MSE2114 - Investment Science
Lecturer Notes XII

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Abstract

In this lecture, we are focusing on interest rate derivates, which are are derivative securities whose payoff depends on interest rates because almost. These securities can be used to control interest rate risk.

Contents

1 Modelling the term structure

Interest rate derivatives are derivative securities whose payoff depends on interest rates. As examples, in this course some interest rate derivatives have been addressed:

1. Interest rate swaps (see Lecture VIII);
2. Interest rate caps and floors;
   - Refers to upper (cap) and lower (floor) bounds on the interest rate;
3. Bond options;
4. Interest rate swaptions (i.e., option on a swap);
5. Bond futures;
6. Bond options;
   - E.g., the option to buy 10-y treasury bonds at a fixed strike price at a given time;
7. Embedded bond options;
   - E.g., callable bonds whose issuer has the right to repurchase the bond in accordance with prespecified terms.

However, those differs from some common concepts such as **bonds** are fixed income securities, not interest rate derivatives and **Mortgages** are fixed income instruments (loans), not securities or derivatives. For example, **mortgage-backed securities** are structured credit instruments (fixed income securities) that pool together many mortgages.

They are issued by a special-purpose company whose only purpose is to own a portfolio of mortgages and distribute the returns and the principal from the mortgages to the security holders in an order of priority of payments. These securities were among the causes of the financial crisis that began in 2007; see article in Investopedia.

The pricing of interest rate derivatives is based on models of the term structure of interest rates. Changes in spot rates are not independent and some simple parallel shift models of the term structure may offer arbitrage opportunities. It is possible to construct an arbitrage-free model for the term structure with a binomial lattice.

The binomial lattice for the short rates is constructed as follows:

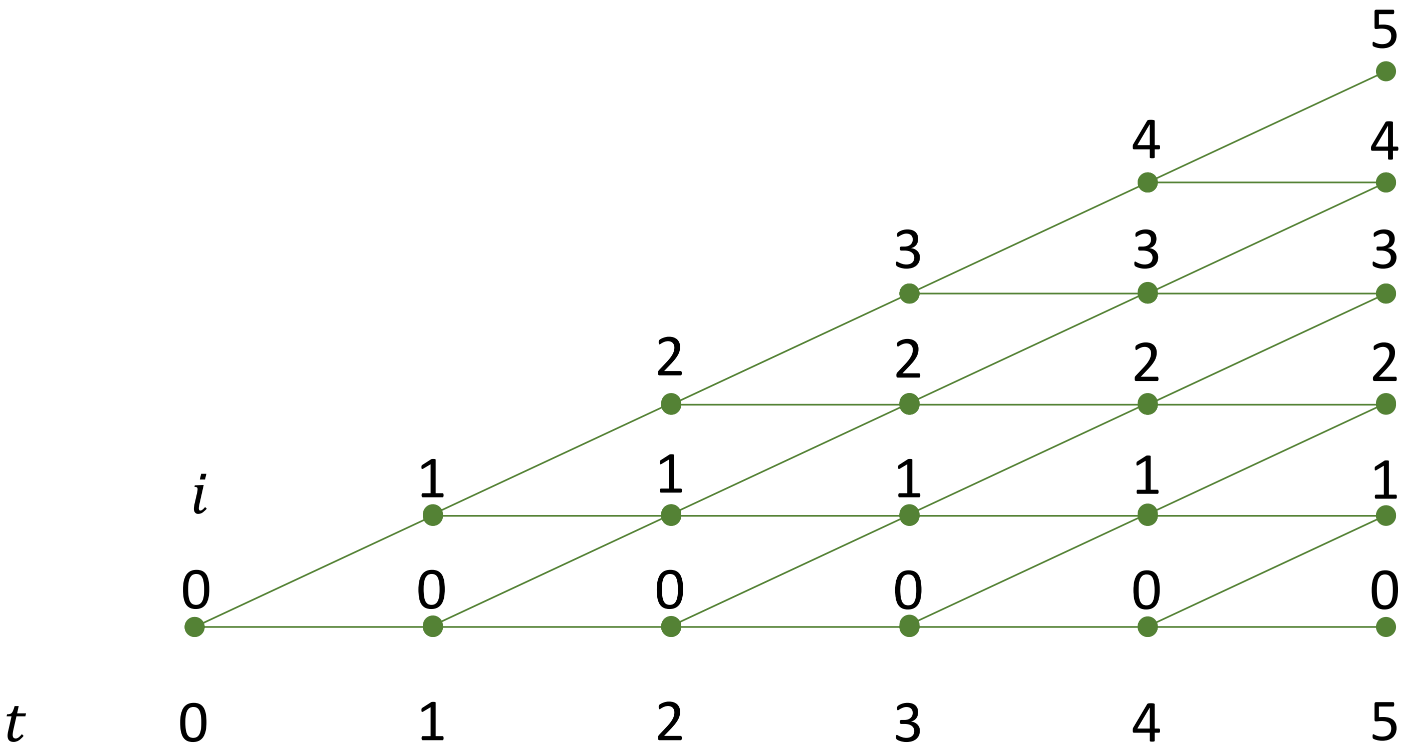
1. Select the period length (e.g., week, month, year);
2. Associate a short rate with each lattice node;
3. Assign a risk-neutral probability q with each arc;

The usual choice of $q = 1/2$ can be assigned for convenience. Normally, the matching of the term structure is done by adjusting the short rates at each node of the lattice, not the risk-neutral probability. The choice of q also influences the bond prices and therefore the term structure.

Still, we can typically find appropriate lattice parameters for $q = 1/2$ so there is no need to use also q to match the term structure. Note that earlier on, we knew the initial stock price S, fixed the u and d parameters and derived the corresponding q (see Lecture IX).

Now we fix q and choose the other lattice parameters to match observed spot rates, which allows us to compute corresponding bond prices. We use the risk-neutral pricing formula the other way around, so to speak.

Index the binomial lattice as (t, i) where t is the period and i is the number of up movements in the lattice:



Let r_{ti} be the short rate in node (t, i). Then the value V_{ti} of the interest rate derivative at (t, i) is:

$$V_{ti} = \frac{1}{1 + r_{ti}} \left(qV_{t+1,i+1} + (1 - q)V_{t+1,i} \right) + D_{ti},$$

where D_{ti} is the dividend paid at node (t, i) and $q = 1/2$ is the risk-neutral probability of the upward movement. Importantly, this formula is **arbitrage-free**. It is implicitly assumed that it is possible to construct an elementary security (a state-price security) for each state in the lattice using available securities (see Lecture X).

If $V_{ti} - D_{ti} < 0 \Rightarrow V_{t+1,i} < 0$ or $V_{t+1,i+1} < 0$: No **type A** arbitrages (i.e., an initial positive payoff cannot lead to non-negative future payoffs only).

$V_{t+1,i}, V_{t+1,i+1} \geq 0$ with strict inequality in one $\Rightarrow V_{ti} > 0$. No **type B** arbitrages (i.e., an initial non-positive payoff cannot lead only to non-negative payoffs with a positive expection).

Consider a bond which yields 1 € at the end of period two. Denote the price of the bond at note (t, i) with $P_{ti}(2)$:

- In node $(1, 0)$ the value of the bond is
$$P_{10}(2) = \frac{1}{1 + r_{10}} \left(\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 \right)$$
- In node $(1, 1)$ the value of the bond is
$$P_{11}(2) = \frac{1}{1 + r_{11}} \left(\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 \right)$$
- In node $(0, 0)$ the value of the bond is
$$\begin{aligned} P_{00}(2) &= \frac{1}{1 + r_{00}} \left(\frac{1}{2} P_{10}(2) + \frac{1}{2} P_{11}(2) \right) \\ &= \frac{1}{1 + r_{00}} \left(\frac{1}{2} \frac{1}{1 + r_{10}} + \frac{1}{2} \frac{1}{1 + r_{11}} \right) \end{aligned}$$

Thus the two period spot rate s_2 can be solved from:

$$P_{00}(2) = \frac{1}{(1 + s_2)^2}$$

This process can be applied to evaluate the price $P_{00}(k)$ for any period k. Thus, risk neutral pricing when the short rates vary according to the binomial lattice generates the entire term structure as in the deterministic case (Lecture III).

Consider a binomial lattice with six periods such that $r_{00} = 0.070$. Assume that the short rate dynamics is constructed using an up factor of $u = 1.3$ and a down factor of $d = 0.9$. Then, the short rates are given by:

0	1	2	3	4	5
					0.260
				0.200	0.180
			0.154	0.138	0.125
		0.118	0.106	0.096	0.086
	0.091	0.082	0.074	0.066	0.060
0.070	0.063	0.057	0.051	0.046	0.041

The four year spot rate is the price of a zero coupon bond which pays its face value in 4 years:

0	1	2	3	4
				1
			0.867	1
		0.792	0.904	1
	0.751	0.848	0.931	1
0.733	0.818	0.891	0.951	1

Values are computed so that, e.g., $P_{33}(4) = (0.5 \cdot 1 + 0.5 \cdot 1)/(1 + 0.154) = 0.867$. Thus the 4 year spot rate is:

$$P_{00}(4) = \frac{1}{(1 + s_4)^4} = 0.733 \Rightarrow s_4 = 0.0806$$

All other spot rates can be computed similarly.

2 Pricing applications: Bond option

Bond option, e.g. assume that the term structure is captured by the binomial lattice of the previous example. The price of a 4 year zero coupon bound with face value 100 € is 73.34 €.

Consider a European call to buy this bond in 3 years with a strike price of $K = 90$ €. What is the value of this call option?

Construct a 3 year binomial lattice for the option value V_{ti} such that the last node has value of the option at expiration:

$$V_{3i} = \max\{0, P_{3i}(4) - K\},$$

where $P_{3i}(4)$ is the value of the bond in year 3 after i up movements.
The option value is computed recursively. Value of the option is 1.602 €.

0	1	2	3
			0.000
		0.169	0.378
	0.821	1.623	3.135
1.602	2.606	3.918	5.145

Bond forward, e.g., consider a forward contract for buying after 4 years a bond which has, upon purchase, 2 years to maturity and whose face value is 100 € with 10% coupon rate.

What is the forward price of this contract? At the time of writing the contract, the price is chosen to make contract of value 0. Use first binomial lattice to price the bond today. Value of the 6-year bond at time 0 is 72.90 €.

0	1	2	3	4	5	6
						110
					97.308	110
				83.561	103.226	110
			76.379	92.691	107.815	110
		73.074	87.058	99.962	111.267	110
	72.196	84.459	95.694	105.534	113.802	110
72.901	83.811	93.724	102.383	109.681	115.634	110

Forward price of any asset with a present value of 72.90 must be the present value accrued by the spot rate.

The 4 year discount factor $d_{0,4} = 0.733$. Forward price (= price paid on delivery):

$$72.90 \cdot (1 + s_4)^4 = \frac{72.90}{d_{0,4}} = \frac{72.90}{0.733} \approx 99.40$$

For **bond future**, we consider a futures contract analogous to the previous example. What is the futures price of this contract?

The price of the bond in 4 years was determined in the previous example. Value of futures contract at the third year depends on the value of the bond in the fourth year.

Just after node $(3, 3)$, the value of the bond is either 83.56 € or 92.69 €. If the futures price is F, the contract yields profit either $83.56 - F$ or $92.69 - F$, which yields the expected value.

$$0.5 \cdot (83.56 - F) + 0.5 \cdot (92.69 - F)$$

The price of the futures contract is set so that the expected value is zero

$$\Rightarrow F = 0.5 \cdot (83.56 + 92.69) = 88.13$$

Hence futures price in a binomial lattice is determined as the expected value <u>without</u> discounting:

0	1	2	3	4
				83.561
			88.126	92.691
		92.226	96.327	99.962
	95.882	99.537	102.748	105.534
99.120	102.357	105.178	107.607	109.681

Price is 99.12 €< 99.40 €. The futures-forward equivalence does not hold with fluctuating interest rates (in contrast to the situation when the expectations hypothesis holds).

3 Forward equation

The binomial lattice determines the term structure completely. The k year spot rate can be computed using the value of a k year zero coupon bond. With backward recursion, every maturity needs to be computed separately:

- Computing spot rates for the k-th year requires $1 + 2 + \cdots + k = k(k+1)/2$ single node evaluations
- Computing the spot rates for n years requires n recursions and $\sum_{k=1}^{n} k(k+1)/2 \approx n^3/6$ single node evaluations

The entire term structure can be determined with a single recursion by utilizing *elementary prices*. **Elementary price** $\psi_{k,s}$ is the price of a security that yields 1 unit of cash flow at time k in state s and zero otherwise (i.e., an **elementary security**, also called a *state-price security* or *Arrow-Debreu security*).

The same as a *state price* mentioned in Lecture X. Any asset governed by a binomial lattice can be priced using elementary prices.

In theory, all elementary prices can be derived using normal backward recursion, but this requires plenty of computations. It turns out that there is a mathematically equivalent but computationally more efficient way of computing all elementary prices.

Forward equation is a shortcut for computing the present values given by the risk-neutral pricing formula in a binomial lattice for an elementary security, utilizing the values of elementary securities in earlier states (these are available as results of earlier computations).

The forward equation is a mechanism for pricing elementary securities in the lattice. These securities pay 1 only in one node and 0 in all the other nodes. However: lattice is mainly full of zeros and there are only one (bottom and top nodes) or two (middle nodes) *immediate predecessor* nodes for the node with the payoff of 1 (i.e., the node for which the elementary price is being calculated).

- Under the risk-neutral pricing formula, only these immediate predecessor nodes have a positive value in the previous time period
- All other nodes for the previous time period have zero value
- Sum of the *time-0 present values* of those immediate predecessors in the previous time period must be the *time-0 present value* of the node in its actual time period

Time-0 present value of a node in time period t is:

(i) the value of the node (as given by the risk-neutral pricing formula) + the cash flow at this node, multiplied by

(ii) the *time-0 present value* of 1 in that node (i.e., the *elementary price* of the node)

If we derive elementary prices by starting from the beginning and continuing towards the end, we always know the elementary prices of the previous nodes. The *elementary price* $\psi_{t,s}$ *of a node* (t, s) (= the *time-0 present value* of 1 at node in time period t and state s) can be computed as the sum over the *time-0 present values* of the immediate predecessor nodes of this node in time period $t-1$.

Calculate the value that the predecessor node would have (in time $t-1$) using the risk-neutral pricing formula:

- $q \cdot 1/(1 + r_{(t-1)(s-1)})$, or
- $(1 - q) \cdot 1/(1 + r_{(t-1)s})$

Time-0 present value of the predecessor node $(t-1, s)$ is the value given by the risk-neutral valuation formula above, multiplied by and the elementary price $\psi_{t-1,s}$ of this predecessor node (the time-0 present value of 1 in this node).

If there are two immediate predecessor nodes, we have:

$$\begin{aligned}\psi_{t,s} = {} & q \cdot \frac{1}{1 + r_{(t-1)(s-1)}} \psi_{t-1,s-1} \\ & + (1-q) \cdot \frac{1}{1 + r_{(t-1)s}} \psi_{t-1,s}\end{aligned}$$

By rearranging the terms and denoting $d_{t,s} = 1/(1 + r_{ts})$, we get:

$$\begin{aligned}\psi_{t,s} = {} & q \cdot d_{t-1,s-1} \psi_{t-1,s-1} \\ & + (1-q) \cdot d_{t-1,s} \psi_{t-1,s}\end{aligned}$$

With $q = 1/2$ we further have:

$$\psi_{t,s} = \frac{1}{2}\left(d_{t-1,s-1}\psi_{t-1,s-1} + d_{t-1,s}\psi_{t-1,s}\right)$$

Consider state $(k+1, s)$, where $s \neq 0$ and $k \neq 1$:

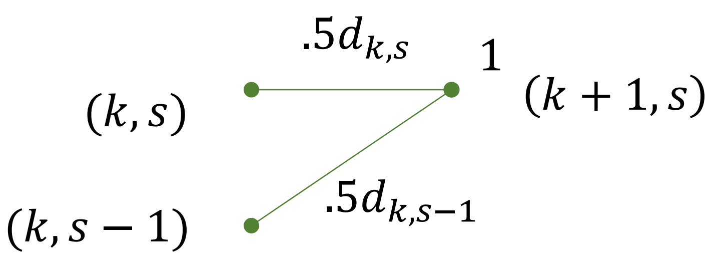
- Elementary price in the node is $\psi_{k+1,s}$
- Node has two predecessors (k, s) and $(k, s-1)$
- Predecessors have elementary prices $\psi_{k,s-1}$ and $\psi_{k,s}$
- The only way to arrive at $(k+1, s)$ is via either (k, s) or $(k, s-1)$ and thus the value of state $(k+1, s)$ at time zero is
$$\psi_{k+1,s} = \frac{1}{2}(d_{k,s-1}\psi_{k,s-1} + d_{k,s}\psi_{k,s}),$$
where $d_{k,s}$ is discount factor from time k to $k+1$ in state s

Node $(k+1, k+1)$ only has a single predecessor:

- Single unit cash flow in $(k+1, k+1)$ is equivalent to getting in node (k, k) a cash flow of $0.5d_{k,k}$ units, and hence
$$\psi_{k+1,k+1} = \frac{1}{2}d_{k,k}\psi_{k,k}$$

Node $(k+1, 0)$ also only has a single predecessor $(k, 0)$ such a single unit cash flow in node $(k+1, 0)$ equivalent to getting in node $(k, 0)$ a cash flow of $0.5d_{k,0}$, and hence:

$$\psi_{k+1,0} = \frac{1}{2}d_{k,0}\psi_{k,0}$$



Node $(k+1, k+1)$ only has a single predecessor and such single unit cash flow in $(k+1, k+1)$ is equivalent to getting in node (k, k) a cash flow of $0.5d_{k,k}$ units, and hence:

$$\psi_{k+1,k+1} = \frac{1}{2}d_{k,k}\psi_{k,k}$$

Node $(k+1, 0)$ also only has a single predecessor $(k, 0)$ such that a single unit cash flow in node $(k+1, 0)$ equivalent to getting in node $(k, 0)$ a cash flow of $0.5d_{k,0}$, and hence:

$$\psi_{k+1,0} = \frac{1}{2}d_{k,0}\psi_{k,0}$$

Parts A, B and C define the three forms of the **forward equation**, which yields the elementary prices of a state using the elementary prices of its predecessor states:

$$\begin{aligned}
\psi_{k+1,s} &= \frac{1}{2}(d_{k,s-1}\psi_{k,s-1} + d_{k,s}\psi_{k,s}), && 0 < s < k+1 \\
\psi_{k+1,k+1} &= \frac{1}{2}d_{k,k}\psi_{k,k}, && s = k+1 \\
\psi_{k+1,0} &= \frac{1}{2}d_{k,0}\psi_{k,0}, && s = 0
\end{aligned}$$

After a single recursion for defining all elementary prices, the prices of zero-coupon bonds (and hence spot rates) can be computed from the elementary prices as:

$$P_{00}(k) = \sum_{s=0}^{k} \psi_{k,s}$$

For the short rates in the previous example, the elementary prices are:

	0	1	2	3	4	5	6
							0.007
Sum = bond values						0.017	0.047
					0.041	0.094	0.130
				0.096	0.175	0.203	0.189
			0.214	0.296	0.276	0.216	0.153
		0.467	0.434	0.305	0.191	0.113	0.065
	1	0.467	0.220	0.104	0.049	0.024	0.011
Sum	1	0.935	0.868	0.801	0.733	0.667	0.602

The spot rates can then be computed in a straightforward manner, e.g.,

$$\frac{1}{(1+s_3)^3} = 0.8006 \Rightarrow s_3 = 0.0769$$

4 Matching the term structure

The short rate lattice needs to be matched with the observed term structure. We first select an appropriate *model of interest rate dynamics* to describe the movements of interest rates in the binomial lattice.

Then the parameters of this model are chosen to match the observed term structure. Often accomplished with the help of elementary prices. An analytic solution for the parameters typically does not exist. Often one minimizes an error measure (e.g., sum of squared errors over relevant time periods). Solving the right parameters is an optimization problem.

In **Ho-Lee model**, short rates are calculated with the linear model:

$$r_{ks} = a_k + b_k s,$$

where a_k and b_k are parameters to be estimated and $s = 0, 1, \ldots, k$ is the number of up movements, such that:

- a_k is an aggregate drift parameter
- b_k is a volatility parameter
  - Usually $b_k = b$ is the same for all time periods
  - It can be shown that $b/2$ is the standard deviation of the one period rate

Black-Derman-Toy model, such that short rates are calculated with the log-linear model:

$$r_{k,s} = a_k e^{b_k s}$$

Consider the 6-year term structure presented below ("observed spot rates"):

Year k	1	2	3	4	5	6
Observed spot-rates	0.077	0.083	0.088	0.093	0.098	0.102

We want to match the Ho-Lee model with this term structure. That is, we want to find the parameters for the Ho-Lee model that yield exactly the observed spot rates.

We simplify the analysis by assuming that $b_k = b = 0.025$. In general, the term b_k can set used to control interest rate volatility implied by the Ho-Lee model.

The desired parameters are found by calculating the following items in the following order (each item implies the next):

1. Short rate lattice
2. Elementary prices for each node
3. Prices of zero-coupon bonds for each maturity
4. Spot rates for each maturity
5. Sum of squared differences of the observed spot rates and the spot rates in item 4

In practice, **optimization** is employed update parameter values for the Ho-Lee model so that the sum of squared differences is minimized, with repeated calculations until the minimum has been found. The spreadsheet calculation is presented in the following two slides with the decision variables and objective cells of the optimization problem highlighted with blue cell color.

Year k	0	1	2	3	4	5	6
Observed spot-rates		0.077	0.083	0.088	0.093	0.098	0.102
a_k		0.076	0.074	0.072	0.067	0.062	
b	0.025						

Short rate lattice

$s\backslash k$	0	1	2	3	4	5
5						0.187
4					0.167	0.162
3				0.147	0.142	0.137
2			0.124	0.122	0.117	0.112
1		0.101	0.099	0.097	0.092	0.087
0	0.077	0.076	0.074	0.072	0.067	0.062

Elementary prices

	s\k	0	1	2	3	4	5	6
	6							0.007
	5						0.018	0.047
	4					0.041	0.092	0.124
	3				0.094	0.169	0.192	0.174
	2			0.211	0.288	0.262	0.200	0.138
	1		0.464	0.427	0.294	0.181	0.105	0.059
	0	1.000	0.464	0.216	0.100	0.047	0.022	0.010
Bond price $P_{00}(k)$		1.000	0.929	0.853	0.776	0.700	0.628	0.560
Implied spot-rate			0.077	0.083	0.088	0.093	0.097	0.102
Squared difference			0.000	0.000	0.000	0.000	0.000	0.000
Sum of sq.-diffs.		0.000						