MSE2114 - Investment Science
Lecturer Notes II

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Contents

1 Introduction

For this lecture, let us understand some useful terminology. Starting with **financial instruments**, which represents any type of monetary contract between parties (people, business, organization, etc.). Examples include cash, evidence of ownership, and other forms of monetary representation. Some of those instruments have a fixed stream of cash flows, generally agreed upon beforehand. Those are called **fixed income instruments.**

Many of those instruments are protected by some layer of **security**, which allows for easy and ready transference in a well-developed market. These securities come by policies and laws generally placed by the government. Similarly, there is correlated security for fixed instruments, a.k. a. **fixed income security**.

Examples can be found in money market instruments, such as trade-able instruments that offer guaranteed interest, certificates of deposit offered by banks and short-term notes by corporations, U.S. government securities and other bonds. Other examples that cannot be usually classified as securities are **mortages** and **bonds**. The former is used by purchasers (buyers) of real property to **raise funds** to buy real estate or by **existing** property owners to raise funds for any purpose while putting a lien on the mortgaged property. Analogously, some owner can use their property as collateral to borrow resources (**reverse mortgages**). The latter provides a series of payments at equal intervals. It can comprehend different forms, such as regular deposits to a savings account, monthly insurance payments, and pension payments.

Another important concept is **bonds**, which are fixed cash flows unless the issuer falls back to default. With such bonds come **ratings**, which are representations of the creditworthiness of corporate or government bonds. **Credit rating agencies** publish the ratings and evaluate a bond issuer's financial strength and capacity to repay the bond's principal and interest according to the contract. Some of the most common agencies are known as the big three: *Moody's*, *Standard & Poor's*and *Fitch*. Some government agencies also define some securities; traditionally, U.S. Treasury securities are almost risk-free.

Fixed-income securities can also be used to define the **time value of money**, given that they have pre-agreed cash flows. For those securities, the policies and directives to trade in the market are very well defined and established throughout the years. At the same time, the prices of these securities on the market reflect the time value of money and offer an insight into how demand and supply are behaving at that moment. Finally, it can also be used to imply two forms of risk premium: **fundamental**, based on the issuer's decision and **market**, which accounts for when the security goes down.

2 Annuities

In investment science, an **annuity** is a series of payments made at equal intervals. Examples of annuities can be as simple as regular deposits to a savings account, monthly home mortgage payments, monthly insurance payments and pension payments. They can also be classified by the frequency of the payments.

Let us consider the following example. A client receives an annual amount of money, A, every year. Associated with this annual amount, there is also an interest rate. This constitutes a **perpetual annuity**. In the same example, to calculate the net present value of this cash flow stream, the previous equation for such is simplified to:

$$
\begin{aligned}
P &= \sum_{k=1}^{\infty} \frac{A}{(1+r)^k} = \frac{A}{1+r} + \frac{1}{1+r} \sum_{k=1}^{\infty} \frac{A}{(1+r)^k} \\
&= \frac{A}{1+r} + \frac{P}{1+r} \\
\Rightarrow P &= \frac{A}{r}
\end{aligned}
$$

Another example is that the annuity is limited by **a given amount of years** (n). The cash flows are also discounted with an interest rate r for this particular case. The present value is also computed using the equation for net present value with perpetual annuity accounting for the cash flows that will not occur after the n-th year.

$$
\begin{aligned}
P &= \sum_{k=1}^{n} \frac{A}{(1+r)^k} \\
&= \underbrace{\sum_{k=1}^{\infty} \frac{A}{(1+r)^k}}_{\frac{A}{r}} \quad - \quad \underbrace{\sum_{k=n+1}^{\infty} \frac{A}{(1+r)^k}}_{\frac{1}{(1+r)^n} \sum_{k=1}^{\infty} \frac{A}{(1+r)^k} = \frac{1}{(1+r)^n} \frac{A}{r}} \\
\Rightarrow P &= \frac{A}{r} \left(1 - \frac{1}{(1+r)^n} \right)
\end{aligned}
$$

3 Bonds

We can define bonds as an example of bonds as examples of fixed income securities. A bond pays its **principal** (also known as **face value** or **par value**) on its **maturity date** (which is the date on which the final payment is due on a loan or other financial instrument). Most bonds pay periodic interest payments (also known as **coupon payments**), which are defined as a percentage of face value. When a bond is issued, the coupon rate usually is similar to interest, even if the bond is issued under a discount.

As mentioned in the previous lecture, the internal rate of return is an interest rate that forces the present value to be zero. For bonds, the equivalent is called yield to maturity (YTM), and it is defined as the internal rate of return of a bond per annum (p.a.) at its current price.

Considering the following notation:

(i) Face value F

(ii) m coupon payments p.a., each payment C/m

(iii) n periods (payments) left to maturity

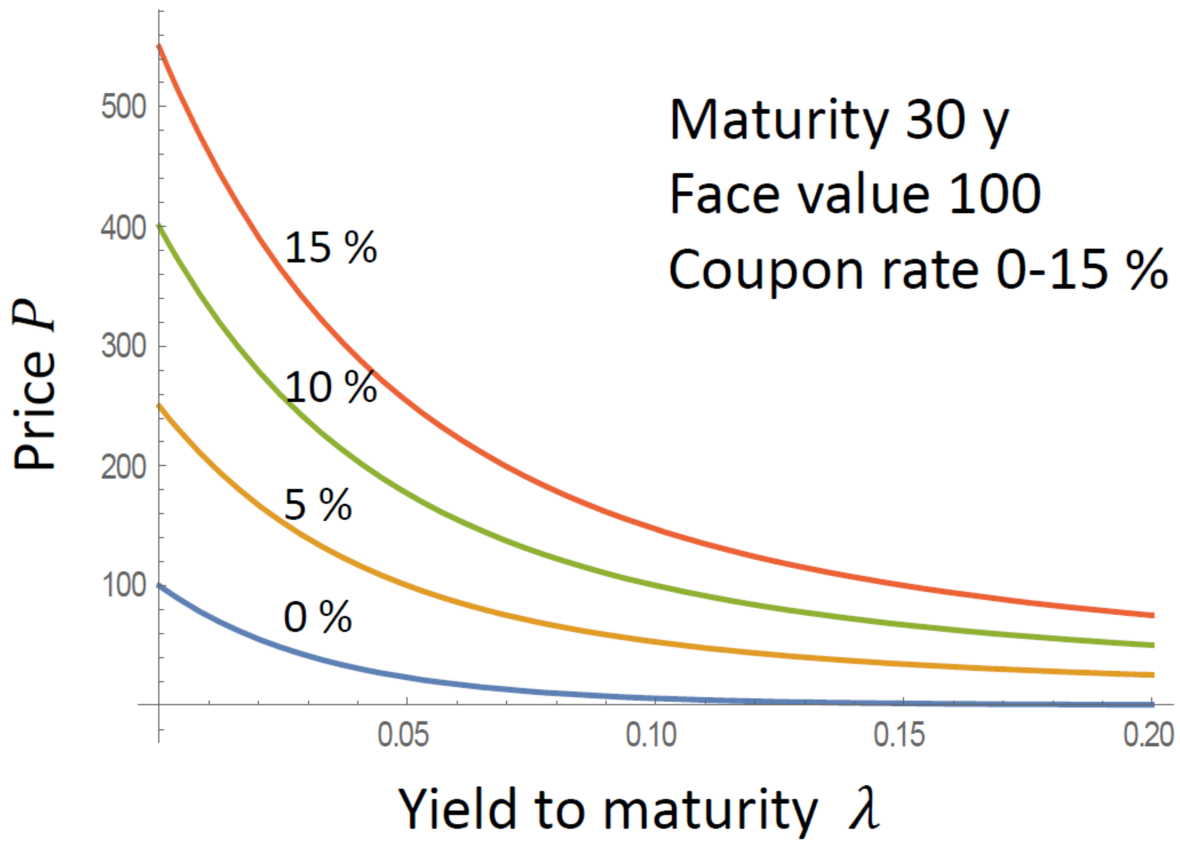
If the current price is $P \Rightarrow$, then YTM is the rate λ such that

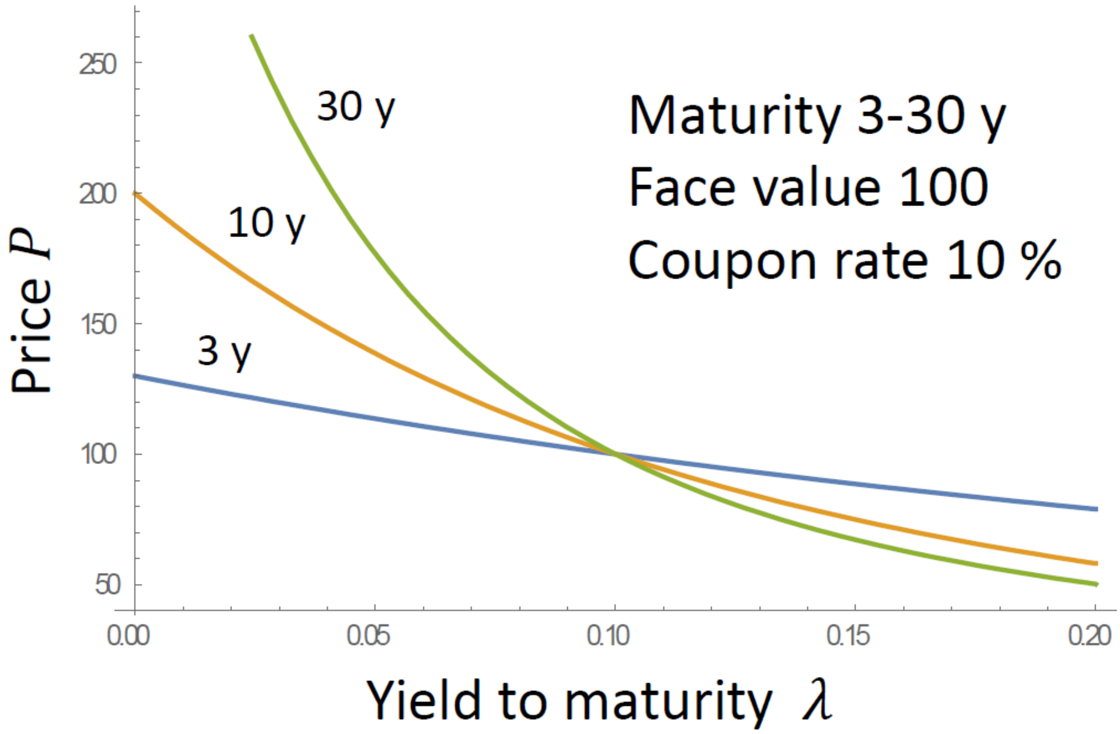
$$
\begin{aligned}
P &= \frac{F}{[1 + (\lambda/m)]^n} + \underbrace{\sum_{k=1}^{n} \frac{C/m}{[1 + (\lambda/m)]^k}}_{\text{Finite life stream}} \\
\Rightarrow P &= \frac{F}{[1 + (\lambda/m)]^n} + \frac{C}{\lambda} \left(1 - \frac{1}{[1 + (\lambda/m)]^n} \right)
\end{aligned}
$$

The formula is best understood if you substitute $r = \lambda/m$ as the periodic (e.g. monthly) IRR, then obtain an annualized rate (= YTM) by linear annualization $\lambda = rm$.

A bond can increase (or decrease) its interest over the time since the last coupon payment. This rate is another version of the interest rate known as accrued interest (AI).

U.S. Treasury Quotes Wednesday, January 13, 2016					
Treasury Notes & Bonds					
Maturity	**Coupon**	**Bid**	**Asked**	**Chg**	**Asked yield**
1/31/2020	1.250	99.45	99.47	0.0469	1.385
1/31/2020	1.375	99.93	99.95	0.0234	1.389
2/15/2020	3.625	108.82	108.84	0.0391	1.393
2/15/2020	8.500	128.23	128.25	0.0547	1.369
2/29/2020	1.250	99.38	99.39	0.0703	1.402
2/29/2020	1.375	99.91	99.93	0.0938	1.393
3/31/2020	1.125	98.84	98.85	0.1250	1.407
3/31/2020	1.375	99.82	99.84	0.0859	1.415
4/30/2020	1.125	98.80	98.81	0.1094	1.411
4/30/2020	1.375	99.76	99.77	0.0703	1.429
5/15/2020	3.500	108.56	108.58	0.0703	1.451
5/15/2020	8.750	130.61	130.63	0.1094	1.437
5/31/2020	1.375	99.66	99.68	0.1172	1.451
Coupon: Annual rate (%)					
Ask and bid prices: % of face value					
Chg: Daily change in asked price					
Asked yield: yield to maturity at asked price					





As seen in the previous figures, $\lambda = 0$ happens when the price is equal to the total cash flow:

$$P = F + \frac{nC}{m} \tag{1}$$

In addition, λ is equal to the coupon rate if the price is equal to face value:

$$P = F \tag{2}$$

If the yield λ increases, the price P of bonds with low coupon rates declines more than that of bonds with high coupon rates. The bigger early coupon payments are less affected by the rising interest rates. For bonds of longer maturity, the steeper price-yield curve reveals that such bonds are susceptible to interest rates.

4 Duration

In investment science, the duration consists of the weighted average of the times until those fixed cash flows are received. Duration can measure how long it takes, in years, for an investor to be repaid a bond's price by the bond's total cash flows. Duration can also measure the sensitivity of a bond's or fixed-income portfolio's price to changes in interest rates.

Considering the cash flow stream $(c_0, c_1, \ldots, c_n)$ which gives c_i at time $t_i, i = 0, \ldots, n$, the **duration** of this cash flow stream can be calculated as:

$$D = \frac{PV(t_0)t_0 + PV(t_1)t_1 + \cdots + PV(t_n)t_n}{PV},$$

where $PV(t_i), i = 0, 1, \ldots, n$ is the present value of c_i and $PV = PV(t_0) + \cdots + PV(t_n)$.

It can also reflect the bond's maturity such that if there are no coupons, the duration is equal to the maturity. If any coupon remains, the duration is smaller than the maturity. For two bonds with the same total cash flow (i.e., coupons + face value), the duration is shorter for the one with a higher coupon rate.

A particular duration case is obtained when YTM is used as the interest rate. For that, the duration calculation is:

$$\Rightarrow \quad D = \frac{\sum_{k=1}^{n} \frac{k}{m} \frac{c_k}{(1+\frac{\lambda}{m})^k}}{PV}, \text{ where}$$
$$PV = \sum_{k=1}^{n} \frac{c_k}{(1+\frac{\lambda}{m})^k}$$

For bonds, it goes even further as:

$$D = \frac{1+y}{my} - \frac{1+y+n(c-y)}{mc[(1+y)^n - 1] + my}, \text{ where}$$
$$m = \text{periods per year} \qquad n = \text{nr of periods left}$$
$$y = \lambda/m = \text{yield per period} \qquad c = \text{coupon rate}$$

Another particular case is modified duration, which expresses the measurable change in the value of a security in response to a change in interest rates. For example, assuming that the present value of cash flow c_k is:

$$PV_k = \frac{c_k}{[1+(\lambda/m)]^k}$$
$$\Rightarrow \frac{dPV_k}{d\lambda} = -\frac{(k/m)c_k}{[1+(\lambda/m)]^{k+1}} = -\frac{(k/m)PV_k}{1+(\lambda/m)}$$

This allows the price of a bond to be calculated as follows:

$$P = \sum_{k=1}^{n} PV_k$$
$$\Rightarrow \frac{dP}{d\lambda} = \sum_{k=1}^{n} -\frac{(k/m)PV_k}{1+(\lambda/m)} = -\frac{1}{1+(\lambda/m)} \frac{\sum_{k=1}^{n}(k/m)PV_k}{P} P$$
$$\Rightarrow \frac{dP}{d\lambda} = -\frac{1}{1+(\lambda/m)} DP = -D_M P,$$

where D_M is the **modified duration** $D_M = D/[1+(\lambda/m)]$

Bonds can also be managed in a portfolio, creating a collection consisting of a set of m bonds. The collective price P is composed of the sum of the individual bond prices $P_1 + P_2 + \cdots P_m$. The following theorem can calculate the duration of a portfolio:

Theorem 4.1. Duration of a portfolio

Suppose there are m fixed-income securities with prices and durations of P_i and D_i, respectively, $i = 1, 2, \ldots, m$, all computed using the same yield. Then, the price P and duration D of the portfolio consisting of the aggregate of these securities are:

$$P = P_1 + P_2 + \cdots + P_m$$
$$D = w_1 D_1 + w_2 D_2 + \cdots + w_m D_m,$$

where $w_i = P_i/P, i = 1, 2, \ldots, m$

Proof. Outline for the case of two securities A and B:

$$D^{A+B} = \sum_{k=0}^{n} \frac{PV_k^{A+B} t_k}{P^{A+B}}$$
$$D^{A} = \sum_{k=0}^{n} \frac{PV_k^{A} t_k}{P^{A}}$$
$$D^{B} = \sum_{k=0}^{n} \frac{PV_k^{B} t_k}{P^{B}}$$
$$\Rightarrow P^A D^A + P^B D^B = \sum_{k=0}^{n} t_k \left(PV_k^A + PV_k^B\right)$$

Divide both sides of equation by $P = P^{A+B} = P^A + P^B$:

$$\Rightarrow \frac{P^A D^A + P^B D^B}{P} = \frac{\sum_{k=0}^{n} t_k \left(PV_k^A + PV_k^B\right)}{P} = D$$
$$\Rightarrow D = \frac{P^A}{P} D^A + \frac{P^B}{P} D^B$$

By definition, the duration of a portfolio of A and B is:

$$D^{A+B} = \sum_{k=0}^{n} \frac{t_k PV_k^{A+B}}{P^{A+B}} = \sum_{k=0}^{n} \frac{t_k PV_k^{A+B}}{P}$$

Thus, $PV_k^{A+B} = PV_k^A + PV_k^B$ implies $D^{A+B} = D$

- Holds when payments from A and B are discounted with the same rate for each period k, assuming the same yield
- This assumption of identical interest rates does not hold for Macaulay duration, which uses YTM for each bond

5 Immunization

Immunization is the development of an investment strategy when either an investor has a liability stream (a cash flow stream the investor has to pay) that is sensitive to interest rates, or they want to construct an investment portfolio to match this liability stream both in terms of present value and interest rate sensibility.

The combination of the portfolio and the liability stream is immune e to (small) interest rate changes. In order to immunize a bond portfolio, it may be difficult or expensive in practice and most likely has to start with shortening other bonds.

The rule of thumb is:

> *Buy a portfolio of equal NPV whose interest rate sensitivity is the same as that of the liability stream being immunized.*

If there are zero coupon bonds with enough maturities, perfect matching cash flows is possible, revealing equivalence between interest rate sensibilities. However, such a task is quite difficult, considering zero coupons are rare. Alternatively, the duration can be used by approximating the interest rate sensitivity.

Consider the goal of immunizing a liability stream with duration D and price P. In addition, the bonds A and B are available for immunization. By buying A and B for total amount V_A and V_B (unit price times units bought), the duration can be calculated as:

$$\begin{aligned} P &= V_A + V_B \\ D &= w_A D_A + w_B D_B, \text{ where} \\ w_i &= \frac{V_i}{P}, i = A, B \end{aligned}$$

With more than two bonds, it helps diversify risk (of default) and create a system with many solutions due to the number of variables exceeding the number of equations.

6 Useful links

Some useful links that might further your experience in this course:

- Bloomberg bond rates
- Statistics on the Finnish central government debt
- Information on credit ratings
- Credit ratings of Finland
- Euro area yield curves

- Russia government bonds
- 10-year government bond spreads
- Debt structure of Stora Enso
- S&P credit rating of Stora Enso
- List of sovereign debt crises
- List of stock market crashes and bear markets