MSE2114 - Investment Science Lecturer Notes II

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1 Introduction

So far, we have explored the time value of money in fixed-income securities (mainly bonds). On the one hand, the coupon rates reflect the issuer's point of view, while the internal return rate justifies the bondholder's goals. However, the prevailing interest rate is implied by the yield to maturity (**YTM**), which is not the same for all bonds.

Based on previous graphics (in Lecture II), for long bonds (in terms of both duration and maturity), **YTM** tends to be higher than that of short ones. This cannot be fully explained by different levels of default risk by issuers (fundamental risk). At the same time, bonds with high duration have greater price volatility than bonds with short duration (market price risk). Therefore, there is a trade-off between fundamental risk vs. market price risk.

In real applications, there are many yield curves based on different instruments. In a particular case, when the coupon rates are zero, the resulting yield curve is referred to as **spot rate curve**.

2 Emergence of the term structure

As mentioned in previous lecturers, bonds can have varying maturities, and the issuer establishes that. There are many explanations beyond that:

- Long duration bonds are more **volatile** ⇒ greater market price risk, high volatility;
- Over a longer time, there is a **greater** risk that something unexpected happens and the issuer defaults for unforeseen reasons ⇒ more fundamental risk. For instance, bankruptcy;
- Correlation characteristics of short and long bonds can be **different** (e.g., long bonds may have a stronger correlation with stock markets ⇒ more systematic risk);
- Also influenced by expectations of changes in short-term interest rates over the life of the bond (i.e., if short-term interest rates continue to rise, this will imply higher interest rates for long bonds);
- Coupon rates might be **fluctuating**;
- Expectation hypothesis - markets believe in rising rates. Hence, preference for liquidity (convert assets to cash or acquire cash—through a loan or money in the bank—to pay its short-term obligations or liabilities):
  - Short bonds are considered more **liquid**: One can easily sell large amounts of bonds with minimal bid-ask spreads even in times of severe market volatility;
  - Also, if the market comes to a halt, then short bonds are better because they can be liquidated **faster**;
  - Keynes proposed :
    1. the transactions motive (=assure basic transactions);
    2. the precautionary motive (=prepare for unusual costs caused by unexpected problems);
    3. the speculative motive (=speculate that bond prices will fall).

3 Spot rates

Spot rates are rates (generally per year) s_t for a zero-coupon bond, considering the interest accumulated from the present until time t. It considers principal (or face value) and interest paid at the time t. Considering the compound (similar to compound interest) effect, the annualization of these rates is as follows:

- Yearly: $(1 + s_t)^t$
- m periods per year: $(1 + s_t/m)^{mt}$

- Continuous: $e^{s_t t}$

Only after analyzing the compounding effect can the correct impact of a spot rate be understood. The rate s_t/m is best understood as the "average" periodic (e.g., monthly) interest rate from the present to time t that is annualized *linearly* (i.e., s_t is multiplied by the constant coefficient $1/m$).

Now, let us focus on the following problem:

Why different compounding conventions?

Generally, coupon rates of bonds are calculated as a fraction of the principal. Hence, no compound interest is observed. In addition, the market convention is that if **YTM** = coupon rate, then the market price of the bond is 100% of face value. To ensure that the **YTM** of a non-coupon bond implies the same IRR as an equal **YTM** of a zero-coupon bond (i.e., spot rate), they must be quoted using the same convention.

The quoting conventions ensure that **YTM** has a well-defined meaning for market participants and that all bonds are compared on a like-for-like basis (typically setting them up to the same criteria).

The actual IRR of the bonds and the amount of interest that cash accrues (interest between coupon payments) annually will differ from non-annually compounded **YTM**s.

Annual compounding is a valuable convention because the spot rate then coincides with the actual IRR of the bonds and the true interest rate on cash.

For practical purposes, it may be sensible to employ annual compounding, making it easier for market participants to compare instruments.

To calculate **spot rates**, two main strategies can be applied. First, it is via **bootstrapping**. In computer science, mathematics and mainly statistics, bootstrapping is a procedure that resamples a single dataset to create many simulated samples. This process allows for calculating standard errors, constructing confidence intervals, and testing hypotheses for numerous sample statistics. In finance, it describes a situation in which an entrepreneur starts a company with little capital, relying on money other than outside investments. An individual is said to be bootstrapping when they attempt to find and build a company from personal finances or the operating revenues of the new company.

For instance:

- Assume we know s_1:
  - Implied by the **YTM** of a 1-year U.S. Treasury bill;
- Consider a 2-year bond with annual coupon payment C and face value F;
- Value of this bond is:

$$P = \frac{C}{1+s_1} + \frac{C+F}{(1+s_2)^2}$$

- When we know the market price of the bond P, we can solve for s_2 to determine the spot rate for 2 years;
- Continue similarly by considering $3, 4, 5, \ldots$ year bonds to determine $s_3, s_4, s_5, \ldots$;

The second method is through thorough **replication**. This procedure is slightly simplistic and allows the creation of a portfolio which pays no <u>net</u> coupon payments. Therefore, the spot rate must be the IRR of that portfolio, considering that there is no other like that in the market.

- Consider the portfolio in which 10 units of B are bought, and 8 units of A are sold;
- The total annual coupon payments 8 000 € received from B ($10 \cdot 10\ 000€ \cdot 8\%$) and paid to A ($8 \cdot 10\ 000€ \cdot 10\%$) cancel out ;
- At maturity in year 10, there is a cash flow of $10 \cdot 10\ 000$ € $- 8 \cdot 10\ 000$ € $= 20\ 000$ € ;

- The initial investment cost is $10 \cdot 8\,589$ € $- 8 \cdot 9\,872$ € $= 6\,914$ € ;

$$6\,914 \cdot (1+s_{10})^{10} = 20\,000 \Rightarrow s_{10} \approx 11.2\%$$

From different bonds, the **spot rates** can also be different. They can be estimated through different methods, including statistics procedures and varying through time. Similarly to **spot rates**, discount rates can be calculated through the same procedures above.

4 Forward rates

Forward rates are interest rates (generally annually) agreed upon today for a loan taken at i with maturity at j (assuming that $i \leq j$). Between forwards and **spot rates**, the following assumptions can be made:

1) Investing at the 2-year spot rate gives the growth factor $(1+s_2)^2$

2) Investing at the 1-year spot rate and then with the forward rate f_{12} gives the growth factor $(1+s_1)(1+f_{12})$

If arbitrage is not allowed, those growths are required to be equal, then:

$$(1+s_1)(1+f_{12}) = (1+s_2)^2 \Rightarrow f_{12} = \frac{(1+s_2)^2}{(1+s_1)} - 1 \tag{1}$$

In addition, the compounding factor can be expressed as:

A) With yearly compounding

$$(1+s_i)^i(1+f_{ij})^{j-i} = (1+s_j)^j$$
$$\Rightarrow f_{ij} = \left(\frac{(1+s_j)^j}{(1+s_i)^i}\right)^{\frac{1}{j-i}} - 1$$

B) With compounding m times per year

$$\left(1+\frac{s_i}{m}\right)^{mi}\left(1+\frac{f_{ij}}{m}\right)^{m(j-i)} = \left(1+\frac{s_j}{m}\right)^{mj}$$
$$\Rightarrow f_{ij} = m\left(\frac{(1+s_j/m)^j}{(1+s_i/m)^i}\right)^{\frac{1}{j-i}} - m$$

C) With continuous compounding

$$e^{s_i i}e^{f_{ij}(j-i)} = e^{s_j j}$$
$$\Rightarrow f_{ij} = \frac{s_j j - s_i i}{j-i}$$

Spot rates are a particular case of **forward rates** such that $i = 0$ and $j = i$, meaning that it is for a loan taken at time $i = 0$ and paid back at time i. Using this notation, we can assume that with n periods, there are $n(n+1)/2$ **forward rates** and n **spot rates**. Also, it allows **forward rates** to be used to forecast **spot rates**.

For instance, assuming the current **spot rates** are defined as $s_1, s_2, \ldots$. If the interest rates follow investors' expectations, the **spot rates** will equal the current **forward rates** in one year. Hence, $s'_1 = f_{12}, s'_2 = f_{13}, \ldots$. However, forecasts are obtained for a horizon one period shorter than the length of the initial time series. This means that in order to predict a value, it is required to lose at least one degree of freedom from the previous time series unless **spot rates** are assumed to be constant or increase/decrease with a given pattern. As with any other form of forecasting and prediction, it may differ from market to market and participant to participant. Nevertheless, due to supply and demand, risk aversion and volatility.

The invariance theorem is a vital theorem to aid in predicting the value for **forward rates** based on **spot rates** (and other interest). See below:

Theorem 4.1. Invariance Theorem

Suppose that interest rates evolve according to expectations dynamics and that interest is compounded annually. Then any sum invested at the interest rate for n years will grow by a factor of $(1+s_n)^n$ regardless of the investment strategy if all funds (including the initial sum and the accrued interest on investments) remain fully invested.

There are two ways to prove such a theorem. First, based on the description in Luenberger's textbook:

- **Proof**. Consider the two-period case $n = 2$:

  A If you invest in a 2-year zero-coupon bond, the growth is $(1 + s_2)^2$

  B If you invest twice in a row in 1-year zero-coupon bonds, the growth is $(1 + s_1)(1 + s_1') = (1 + s_2)^2$

  – By the expectation hypothesis $s_1' = f_{12}$

  – By the definition of **forward rates**, $(1 + s_1)(1 + f_{12}) = (1 + s_2)^2$

  ⇒ The investments A and B have the same growth factors

  – Note: Any other fixed-income investment can be formed as a combination of these two strategies

    * E.g, value of cash flow stream of a 2-year bond with coupon payments can be obtained by adding present values of two investments of type A and B so that the cash flows are:
      (i) the first B coupon after 1 year;
      (ii) the second B coupon and both face values after 2 years.

- The logic can be extended to any number of periods n

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Alternatively, we can use the definition of **PV** and **FV** as below:

- **Proof**.
  – Suppose that the investor invests the sum P in fixed income securities to obtain the cash flow stream $\mathbf{x} = (x_1, x_2, ..., x_n)$

  – We know that:

$$PV(\mathbf{x}) = \sum_{k=1}^{n} \frac{x_k}{(1 + s_k)^k} = P$$

  – When (i) the interest rates follow the expectation dynamics and (ii) the investor always invests all cash received (remains fully invested), the amount of cash at period n from this stream will be its **future value** using appropriate **forward rates**:

$$FV(\mathbf{x}) = x_n + \sum_{k=1}^{n-1} x_k(1 + f_{kn})^{(n-k)}$$

  – We know that the equation relates future value and present value:

$$FV(\mathbf{x}) = PV(\mathbf{x}) \cdot (1 + s_n)^n = P \cdot (1 + s_n)^n$$

  – (This can double-checked by solving the above two equations.)

  – Thus, the cash in period n is equal to $P \cdot (1 + s_n)^n$ regardless of what individual cash flows in $\mathbf{x}$ are.

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5 Short rates

Short rates are a particular type of forward rate in which the period is singular. For instance:v

$$r_k = f_{k,k+1} \tag{2}$$

Thus, it can be derived as:

$$
\begin{aligned}
1 + s_1 &= 1 + f_{01} = 1 + r_0 \\
(1 + s_2)^2 &= (1 + s_1)(1 + f_{12}) = (1 + r_0)(1 + r_1) \\
&\vdots \\
(1 + s_k)^k &= (1 + r_0)(1 + r_1) \cdots (1 + r_{k-1})
\end{aligned}
$$

In the following table, built with **forward rates**, different rates can be identified. The first row gives **spot rates**. Using those **spot rates**, the **forward rates** on the other rows can be calculated, while the rates in the diagonal values correspond to short rates.

		j				
	f ij	**1**	**2**	**3**	**4**	**5**
	0	3.00 %	3.40 %	3.60 %	3.70 %	3.80 %
	1		3.80 %	3.90 %	3.93 %	4.00 %
i	**2**			4.00 %	4.00 %	4.07 %
	3				4.00 %	4.10 %
	4					4.20 %

The present value can be updated by considering different lengths of n periods. Let $PV(k)$ = Present value (where present = period k) be the cash flows in the cash flow stream that occur in period k or later. Considering that $PV(n) = x_n$, future $PV(k)$ can be calculated via backward recursion as:

$$PV(k) = x_k + \frac{PV(k+1)}{1 + r_k},$$

where r_k = short rate at time k.

This **running present value** can also be written by using the discount factor $d_k = 1/(1 + r_k)$ as:

$$PV(k) = x_k + d_k PV(k+1)$$

6 Term structure and duration

In order to immunize a given investment, analyzing the parallel shifts in the yield rate curve can provide an alternative with small values of λ. In the following graph, the original spot rate curve is represented in the middle, while perturbance created by λ shows two alternatives (above and below). With that, it is possible to immunize the liabilities of an investor against such shifts for small values of λ.

Considering the definition of **duration** (as an "average" of the maturity of bonds), analogous results can be derived for fluctuating **spot rates**:

$$s_k \to s_k + \lambda \text{ for all } k,$$

Considering that s_t is a spot rate curve with $t_0 \leq t \leq t_n$, the duration of this cash flow can be calculated as:

$$D_{FW} = \frac{1}{PV}\sum_{i=1}^{n} t_i x_{t_i} e^{-s_{t_i} t_i}, \text{ where}$$
$$PV = \sum_{i=0}^{n} x_{t_i} e^{-s_{t_i} t_i},$$

This is known as the **Fisher-Weil duration**. For a particular case where the spot rate $s_k \to s_k + \lambda$, the duration can be expressed as follows:

$$
\begin{aligned}
P(\lambda) &= \sum_{i=0}^{n} x_{t_i} e^{-\left(s_{t_i}+\lambda\right)t_i} \\
\Rightarrow \left.\frac{dP(\lambda)}{d\lambda}\right|_{\lambda=0} &= -\sum_{i=0}^{n} t_i x_{t_i} e^{-s_{t_i} t_i} \\
&= -\frac{\sum_{i=0}^{n} t_i x_{t_i} e^{-s_{t_i} t_i}}{P(0)} P(0) \\
\Rightarrow \frac{dP(0)}{d\lambda} &= -D_{FW} P(0) \Leftrightarrow \frac{1}{P(0)}\frac{dP(0)}{d\lambda} = -D_{FW}
\end{aligned}
$$

For periodic compounding, we get the **quasi-modified duration** D_Q, where:

$$
\begin{aligned}
-P(\lambda) &= \sum_{k=0}^{n} x_k \left(1+\frac{s_k+\lambda}{m}\right)^{-k} \\
\Rightarrow \left.\frac{dP(\lambda)}{d\lambda}\right|_{\lambda=0} &= -\sum_{k=1}^{n} \frac{k}{m} x_k \left(1+\frac{s_k}{m}\right)^{-(k+1)}
\end{aligned}
$$

Dividing by $-P(0)$, we get the definition for D_Q as:

$$D_Q = -\frac{1}{P(0)}\frac{dP(0)}{d\lambda} = \frac{\sum_{k=0}^{n} \frac{k}{m} x_k \left(1+\frac{s_k}{m}\right)^{-(k+1)}}{\sum_{k=0}^{n} x_k \left(1+\frac{s_k}{m}\right)^{-k}}$$

For both duration formulas above, the duration of a portfolio remains as seen in the previous lectures:

$$
\begin{aligned}
P &= P_1 + P_2 + \cdots + P_m \\
D &= w_1 D_1 + w_2 D_2 + \cdots + w_m D_m,
\end{aligned}
$$

where $w_i = P_i/P, i = 1, 2, \ldots, m$ and P_i is price of bond i.

However, with **Macaulay duration** (presented in Lecture II), this property holds only on condition that the yield is the same for all bonds, which is one of its main limitations. On the other hand, quasi-modified duration is the most helpful duration metric in practice since most bonds do not pay continuously compounding interest and **spot rates** are typically quoted using the periodic frequency that matches the frequency of coupon payments of the bonds.