Aalto University
School of Science

MS-E2114 Investment Science
Lecture III: Term structure of interest rates

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Overview

This lecture

- We have determined the time value of money from fixed income securities (e.g., bonds)
  - The prevailing interest rate is implied by the yield to maturity (YTM)
- Yet YTM is not the same for all bonds
  - YTM of long bonds (in terms of both duration and maturity) tends to be higher than that of short ones
  - This cannot be fully explained by different levels of default risk by issuers (fundamental risk)
  - Bonds with high duration have greater price volatility than bonds with short duration (market price risk)
- The **yield curve** shows YTM as a function of time to maturity
  - In practice, there are many yield curves based on different instruments (e.g., German government bonds, bonds with lower rating, etc.)
  - The yield curve which is implied by zero-coupon bonds is called **spot rate curve**.

Overview

Term structure

Spot rates

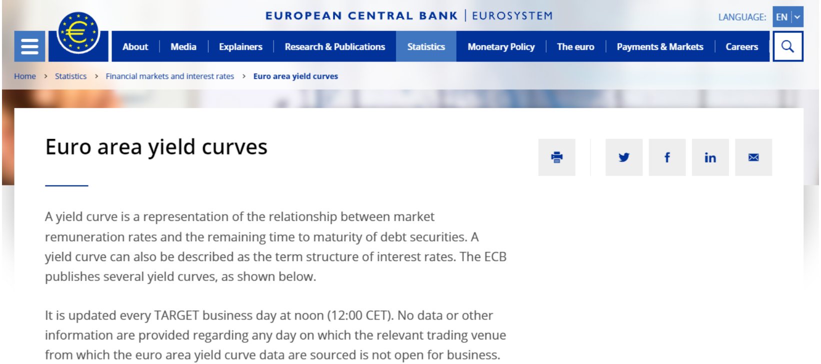
Forward rates

Short rates

Term structure and duration

Immunization

Euro area yield curves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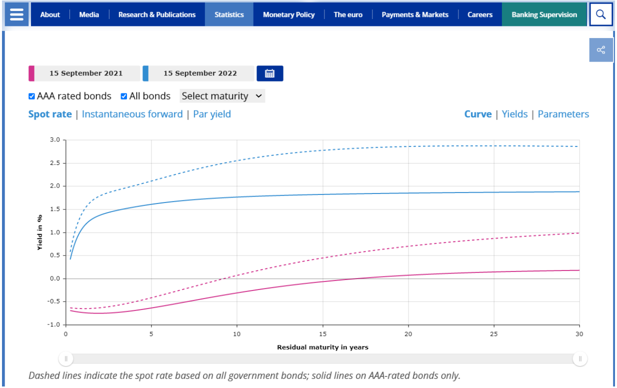
Home > Statistics > Financial markets and interest rates > **Euro area yield curves**

Euro area yield curves

A yield curve is a representation of the relationship between market remuneration rates and the remaining time to maturity of debt securities. A yield curve can also be described as the term structure of interest rates. The ECB publishes several yield curves, as shown below.

It is updated every TARGET business day at noon (12:00 CET). No data or other information are provided regarding any day on which the relevant trading venue from which the euro area yield curve data are sourced is not open for business.

Euro area yield curves



U.S. treasury bond yield curve

Select type of Interest Rate Data

Daily Treasury Par Yield Curve Rates

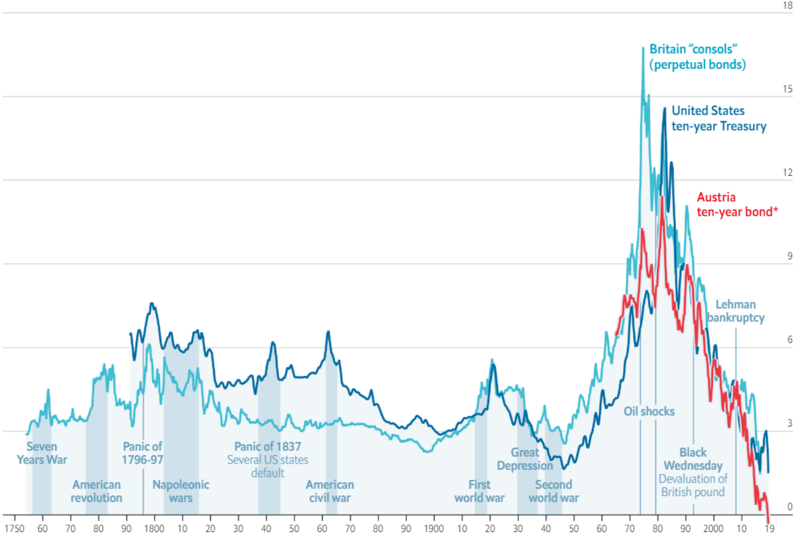
Select Time Period

Current Month

Apply

Date	1 Mo	2 Mo	3 Mo	6 Mo	1 Yr	2 Yr	3 Yr	5 Yr	7 Yr	10 Yr	20 Yr	30 Yr
09/01/2022	2.53	2.80	2.97	3.34	3.51	3.51	3.54	3.39	3.36	3.26	3.64	3.37
09/02/2022	2.49	2.79	2.94	3.33	3.47	3.40	3.44	3.30	3.29	3.20	3.61	3.35
09/06/2022	2.44	2.82	3.04	3.40	3.61	3.50	3.55	3.43	3.41	3.33	3.74	3.49
09/07/2022	2.30	2.80	3.07	3.42	3.60	3.45	3.50	3.37	3.35	3.27	3.67	3.42
09/08/2022	2.57	2.86	3.06	3.44	3.60	3.48	3.54	3.39	3.37	3.29	3.69	3.45
09/09/2022	2.57	2.88	3.08	3.52	3.67	3.56	3.61	3.45	3.42	3.33	3.71	3.47
09/12/2022	2.62	2.93	3.17	3.56	3.70	3.58	3.60	3.47	3.45	3.37	3.76	3.53
09/13/2022	2.55	2.95	3.28	3.75	3.92	3.75	3.75	3.58	3.53	3.42	3.75	3.51
09/14/2022	2.54	2.95	3.24	3.76	3.95	3.78	3.79	3.60	3.52	3.41	3.73	3.47
09/15/2022	2.76	3.03	3.22	3.78	4.00	3.87	3.85	3.66	3.59	3.45	3.75	3.48
09/16/2022	2.68	3.01	3.20	3.77	3.96	3.85	3.81	3.62	3.56	3.45	3.79	3.52

Government bond interest rates



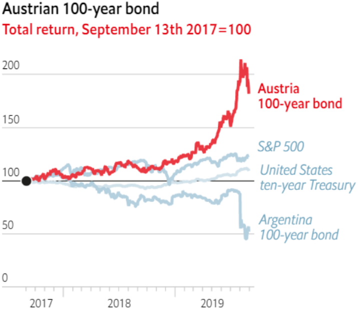
*Mix of different terms prior to 1990

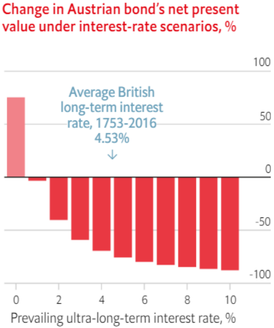
Austria’s 100-year bond has delivered stunning returns

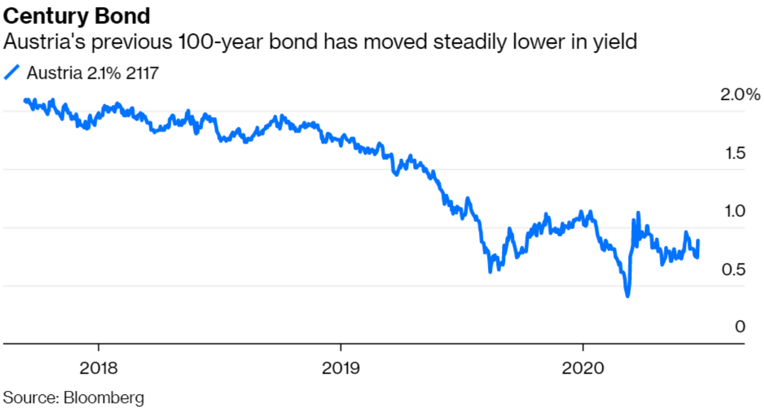
Its price will crash if interest rates rise. But most buyers won’t live long enough to regret it

Print edition | Graphic detail › Sep 12th 2019

Buyers of Austria’s 100-year bond are betting on a century of rock-bottom interest rates






Century Bond
Austria's previous 100-year bond has moved steadily lower in yield
Austria 2.1% 2117
2.0%
1.5
1.0
0.5
0
2018
2019
2020
Source: Bloomberg

An AA+ Rated Government Bond Down 55% Shows the Pain of Higher Rates

Austria's 100-year bonds are down 55% from their peak. *Photographer: Andrei Pungovschi/Bloomberg*

By Tracy Alloway
30. maaliskuuta 2022 klo 16.33 UTC+3

Investors in Austria's 100-year bonds are getting a crash course in duration risk.

Emergence of the term structure

- Why do the bond yields of the same issuer differ for different maturities?
- Possible explanations (several may apply):
  - Long duration bonds are more volatile ($\Rightarrow$ greater market price risk)
  - Over a longer time period, there is a greater risk that something unexpected happens and the issuer defaults for unforeseen reasons ($\Rightarrow$ more fundamental risk)
  - Correlation characteristics of short and long bonds can be different (e.g., long bonds may have a stronger correlation with stock markets $\Rightarrow$ more systematic risk)
  - Also influenced by expectations of changes in short-term interest rates over the life of the bond (i.e., if short term interest rates continue to rise, this will imply higher interest rates for long bonds)

Emergence of the term structure

- Possible explanations (continued, cf. course book):
  A. Expectation hypothesis - markets believe in rising rates
  B. Preference for liquidity
    - Short bonds are considered more **liquid**: One can easily sell large amounts of bonds with minimal bid-ask spreads even in times of severe market volatility
    - Also, if the market comes to a halt, then short bonds are better, because you get your money back sooner
    - Keynes proposed (i) the transactions motive (=assure basic transactions), (ii) the precautionary motive (=prepare for unusual costs caused by unexpected problems) and (iii) speculative motive (=speculate that bond prices will fall).

  C. Market segmentation
    - Market is segmented by bond maturities $\Longrightarrow$ Investors with different preferences buy and sell bonds in different market segments
    - E.g., investors who buy and sell 2-year bonds differ from those who are interested in 10-year bonds

Overview

Spot rates

- **Spot rate** s_t = rate of interest (p.a.) for a zero-coupon bond from the present to time t
  - Principal and interest paid at time t
- Annualization with different compounding intervals gives the following growth factors (t is the time in years)
  - Yearly: $(1 + s_t)^t$
  - m periods per year: $(1 + s_t/m)^{mt}$
  - Continuous: $e^{s_t t}$
- If you see a spot rate quoted in the market, you must know its compounding convention to know what it really means

Example: Periodically compounded spot rates

- The rate s_t/m is best understood as the "average" periodic (e.g., monthly) interest rate from the present to time t that is annualized *linearly* (i.e., s_t is multiplied by the constant coefficient $1/m$)
- For example, suppose that compounding is biannual ($m = 2$) and $s_1 = 8\%$
  - This means that, on average, cash accrues 4% interest every 6 months and thus 100 euro will accrue to $100 \cdot 1.04^2 = 108.16$ euro over one year
  - However, the spot for rate for the last half year, observed six months into the year, need not be 4% if interest rates change
- Thus biannually compounded spot rate of 8% is the same as annually compounded spot rate of 8.16%

Why different compounding conventions?

- Recall that, by definition, YTM is quoted based on periodic compounding
- This is because:
  - Coupon rates of bonds are calculated as a fraction of the principal
  - The market convention is that if YTM = coupon rate, then the market price of the bond is 100% of face value
- If spot rates were quoted with a convention differing from that of the bonds, a non-zero-coupon bond whose YTM equals the spot rate s_t might have a different IRR than the zero-coupon bond implying s_t

Why different compounding conventions?

- The quoting conventions ensure that YTM has a well-defined meaning for market participants and that all bonds are compared on a like-for-like basis
- Annual compounding is a useful convention, because the spot rate then coincides with the actual IRR of the bonds and the true interest rate on cash
- For practical purposes, it may be sensible to employ annual compounding which makes it easier for market participants to compare instruments

Determining the spot rates

- **Method A**: *Bootstrapping*
  - Assume we know s_1
    - Implied by the YTM of a 1-year U.S. Treasury bill
  - Consider a 2-year bond with annual coupon payment C and face value F
  - Value of this bond is

$$P = \frac{C}{1+s_1} + \frac{C+F}{(1+s_2)^2}$$

  - When we know the market price of the bond P, we can solve for s_2 to determine the spot rate for 2 years
  - Continue similarly by considering $3, 4, 5, \ldots$ year bonds to determine $s_3, s_4, s_5, \ldots$

Determining the spot rates

- **Method B**: *Replication*
  - Idea: Construct a portfolio which pays no net coupon payments
    $\Rightarrow$ Spot rate must be the IRR of this portfolio
  - This is the same as constructing a (synthetic) zero-coupon bond if there is not one immediately available in the markets
- Example:

Bond	**Maturity (y)**	**Face value**	**Coupon rate**	**Price (€)**
A	10	10 000	10%	9 872
B	10	10 000	8%	8 589

Determining the spot rates

- Example (cont'd):

Bond	Maturity (y)	Face value	Coupon rate	Price (€)
A	10	10 000	10%	9 872
B	10	10 000	8%	8 589

  - Consider the portfolio in which 10 units of B are bought and 8 units of A are sold
  - The total annual coupon payments 8 000 € received from B ($10 \cdot 10\,000$€ $\cdot\, 8\%$) and paid to A ($8 \cdot 10\,000$€ $\cdot\, 10\%$) cancel out
  - At maturity in year 10, there is a cash flow of $10 \cdot 10\,000$ € $-\, 8 \cdot 10\,000$ € $= 20\,000$ €
  - The initial investment cost is $10 \cdot 8\,589$ € $-\, 8 \cdot 9\,872$ € $= 6\,914$ €

$$6\,914 \cdot (1 + s_{10})^{10} = 20\,000 \Rightarrow s_{10} \approx 11.2\%$$

Determining the spot rates

- Spot rates derived from different bonds can differ from each other
  - Spot rates can be estimated with many methods and bonds
  - Averaging and other statistical methods can be used for improved accuracy

Time-dependent discounting factors

- Spot rates are different for different periods
  $\Rightarrow$ Discount rates are different, too
- PV of (annual) cash flow stream $(x_0, x_1, \dots, x_n)$ is

$$PV = x_0 + d_1 x_1 + d_2 x_2 + \cdots + d_n x_n,$$

  where the discounts factors are

$$\begin{aligned} d_t &= \frac{1}{(1+s_t)^t} && \text{yearly} \\ d_t &= \frac{1}{(1+s_t/m)^{mt}} && m \text{ times per year} \\ d_t &= e^{-s_t t} && \text{continuous} \end{aligned}$$

When you see a spot rate s_t, you have to know also its compounding convention to pick the right formula. Otherwise, the meaning of the spot rate is not well-defined. It could be based on any one of the above formulas.

Example: Spot rates

- What is the value of a 5-year bond with 8% coupon rate and face value 100 € ?

=1/(1+0.05571)

Time t	s_t	d_t	x_t	PV
1	0.05571	0.9472	8	7.57784
2	0.06088	0.8885	8	7.10816
3	0.06555	0.8266	8	6.61254
4	0.06978	0.7635	8	6.10818
5	0.07361	0.7011	108	75.7188
Total				103.1255

Overview

Term structure

Spot rates

Forward rates

Short rates

Term structure and duration

Immunization

Forward rates

- **Forward rate** f_{ij} = interest rate (p.a.) agreed upon today for a loan taken at time i and paid back at time $j, i < j$
- Spot rates imply corresponding forward rates
  1) Investing at the 2-year spot rate gives the growth factor $(1+s_2)^2$
  2) Investing at the 1-year spot rate and then with forward rate f_{12} gives the growth factor $(1+s_1)(1+f_{12})$
  - Because there cannot be arbitrage, these growth factors must be equal
    $\Rightarrow (1+s_1)(1+f_{12}) = (1+s_2)^2$

Forward rates

- Thus we have

$$f_{12} = \frac{(1+s_2)^2}{(1+s_1)} - 1$$

- The forward rate f_{ij} depends on the compounding convention:

A) With yearly compounding

$$(1+s_i)^i(1+f_{ij})^{j-i} = (1+s_j)^j$$
$$\Rightarrow f_{ij} = \left(\frac{(1+s_j)^j}{(1+s_i)^i}\right)^{\frac{1}{j-i}} - 1$$

Forward rates

B) With compounding m times per year

$$\left(1+\frac{s_i}{m}\right)^{mi}\left(1+\frac{f_{ij}}{m}\right)^{m(j-i)}=\left(1+\frac{s_j}{m}\right)^{mj}$$

$$\Rightarrow f_{ij}=m\left(\frac{(1+s_j/m)^j}{(1+s_i/m)^i}\right)^{\frac{1}{j-i}}-m$$

C) With continuous compounding

$$e^{s_i i}e^{f_{ij}(j-i)}=e^{s_j j}$$

$$\Rightarrow f_{ij}=\frac{s_j j-s_i i}{j-i}$$

Spot vs. forward rates

- The spot rate s_i is a special case of a forward rate: $s_i = f_{0i}$
- With n periods, there are $n(n+1)/2$ forward rates and n spot rates
  - For $n = 3$, the spot rates s_i and forward rates f_{ij} are

f_{01}	f_{02}	f_{03}
	f_{12}	f_{13}
		f_{23}

Example: Forward rates

- From the spot rates below, we determine the forward rates as follows:

i	1	2	3	4
s_i	3.00 %	3.40 %	3.60 %	3.70 %

$$f_{i,j} = \left(\frac{(1+s_j)^j}{(1+s_i)^i}\right)^{\frac{1}{j-i}} - 1$$

		j				
	f_{ij}	1	2	3	4	5
	0	3.00 %	3.40 %	3.60 %	3.70 %	3.80 %
	1		3.80 %	3.90 %	3.93 %	4.00 %
i	2			4.00 %	4.00 %	4.07 %
	3				4.00 %	4.10 %
	4					4.20 %

Forecasting spot rates with forward rates

- Let the current spot rates be $s_1, s_2, \ldots$
- What will the spot rates $s'_1, s'_2, \ldots$ in a year's time?
- If the interest rates follow investors' expectations, in one year the spot rates will equal the current forward rates!
- Hence $s'_1 = f_{12}, s'_2 = f_{13}, \ldots$

Forecast for next years' forward rate table

		j				
	f_{ij}	**1**	**2**	**3**	**4**	**5**
	0	~~3.00 %~~	~~3.40 %~~	~~3.60 %~~	~~3.70 %~~	~~3.80 %~~
	1		3.80 %	3.90 %	3.93 %	4.00 %
i	**2**			4.00 %	4.00 %	4.07 %
	3				4.00 %	4.10 %
	4					4.20 %

Forecasting spot rates with forward rates

- Yet the updated table has one column/row less than the previous one
  $\Rightarrow$ Spot forecast derived from $s_1, s_2, \ldots, s_k$ gives $s'_1, s'_2, \ldots, s'_{k-1}$
  - Forecasts are obtained for a horizon which is one period shorter than the length of the initial time series
- But also other forecasts could be made e.g. by assuming that
  - The spot rates stay constant
  - The spot rates increase by a constant/proportional amount
- Yet arbitrage opportunities must be eliminated
- Forecasts for spot rates by some market participants and those implied by forward rates may differ
- But there are reasons for why the rates reach an equilibrium even if many forecasts differ from forward rates (e.g., supply and demand, risk aversion)

Invariance theorem

Theorem

*(**Invariance theorem**) Suppose that interest rates evolve according to expectations dynamics and that interest is compounded annually. Then any sum invested at the interest rate for n years will grow by a factor of $(1 + s_n)^n$ regardless of the investment strategy if all funds (including the initial sum and the accrued interest on investments) remain fully invested.*

Invariance theorem - Proof from Luenberger's textbook

- **Proof**. Consider the two period case $n = 2$:
  - A If you invest in a 2-year zero-coupon bond, the growth is $(1 + s_2)^2$
  - B If you invest twice in a row in 1-year zero-coupon bonds, the growth is $(1 + s_1)(1 + s_1') = (1 + s_2)^2$
  - By the expectation hypothesis $s_1' = f_{12}$
  - By the definition of forward rates, $(1 + s_1)(1 + f_{12}) = (1 + s_2)^2$
  - $\Rightarrow$ The investments A and B have the <u>same</u> growth factors
  - Note: Any other fixed income investment can be formed as a combination of these two strategies
    - E.g, value of cash flow stream of a 2-year bond <u>with</u> coupon payments can be obtained by adding present values of two investments of type A and B so that the cash flows are:
      (i) the first B coupon after 1 year
      (ii) the second B coupon and both face values after 2 years
- The logic can be extended to any number of periods n $\square$

Invariance theorem - Alternative proof

- **Proof**.
  - Suppose that the investor invests the sum P in fixed income securities to obtain the cash flow stream $\mathbf{x} = (x_1, x_2, ..., x_n)$
  - We know that

$$PV(\mathbf{x}) = \sum_{k=1}^{n} \frac{x_k}{(1+s_k)^k} = P$$

  - When (i) the interest rates follow the expectation dynamics and (ii) the investor always invests all cash received (remains fully invested), the amount of cash at period n from this stream will be its **future value** using appropriate forward rates:

$$FV(\mathbf{x}) = x_n + \sum_{k=1}^{n-1} x_k (1 + f_{kn})^{(n-k)}$$

Invariance theorem - Alternative proof

- **Proof (cont'd)**.
  - We know that future value and present value are related by the equation

$$FV(\mathbf{x}) = PV(\mathbf{x}) \cdot (1 + s_n)^n = P \cdot (1 + s_n)^n$$

  - (You can double-check by solving the above two equations.)
  - Thus, the cash in period n is equal to $P \cdot (1 + s_n)^n$ regardless of what individual cash flows in $\mathbf{x}$ are. $\square$

Overview

Short rates

- **Short rate** r_k **=** is the forward rate for a single period
  - $r_k = f_{k,k+1}$
- Short rates can be derived from

$$
\begin{aligned}
1 + s_1 &= 1 + f_{01} = 1 + r_0 \\
(1 + s_2)^2 &= (1 + s_1)(1 + f_{12}) = (1 + r_0)(1 + r_1) \\
&\vdots \\
(1 + s_k)^k &= (1 + r_0)(1 + r_1) \cdots (1 + r_{k-1})
\end{aligned}
$$

Spot vs short vs forward rates

- Consider the forward rate table
  - The first row of forward rate table gives spot rates
  - Spot rates define the forward rates on the other rows of the table
  - Short rates are on the diagonal of the forward rate table

		j				
	f_{ij}	**1**	**2**	**3**	**4**	**5**
	0	3.00 %	3.40 %	3.60 %	3.70 %	3.80 %
	1		3.80 %	3.90 %	3.93 %	4.00 %
i	**2**			4.00 %	4.00 %	4.07 %
	3				4.00 %	4.10 %
	4					4.20 %

Running present value

- Present value of a cash flow stream with a length of n periods can be computed as follows:
  - $PV(k)$ = Present value (where present = period k) of all cash flows in the cash flow stream that occur in period k or later
  - Set $PV(n) = x_n$
  - For $k = n-1, n-2, \ldots, 1, 0$, calculate the backward recursion

$$PV(k) = x_k + \frac{PV(k+1)}{1+r_k},$$

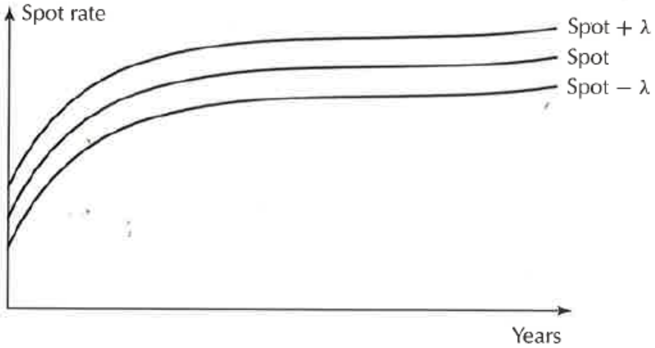
    where r_k = short rate at time k

- This **running present value** can also be written by using the discount factor $d_k = 1/(1+r_k)$ as

$$PV(k) = x_k + d_k PV(k+1)$$

Overview

Parallel shifts in spot rate curve



- The original spot rate curve is the middle curve
- This curve is shifted upward and downward by an amount λ to obtain the other curves
- It is possible to immunize liabilities of an investor against such shifts for small values of λ

Fisher-Weil duration

- Lecture 2: Duration is a measure of the sensitivity to interest rate changes
- We next derive analogous results for changing spot rates

$$s_k \to s_k + \lambda \text{ for all } k,$$

- We first consider the case of continuous compounding
- Given the spot rate curve $s_t, t_0 \leq t \leq t_n$, the **Fisher-Weil duration** for the cash flow stream $(x_{t_0}, x_{t_1}, \ldots, x_{t_n})$ is

$$D_{FW} = \frac{1}{PV} \sum_{i=1}^{n} t_i x_{t_i} e^{-s_{t_i} t_i}, \text{ where}$$

$$PV = \sum_{i=0}^{n} x_{t_i} e^{-s_{t_i} t_i},$$

Fisher-Weil duration

- When $s_k \to s_k + \lambda$, we have

$$
\begin{aligned}
P(\lambda) &= \sum_{i=0}^{n} x_{t_i} e^{-\left(s_{t_i}+\lambda\right)t_i} \\
\Rightarrow \left.\frac{dP(\lambda)}{d\lambda}\right|_{\lambda=0} &= -\sum_{i=0}^{n} t_i x_{t_i} e^{-s_{t_i} t_i} \\
&= -\frac{\sum_{i=0}^{n} t_i x_{t_i} e^{-s_{t_i} t_i}}{P(0)} P(0) \\
\Rightarrow \frac{dP(0)}{d\lambda} &= -D_{FW} P(0) \Leftrightarrow \frac{1}{P(0)} \frac{dP(0)}{d\lambda} = -D_{FW}
\end{aligned}
$$

Quasi-modified duration

- For periodic compounding, we get the **quasi-modified duration** D_Q
- Let the spot rate in period k be s_k and the cash flow stream be $(x_0, x_1, \ldots, x_n)$ (with periodical indexing)

$$-P(\lambda) = \sum_{k=0}^{n} x_k \left(1 + \frac{s_k + \lambda}{m}\right)^{-k}$$

$$\Rightarrow \left.\frac{dP(\lambda)}{d\lambda}\right|_{\lambda=0} = -\sum_{k=1}^{n} \frac{k}{m} x_k \left(1 + \frac{s_k}{m}\right)^{-(k+1)}$$

- Dividing by $-P(0)$, we get the definition for D_Q as

$$D_Q = -\frac{1}{P(0)} \frac{dP(0)}{d\lambda} = \frac{\sum_{k=0}^{n} \frac{k}{m} x_k \left(1 + \frac{s_k}{m}\right)^{-(k+1)}}{\sum_{k=0}^{n} x_k \left(1 + \frac{s_k}{m}\right)^{-k}}$$

Duration of a portfolio

- A convenient property of Fisher-Weil duration and quasi-modified duration is that the duration of a portfolio D is equal to the present-value-weighted sum of the durations D_i of the bonds in the portfolio

$$P = P_1 + P_2 + \cdots + P_m$$
$$D = w_1 D_1 + w_2 D_2 + \cdots + w_m D_m,$$

  where $w_i = P_i/P, i = 1, 2, \ldots, m$ and P_i is price of bond i.
- With Macaulay duration (see Lecture 2, slide 28), this property holds only on condition that the yield is the same for all bonds
- Quasi-modified duration is the most useful duration metric in practice, since most bonds do not pay continuously compounding interest and spot rates are typically quoted using the periodic frequency that matches the frequency of coupon payments of the bonds

Overview

Term structure

Spot rates

Forward rates

Short rates

Term structure and duration

Immunization

Example: Immunization

- A firm has to pay 1 million € in 5 years
- It seeks protection from interest rate risk through immunization
  - Both bonds A and B have face values of 100 €

Bond	Maturity (y)	Coupon rate	Price (€)
A	12	6 %	65.95
B	5	10 %	101.66

Bonds priced according to spot rates

S 1	S 2	S 3	S 4	S 5	S 6
7.67	8.27	8.81	9.31	9.75	10.16
S 7	S 8	S 9	S 10	S 11	S 12
10.52	10.85	11.15	11.42	11.67	11.89

Example: Immunization

TABLE 4.4
Worksheet for Immunization Problem

Year	Spot	d	B_1	PV_1	$-PV'_1$	B_2	PV_2	$-PV'_2$
1	7 67	929	6	5 57	5 18	10	9.29	8.63
2	8 27	853	6	5 12	9 45	10	8.53	15 76
3	8 81	776	6	4 66	12.84	10	7 76	21 40
4	9 31	700	6	4.20	15.38	10	7 00	25.63
5	9 75	628	6	3 77	17.17	110	69 08	314 73
6	10 16	560	6	3.36	18.29			
7	10 52	496	6	2.98	18.87			
8	10 85	439	6	2.63	18.99			
9	11 15	386	6	2.32	18.76			
10	11 42	339	6	2 03	18.26			
11	11 67	297	6	1.78	17.55			
12	11 89	260	106	27.53	295 26			
Total				65.95	466 00		101 66	386 15
Duration					7 07			3 80

The present values and durations of two bonds are found as transformations of cash flows

Example: Immunization

- Based on spot rates, the quasi-modified durations can be computed as

	NPV	D_Q
Liability L	628 000 €	4.56
Bond A	65.95 €	7.07
Bond B	101.66 €	3.80

- For immunization, one needs to buy z_A units of bond A and z_B units of bond B so that

$$\begin{cases} P_A z_A + P_B z_B & = PV_L \\ D_A P_A z_A + D_B P_B z_B & = D_L PV_L \end{cases}$$

$$\Rightarrow z_A = 2208, \quad z_B = 4745 \text{ rounded}$$

Example: Immunization

TABLE 4.5
Immunization Results

	Lambda		
	0	1%	−1%
Bond 1			
Shares	2,208.00	2,208.00	2,208.00
Price	65.94	51.00	70.84
Value	145,602.14	135,805.94	156,420.00
Bond 2			
Shares	4,744.00	4,744.00	4,744.00
Price	101.65	97.89	105.62
Value	482,248.51	464,392.47	501,042.18
Obligation value	627,903.01	600,063.63	657,306.77
Bonds minus obligation	−$52.37	$134.78	$155.40

The overall portfolio of bonds and obligations is immunized against parallel shifts in the spot rate curve.

Overview

Term structure

Spot rates

Forward rates

Short rates

Term structure and duration

Immunization