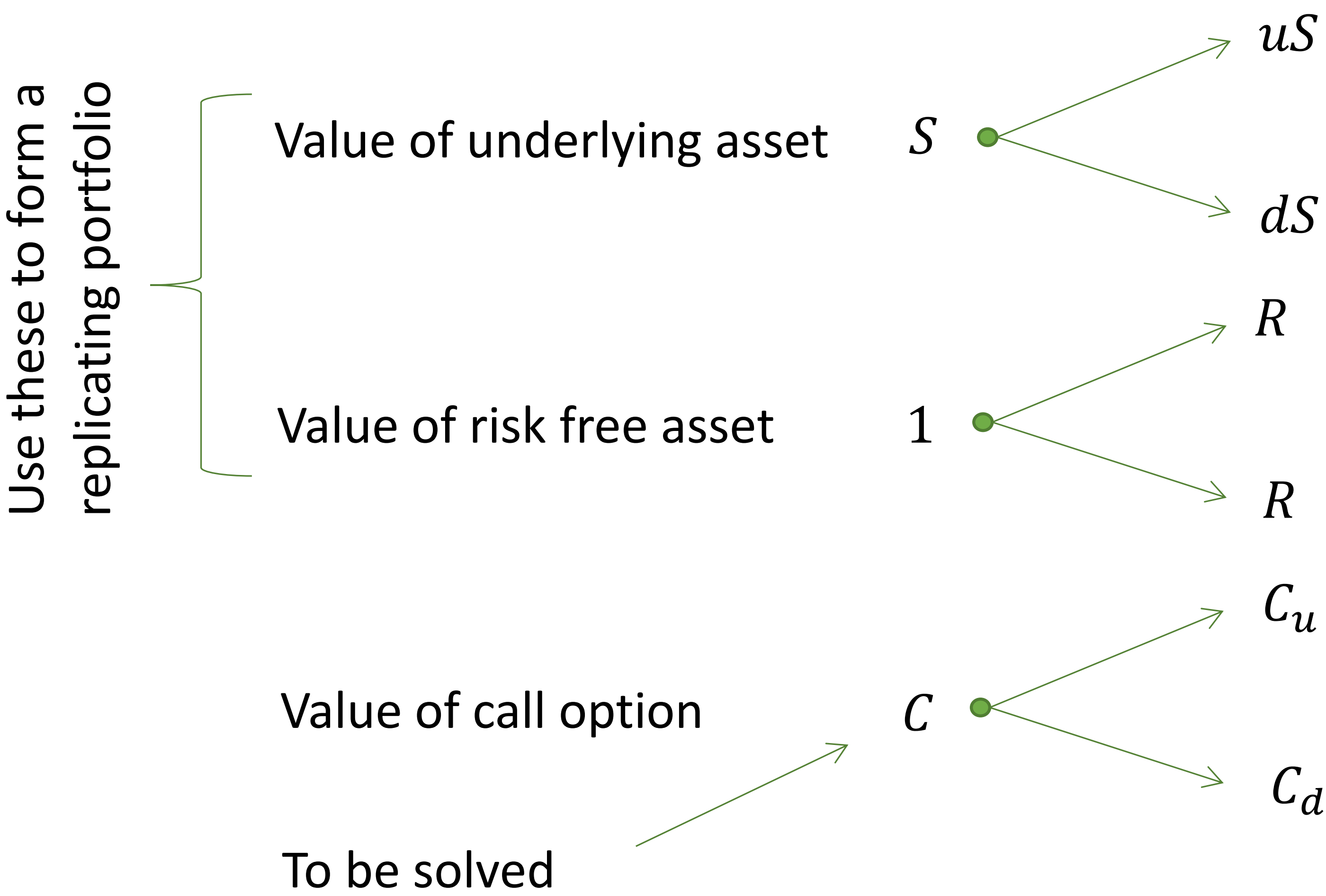
1 Options Price Theory

We first consider the pricing of one-period call options and then extend the analysis to multiple periods and put options. The valuation principle is to form a **replicating portfolio** which consists of two components: the underlying asset and the risk free investment so that the portfolio gives the same cash flow as the option.

Let S be the value of the underlying asset at the start the of period. At the end of the single period the value of this asset is:

$$uS \text{ with probability } p$$
$$dS \text{ with probability } 1-p$$

where $1 > p > 0$ and $u > d > 0$. For this, the risk free interest rate r $(R = 1 + r)$ and it follows that $u > R > d$ (otherwise, there would be arbitrage opportunities).



A **replicating portfolio** is a scenerario where an investor nvest x € in the underlying asset and b € in the risk free asset. The goal is to match the values of both states to obtain:

$$\begin{cases} ux + Rb &= C_u \\ dx + Rb &= C_d \end{cases} \Rightarrow \begin{cases} x &= \frac{C_u - C_d}{u-d} \\ b &= \frac{uC_d - dC_u}{R(u-d)} \end{cases}$$

The cash flows of the call option and the replicating portfolio are identical in **both** states. The values of the call option and the replicating portfolio must be the same:

$$C = x + b = \frac{C_u - C_d}{u-d} + \frac{uC_d - dC_u}{R(u-d)}$$
$$\Rightarrow C = \frac{1}{R}\left(\frac{R-d}{u-d}C_u + \frac{u-R}{u-d}C_d\right)$$

Risk-neutral probabilities q and $1-q$ are defined such that:

$$q = \frac{R-d}{u-d}$$

Misnomer: A risk-neutral probability neither (i) relates to a risk-neutral investor nor (ii) is it a probability of an event. In this scenario, q relates to Arrow-Debreu state prices rather than probabilities. A risk-neutral DM discounts cash flows at the highest expected rate of return in the market, not the risk-free interest rate.

Quantities q/R and $(1-q)/R$ are **state prices** (i.e, the present value of 1 euro in the respective state) implied by **complete markets**. Markets are **complete** when it is possible construct a **state-price security** for each state (also called an *Arrow-Debreu security*) with available securities.A security that pays 1 euro in one state and 0 euro in all other states.

All assets in complete markets can be priced as a linear combination of such state-price securities. The sum of all state prices must be equal to $1/R$, because a portfolio with one unit of each state-price security is riskless and yields 1 for sure.

In general, if π_s is the state price of state s, we must have:

$$\sum_{s \in S} \pi_s = 1/R$$

If we multiply this by R and denote $q_s = R\pi_s$, we get:

$$\sum_{s \in S} q_s = 1$$

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The quantities q_s may seem to look like probabilities, which are also non-negative quantities that sum up to 1. However, q_s's have nothing to do with probabilities. Neither do portfolio weights and other things that just happen to sum up to 1.

Here, we have 2 securities (the stock and the risk-free asset) and 2 states (u and d), and therefore we can construct securities that pay 1 in one state and 0 in all the other states (there is just one other state, though). hat is, in a binomial lattice, we have:

$$\pi_u = q/R$$
$$\pi_d = (1-q)/R$$
$$\pi_u + \pi_d = 1/R$$

Definition 1.1. Options pricing formula

he value of a one-period call option on a stock governed by a binomial lattice is

$$C = \frac{1}{R}\left[qC_u + (1-q)C_d\right].$$

where

$$q = \frac{R-d}{u-d}.$$

Remark: This pricing formula works for *any* asset governed by the binomial lattice, not just a call option. C_u and C_d are values of the asset is states u and d, respectively.

he options pricing formula is:

$$C = \frac{1}{R}\left[qC_u + (1-q)C_d\right]$$

The value of the call option is the discounted expected value with respect to the risk-neutral probabilities q and $1-q$. That is, state prices are q/R for state u and $(1-q)/R$ for state d. No arbitrage, which means: $u > R > d \Rightarrow 1 > q > 0$.

The single-period solution method can be extended to multiperiod options by working backward one step at a time. The value of the option at the last time period (expiration) is known.

- For a call option: $C = \max\{S-K, 0\}$
- For a put option: $P = \max\{K-S, 0\}$

The value of the option at other time periods can be calculated by using the options pricing formula recursively by starting from the next to last period and rolling back to period 0.

For a European option with no intermediate decision opportunities, the value in each node is given by:

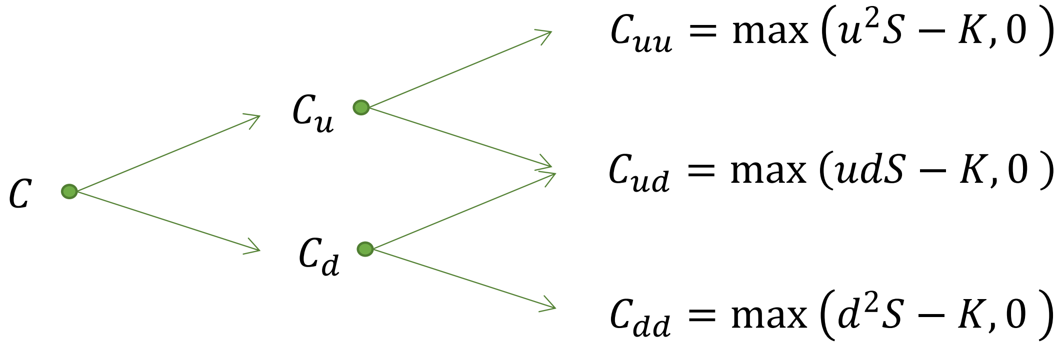
$$C = \frac{1}{R}\left[qC_u + (1-q)C_d\right]$$

For an American option, in each node, choose between exercising and not exercising the option early depending on which alternative has higher value, e.g., a call:

$$C = \max\left\{\frac{1}{R}\left[qC_u + (1-q)C_d\right], S - K\right\}$$

Proceed recursively by choosing the alternative with the highest value in each node.

For example, in two periods we have the following lattice:



The first period values are calculated recursively:

$$C = \frac{1}{R}\left[qC_u + (1-q)C_d\right], \qquad \begin{aligned} C_u &= \frac{1}{R}\left[qC_{uu} + (1-q)C_{ud}\right] \\ C_d &= \frac{1}{R}\left[qC_{ud} + (1-q)C_{dd}\right] \end{aligned}$$

Example:

Stock price is 80 € and the standard deviation of logarithmic price changes is $\sigma = 0.40$. Consider a European call which expires in four months with the strike price 85 €Ẇhat is the price of this call when risk free rate is 8% and no dividends are paid during this time?

Build the binomial lattice:

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.40\sqrt{1/12}} = 1.122$$
$$d = e^{-\sigma\sqrt{\Delta t}} = e^{-0.40\sqrt{1/12}} = 0.891$$

No need to calculate the probability p of going up in the lattice, since it turns out it does not influence the calculations (due to *complete markets*). Binomial lattice of stock price:

0	1	2	3	4
80 €	89.79 €	100.78 €	113.12 €	126.96 €
	71.28 €	80.00 €	89.79 €	100.78 €
		63.50 €	71.28 €	80.00 €
			56.58 €	63.50 €
				50.41 €

Value of the call can be determined recursively. In period 4, the value of the call at expiration is $\max\{S-K, 0\}$. In periods $0-3$, the value of the call is $\frac{1}{R}\left[qC_u + (1-q)C_d\right]$.

$$r = 0.080 \Rightarrow R = 1 + \frac{r}{12} = 1.007$$
$$q = \frac{R-d}{u-d} = \frac{1.007 - 0.891}{1.122 - 0.891} = 0.50$$

0	1	2	3	4
6.40 €	10.94 €	18.14 €	28.68 €	41.96 €
	1.93 €	3.89 €	7.84 €	15.78 €
		0.00 €	0.00 €	0.00 €
			0.00 €	0.00 €
				0.00 €

Therefore, the call is worth 6.40 €˙

The valuation formula for the price C for an American call is:

$$C = \max\left\{\frac{1}{R}\left[qC_u + (1-q)C_d\right], S - K\right\}$$

It turns out that, if the risk-free interest rate is positive, it is not optimal to exercise an American call option on a non-dividend-paying stock before the expiration date. If at any time t:

A: $S_t < K$, then it would be cheaper to buy the underlying asset from the market

B: $S_t > K$, then exercise at time $t < T$ could seem reasonable but it turns out that the value of keeping the call option is higher than the price gained from early exercise IF the stock does not pay dividends and the risk-free interest rate is positive $\Rightarrow$ Early exercise is not optimal

Theorem 1.2. No early exercise

he early exercise of an American call option on a stock that pays no dividends prior to expiration is never optimal, provided that prices are such that no arbitrage is possible and the risk-free interest rate is positive.

Proof: The result is proven in a two-period case, from which the result could be extended to next periods. Two period lattice:

$$C_{uu} = \max\left\{u^2S - K, 0\right\} \geq u^2S - K,$$
$$C_{ud} = \max\left\{udS - K, 0\right\} \geq udS - K,$$
$$C_{dd} = \max\left\{d^2S - K, 0\right\} \geq d^2S - K$$

$$C_u = \frac{1}{R}\left[qC_{uu} + (1-q)C_{ud}\right] \geq \frac{1}{R}\left[qu^2S + (1-q)udS - K\right]$$
$$\Rightarrow C_u \geq \frac{uS}{R}\left[qu + (1-q)d\right] - \frac{K}{R}.$$

Because

$$q = \frac{R-d}{u-d} \Rightarrow R = qu + (1-q)d,$$

we have

$$C_u \geq \frac{uS}{R}R - \frac{K}{R} = uS - \frac{K}{R} > uS - K,$$

if $R > 1$. Likewise, when $R > 1$, it can be shown that

$$C_d > dS - K$$

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Example II:

The valuation of put options is analogous to that of call options with the terminal values for the option being different. Consider a European put with a strike price 85 € on the same stock as in prevous slides. In period 4, the value of the put at expiration is $\max\{K - S, 0\}$. In periods $0 - 3$, the value of the put is $\frac{1}{R}\left[qP_u + (1-q)P_d\right]$.

0	1	2	3	4
9.17 €	4.47 €	1.23 €	0.00 €	0.00 €
	13.98 €	7.77 €	2.48 €	0.00 €
		20.37 €	13.16 €	5.00 €
			27.86 €	21.50 €
				34.59 €

Therefore, value of the put is 9.17 €˙

The corresponding American put option may be exercised early. E.g., if the price of underlying asset goes down in every period, then the option should be exercised at the beginning of the third period, because 85.00 - 63.50 = 21.50 > 20.37 .The valuation formula for the price P for an American put is:

$$P = \max\left\{ \frac{1}{R}\left[qP_u + (1-q)P_d\right], K - S \right\}$$

Consider an American put with a strike price 85 € on the stock of the previous examples. The valuation formula gives the following values:

0	1	2	3	4
9.58 €	4.61 €	1.23 €	0.00 €	0.00 €
	14.68 €	8.05 €	2.48 €	0.00 €
		21.50 €	13.72 €	5.00 €
			28.42 €	21.50 €
				34.59 €

The value of the American put is 9.58 €Ṫhe possibility of being able to exercise the option early yields $9.58 - 9.17 = 0.41$ € of additional value compared to the European put.

2 Real Options Pricing

The principles of options pricing can be applied to determine the correct price when the underlying asset is not a financial instrument, but some real investment:

- Natural resources (e.g. oil, gas, lumber, minerals)
- Real estate
- R&D-projects
- Intellectual property rights (IPR)

Options are related to:

1. Investment size (option to expand or contract)
2. Investment timing (option to postpone, abandon, sequence)
3. Investment management (option of using alternative resources)

It is possible to extract from a mine at most 10 000 ounces of gold per year for a cost of 200 € /ounce .The current price of gold is 400 € /ounce; this is estimated to increase each year by 20% ($u = 1.2$) with probability 0.75 and decrease by 10% ($d = 0.9$) with probability 0.25.

The risk free rate r is 10%. What is the value of a 10 year lease of the mine?

The binomial lattice for the price S of gold:

0	1	2	3	4	5	6	7	8	9	10
400	480	576	691.2	829.44	995.33	1194.4	1433.3	1719.9	2063.9	2476.7
	360	432	518.4	622.08	746.5	895.8	1075	1290	1547.9	1857.5
		324	388.8	466.56	559.87	671.85	806.22	967.46	1161	1393.1
			291.6	349.92	419.9	503.88	604.66	725.59	870.71	1044.9
				262.44	314.93	377.91	453.5	544.2	653.03	783.64
					236.2	283.44	340.12	408.15	489.78	587.73
						212.58	255.09	306.11	367.33	440.8
							191.32	229.58	275.5	330.6
								172.19	206.62	247.95
									154.97	185.96
										139.47

At the end of the last year, the lease is worthless, because no more gold can be extracted. Each year, there is a possibility to make a profit by exercising the option to mine gold. Mining is profitable only if $S > 200$, hence the profit made from the lease in each period is:

$$P = \max\left\{10\ 000 \cdot \frac{S-200}{1.10},\ \ 0\right\}$$

Value V of the lease contract can be computed recursively as the sum of the profit that can be made in a given year plus the risk-neutral expected value of lease in next period:

$$V = P + \frac{1}{R}\left[qV_u + (1-q)V_d\right]$$

where $q = \frac{R-d}{u-d} = \frac{1.1-0.9}{1.2-0.9} = 0.667$

Computations in the binomial lattice give the lease value $V = 24.1$ M€Ė.g., on the the top row at the beginning of the last year $10000(2063.9 - 200)/1.1 = 16.9 \times 10^6$.

0	1	2	3	4	5	6	7	8	9	10
24.1	27.8	31.2	34.2	36.5	37.7	37.1	34.1	27.8	16.9	0
	17.9	20.7	23.3	25.2	26.4	26.2	24.3	20	12.3	0
		12.9	15	16.7	17.9	18.1	17	14.1	8.7	0
			8.8	10.4	11.5	12	11.5	9.7	6.1	0
				5.6	6.7	7.4	7.4	6.4	4.1	0
					3.2	4	4.3	3.9	2.6	0
						1.4	2	2.1	1.5	0
							0.4	0.7	0.7	0
								0	0.1	0
									0	0
										0

The company gets an option to buy a new mining machine which increases extraction capacity to 12 500 ounces per year, but which costs 4 million € increases extraction costs to 240 € /ounce. Should the new machine be bought? If yes, then when?

We first calculate the lattice for the value of the lease when the new machine is available (i.e., bought immediately). Each cell of the lattice contains the value of extracting gold at that period and in the following periods with the new machine. The purchase price is not included in this lattice.

If the new machine is purchased immediately, the binomial lattice of the value V' (in M €) of the lease is given below. Note that it is not worthwhile to exercise this real option immediately, because $27.0 - 4.0 = 23.0 < 24.1$.

0	1	2	3	4	5	6	7	8	9	10
27	31.8	36.4	40.4	43.5	45.2	44.8	41.4	33.9	20.7	0
	19.5	23.3	26.6	29.3	31	31.2	29.2	24.1	14.9	0
		13.5	16.3	18.7	20.4	21	20	16.8	10.5	0
			8.6	10.8	12.5	13.4	13.2	11.3	7.2	0
				4.9	6.5	7.7	8	7.2	4.7	0
					2.3	3.4	4.1	4.1	2.8	0
						0.8	1.3	1.8	1.4	0
							0.1	0.2	0.4	0
								0	0	0
									0	0
										0

There are three alternatives at each node in the option to select the new machine:

1. Buy the machine (value determined by the corresponding node in previous lattice minus the investment 4 M€)
2. Extract with the old machine (value of gold extracted now plus discounted risk-neutral expected value of sequel nodes)
3. Do not extract (risk-neutral expected value of sequel nodes)

In each node, take the alternative that yields highest value.

The value V of the the lease + option for new machine can be computed recursively from:

$$
\begin{aligned}
V = \max\{&V' - 4\,000\,000, \\
&P + \frac{1}{R}\left[qV_u + (1-q)V_d\right], \\
&\frac{1}{R}\left[qV_u + (1-q)V_d\right]\},
\end{aligned}
$$

where $P = 10\,000 \cdot \frac{S-200}{1.10}$.

The binomial lattice for value V (in M €) of the lease + option for new machine is given below. New machine is bought in the leftmost blue cells (from which point onwards the new machine will be used).

0	1	2	3	4	5	6	7	8	9	10
24.6	28.6	32.6	**36.4**	**39.5**	**41.2**	**40.8**	**37.4**	**29.9**	16.9	0
	18	20.9	23.5	25.6	**27**	**27.2**	**25.2**	**20.1**	12.3	0
		12.9	15	16.7	17.9	18.1	17	14.1	8.7	0
			8.8	10.4	11.5	12	11.5	9.7	6.1	0
				5.6	6.7	7.4	7.4	6.4	4.1	0
					3.2	4	4.3	3.9	2.6	0
						1.4	2	2.1	1.5	0
							0.4	0.7	0.7	0
								0	0.1	0
									0	0
										0