MSE2114 - Investment Science
Lecturer Notes V

Fernando Dias

October 9, 2023

Contents

1 Introduction

In this course so far, we have studied how to construct efficient portfolios, especially via portfolio optimisation applied to one and fund theorems. Those directly applied the optimisation model described by Nobel Prize winner Harry Markowitz. In addition, we found out how to obtain optimal values for w_i under the assumption of random returns.

In this lecture, we expand the mean-variance portfolio framework by adding a capital asset pricing model (CAPM).

2 Capital asset pricing model

Recalling that, according to **one-fund theorem**, whenever there is a risk-free asset, a mean-variance investor invests a combination of a fund F and the risk-free asset, which is mathematically proved to be the most efficient.

Assuming such a condition, the expected rate $\bar{r}$ can be expressed as a function of the interest rate from the fund F and the standard variation of the random returns in a portfolio:

$$\bar{r} = r_f + \frac{\bar{r}_F - r_f}{\sigma_F}\sigma$$

In CAPM, the main assumption is that the **whole market** is treated as a fund, and all investors invest in that **single** and **particular** fund. This market fund M is also known as **market portfolio**, or simply, **market**.

If the following assumptions hold, all investors in the market build the same one-fund:

1. All investors are one-period mean-variance investors
2. All investors have the same estimates of means, variances and covariances for each asset
3. All investors can borrow and lend at the risk-free interest rate without limit

However, not all estimates can be identical. For example, supply and demand affect assets and prices, leading to changes in estimations in a domino effect. Larger institutions are important because they slowly drive the market into **equilibrium**. In this scenario, we assume that, in the one-fund theorem, $F = M$.

This assumption leads to many extra consequences. First, any efficient portfolio is a **combination** of the risk-free asset and the market. Therefore, the interest rate (**capital market line**) is calculated as:

$$\bar{r} = r_f + \frac{\bar{r}_M - r_f}{\sigma_M}\sigma$$

The ratio $\frac{\bar{r}_M - r_f}{\sigma_M}$ is called the (market) **price of risk**.

In any portfolio, each asset has its weight. Considering the scenario of equilibrium, the weights are capitalised to reflect how large is a company's market capitalisation compared to the entirety of the market.

In the one-fund theorem, the problem can be solved for the optimal portfolio weights by knowing the expected returns, variances and covariances. However, with optimal weights, the earlier parameters can be obtained. Specifically, when the problem is solved for the expected returns based on capitalisation weights, variances and covariances, the resulting formulation is known as **Capital Asset Pricing Model** (**CAPM**) (a reversed direction in the one-fund theorem).

Consider the following theorem:

Theorem 2.1. Capital asset pricing model

If the market portfolio M is efficient, the expected return $\bar{r}_i$ of any asset i satisfies

$$\bar{r}_i - r_f = \beta_i(\bar{r}_M - r_f),$$

where

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}.$$

The term $\bar{r}_i - r_f$ is the **expected excess rate of return** of asset i. Likewise, the term $\bar{r}_M - r_f$ is the expected excess rate of return of the market portfolio. The multiplier:

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

is called the asset's **beta**.

For the model, consider the proof proposed by Luenberger: For any α, consider investing a portion α in asset i and $1-\alpha$ in the market portfolio M. The expected return and volatility of this portfolio are:

$$\begin{aligned}
\bar{r}_\alpha &= \alpha\bar{r}_i + (1-\alpha)\bar{r}_M \\
\sigma_\alpha &= \sqrt{\alpha^2\sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha)^2\sigma_M^2}
\end{aligned}$$

Curve depicted by the pairs $(\sigma_\alpha, \bar{r}_\alpha), \alpha \geq 0$ that parallel the capital market line at $\alpha = 0$. The tangent of this line can be computed from the following differentials:

$$\begin{aligned}
\frac{d\bar{r}_\alpha}{d\alpha} &= \frac{d}{d\alpha}\left(\alpha\bar{r}_i + (1-\alpha)\bar{r}_M\right) = \bar{r}_i - \bar{r}_M \\
\frac{d\sigma_\alpha}{d\alpha} &= \frac{1}{2\sigma_\alpha}\left(2\alpha\sigma_i^2 + 2(1-2\alpha)\sigma_{iM} - 2(1-\alpha)\sigma_M^2\right) \\
\left.\frac{d\sigma_\alpha}{d\alpha}\right|_{\alpha=0} &= \frac{1}{2\sigma_\alpha}\left(2\sigma_{iM} - 2\sigma_M^2\right) = \frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}
\end{aligned}$$

When $\alpha = 0$, the curve must be parallel to the capital market line:

$$\begin{aligned}
\left.\frac{d\bar{r}_\alpha}{d\sigma_\alpha}\right|_{\alpha=0} &= \left.\frac{d\bar{r}_\alpha}{d\alpha}\frac{d\alpha}{d\sigma_\alpha}\right|_{\alpha=0} = (\bar{r}_i - \bar{r}_M)\left(\frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}\right)^{-1} = \frac{\bar{r}_M - r_f}{\sigma_M} \\
\Rightarrow \bar{r}_i - \bar{r}_M &= \frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}\frac{\bar{r}_M - r_f}{\sigma_M} = \left(\frac{\sigma_{iM}}{\sigma_M^2} - 1\right)(\bar{r}_M - r_f) \\
&= \frac{\sigma_{iM}}{\sigma_M^2}(\bar{r}_M - r_f) - \bar{r}_M + r_f \\
\Rightarrow \bar{r}_i - r_f &= \frac{\sigma_{iM}}{\sigma_M^2}(\bar{r}_M - r_f),
\end{aligned}$$

which completes the proof. 😎

An alternative proof would be to recall the optimality conditions for the one-fund theorem:

$$\begin{aligned}
0 &= \frac{\partial}{\partial w_k}\frac{\sum_{i=1}^n w_i(\bar{r}_i - r_f)}{\sqrt{\sum_{i=1}^n\sum_{j=1}^n w_i w_j \sigma_{ij}}}, \quad k = 1, 2, \ldots, n \\
\Rightarrow \bar{r}_k - r_f &= \frac{\sum_{i=1}^n w_i(\bar{r}_i - r_f)}{\sum_{i=1}^n\sum_{j=1}^n w_i w_j \sigma_{ij}}\sum_{i=1}^n w_i\sigma_{ik}, \quad k = 1, 2, \ldots, n
\end{aligned}$$

Since the market fund is:

$$r_M = \sum_{i=1}^n w_i r_i$$

the following can be calculated:

$$\begin{aligned}
\bar{r}_M &= \sum_{i=1}^n w_i\bar{r}_i, \quad \bar{r}_M - r_f = \sum_{i=1}^n w_i(\bar{r}_i - r_f) \\
\sigma_M^2 &= Var\left[r_M\right] = Var\left[\sum_{i=1}^n w_i r_i\right] = \sum_{i=1}^n\sum_{j=1}^n w_i w_j \sigma_{ij} \\
\sigma_{kM} &= Cov\left[r_k, r_M\right] = Cov\left[r_k, \sum_{i=1}^n w_i r_i\right] = \sum_{i=1}^n w_i\sigma_{ik}
\end{aligned}$$

And,

$$\bar{r}_k - r_f = \frac{\sum_{i=1}^{n} w_i(\bar{r}_i - r_f)}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}} \sum_{i=1}^{n} w_i \sigma_{ik}, \quad k = 1, 2, \ldots, n$$

$$\Rightarrow \bar{r}_k - r_f = \frac{\bar{r}_M - r_f}{\sigma_M^2} \sigma_{kM}, \quad k = 1, 2, \ldots, n$$

By rearranging the terms and denoting:

$$\beta_k = \frac{\sigma_{kM}}{\sigma_M^2},$$

The final equation is:

$$\bar{r}_k - r_f = \beta_k \left(\bar{r}_M - r_f\right), \quad k = 1, 2, \ldots, n$$ 😎

Keys assumptions to be remembered in the capital asset pricing model:

1. Investors consider only the expected return and the variance of returns;
2. All investors have the same estimates concerning asset parameters (expected returns, covariances);
3. Unlimited lending/borrowing at the risk-free rate;
4. There are no transaction costs;
5. There are no taxes;
6. Any quantity of an asset can be purchased;
7. Individual investors cannot influence prices;
8. Unlimited shorting of risky assets is possible;
9. All assets have efficient markets.

The beta can be interpreted as standard-deviation-scaled correlation to market:

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2} = \rho_{iM} \frac{\sigma_i}{\sigma_M}$$

where $\rho_{iM} = \sigma_{iM}/(\sigma_i \sigma_M)$ is the correlation coefficient between asset i and the market.

If the return of the market portfolio r_M is interpreted as a **factor** influencing the return on the asset r_i, then the beta can be also interpreted as a **factor loading** of r_M (i.e, the multiplier used with factor r_M):

$$r_i = r_f + \beta_i(r_M - r_f) + \varepsilon_i = r_f(1 - \beta_i) + \beta_i r_M + \varepsilon_i$$

The beta can be interpreted as a standard-deviation-scaled correlation to the market:

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2} = \rho_{iM} \frac{\sigma_i}{\sigma_M}$$

where $\rho_{iM} = \sigma_{iM}/(\sigma_i \sigma_M)$ is the correlation coefficient between asset i and the market.

If the return of the market portfolio r_M is interpreted as a **factor** influencing the return on the asset r_i, then the beta can be also interpreted as a **factor loading** of r_M (i.e, the multiplier used with factor r_M):

$$r_i = r_f + \beta_i(r_M - r_f) + \varepsilon_i = r_f(1 - \beta_i) + \beta_i r_M + \varepsilon_i$$

The beta of an asset that is uncorrelated with other assets ($\sigma_{ij} = 0, i \neq j$) is proportional to its variance σ_i^2 and weight w_i in the market:

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2} = w_i \frac{\sigma_i^2}{\sigma_M^2}$$

The beta of the asset is **not** 0 because the asset is part of the market portfolio, hence the dependence on its weight. A zero-beta asset must be negatively correlated with other assets in the market portfolio. An uncorrelated asset with a return equal to the risk-free rate but a positive variance is strictly inferior to the risk-free asset. It thus cannot be part of the optimal portfolio for a mean-variance optimiser.

The beta of an uncorrelated asset must be higher than 0 if it is part of the market portfolio. The return of an uncorrelated asset must be higher than the risk-free interest rate (assuming positive weight). If an uncorrelated asset has a return that is very close to the risk-free interest rate, then either its weight in the market fund must be very small (variance is diversified away), or its variance must be very small (getting close to the risk-free asset).

In addition, the beta β_i has a seemingly surprising impact on price:

- If β_i is large, there is a large risk premium:
  - The asset is not suitable for diversification;
  - Possibly a large weight in the market;
  - If $\beta_i > 1$, we know that $\sigma_i > \sigma_M$, because correlation is limited to 1.
- If $\beta_i = 0$, there is no risk premium:
  - All asset risk can be eliminated, including the fact that the asset is part of the market;
- If $\beta_i < 0$, the risk premium is negative:
  - Correlation with the market is negative: The value of the asset tends to increase when the markets go down;
  - The asset has insurance value, even if the variance is high;
  - If $\beta_i < -1$, we know that $\sigma_i > \sigma_M$, because correlation is limited to -1.

The value of beta and alpha are not the only guidance to decide an investment selection. The theory implies that an investor should always invest in a combination of the market fund and the risk-free asset; this will be optimal. When buying a fund or asset, the theory assumes that an investor will make additional investments to have the market portfolio in the end, which ultimately justifies the expected returns of assets based on their betas. Otherwise, a portfolio will not be efficient and will not fall on the capital market line.

CAPM provides a framework for the reasonable pricing of individual assets, which may help identify mispriced assets in practice and define the pricing of new assets introduced to the market. As a factor model, CAPM explains the relation between asset prices and their correlation with the market: many stocks (but not all) tend to correlate with the market. Historically, this was one of the results that made CAPM interesting.

For a portfolio, β can be expressed as:

$$
\begin{aligned}
Cov\left[\sum_{i=1}^{n} w_i r_i, r_M\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{n} w_i r_i - \sum_{i=1}^{n} w_i \bar{r}_i\right)(r_M - \bar{r}_M)\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{n} w_i (r_i - \bar{r}_i)(r_M - \bar{r}_M)\right] = \sum_{i=1}^{n} w_i \mathbb{E}\left[(r_i - \bar{r}_i)(r_M - \bar{r}_M)\right] \\
\Rightarrow Cov\left[\sum_{i=1}^{n} w_i r_i, r_M\right] &= \sum_{i=1}^{n} w_i Cov[r_i, r_M] \\
\Rightarrow \beta &= \frac{Cov\left[\sum_{i=1}^{n} w_i r_i, r_M\right]}{\sigma_M^2} = \sum_{i=1}^{n} w_i \frac{Cov\left[r_i, r_M\right]}{\sigma_M^2} \\
\Rightarrow \beta &= \sum_{i=1}^{n} w_i \beta_i
\end{aligned}
$$

3 Systematic and non-systematic risk

Considering the random return r_i of given asset i, the following error is observed.

$$r_i = r_f + \beta_i(r_M - r_f) + \varepsilon_i,$$

Where r_M is the random rate of return of the market, and ε_i is a new random variable that makes the equation be verified. This is the **factor model** form. This is done by defining a new random variable ε_i and by adding and deducting $r_f + \beta_i(r_M - r_f)$ from r_i:

$$\begin{aligned} r_i &= r_i + (r_f + \beta_i(r_M - r_f)) - (r_f + \beta_i(r_M - r_f)) \\ \Rightarrow r_i &= r_f + \beta_i(r_M - r_f) + \underbrace{(r_i - r_f - \beta_i(r_M - r_f))}_{\varepsilon_i} \end{aligned}$$

The new random variable ε_i has the following properties under CAPM:

A Expected return of the asset i must follow the security market line so that:

$$\begin{aligned} \mathbb{E}[r_i] &= \mathbb{E}[r_f + \beta_i(r_M - r_f) + \varepsilon_i] \\ &= r_f + \beta_i\left(\mathbb{E}[r_M] - r_f\right) + \mathbb{E}[\varepsilon_i] \\ \Rightarrow \mathbb{E}[\varepsilon_i] &= 0 \end{aligned}$$

Equivalently:

$$\begin{aligned} \mathbb{E}[\varepsilon_i] &= \mathbb{E}[r_i - r_f - \beta_i(r_M - r_f)] \\ &= \mathbb{E}[r_i] - r_f - \beta_i(\mathbb{E}[r_M] - r_f) \\ &= r_f + \beta_i(\bar{r}_M - r_f) - r_f - \beta_i(\bar{r}_M - r_f) = 0 \end{aligned}$$

B Linearity of covariance concerning one variate yields:

$$\begin{aligned} Cov[r_i, r_M] &= Cov[r_f + \beta_i(r_M - r_f) + \varepsilon_i, r_M] \\ &= Cov[r_f + \beta_i(r_M - r_f), r_M] + Cov[\varepsilon_i, r_M] \\ &= \beta_i\sigma_M^2 + Cov[\varepsilon_i, r_M] = Cov[r_i, r_M] + Cov[\varepsilon_i, r_M] \\ \Rightarrow Cov[\varepsilon_i, r_M] &= 0 \end{aligned}$$

$$\begin{aligned} \varepsilon_i &= r_i - r_f - \beta_i(r_M - r_f) = (\beta_i - 1)r_f + r_i - \beta_i r_M \\ Cov[\varepsilon_i, r_M] &= Cov[(\beta_i - 1)r_f + r_i - \beta_i r_M, r_M] \\ &= 0 + \underbrace{Cov[r_i, r_M]}_{\beta_i Var[r_M]} - \beta_i Var[r_M] = 0 \end{aligned}$$

Using the formula for the variance of a sum, the variance of the return r_i can be written as:

$$\begin{aligned} Var[r_i] &= Var[r_f + \beta_i(r_M - r_f) + \varepsilon_i] = Var[\beta_i r_M + \varepsilon_i] \\ &= \beta_i^2 Var[r_M] + 2\beta_i \underbrace{Cov[r_M, \varepsilon_i]}_{=0} + Var[\varepsilon_i] \\ \Rightarrow \sigma_i^2 &= \underbrace{\beta_i^2\sigma_M^2}_{\substack{\text{Systematic} \\ \text{(market) risk}}} + \underbrace{Var[\varepsilon_i]}_{\substack{\text{Non-systematic} \\ \text{risk}}} \end{aligned}$$

Based on the investment, it can be classified as:

- Systematic risk $\beta_i^2 Var[r_M]$: This is the risk that correlates with the market portfolio and cannot be diversified;
- Non-systematic risk $Var[\varepsilon_i]$: This does not correlate with the market portfolio and can therefore be diversified;

4 Assessing historical performance with CAPM

In order to verify the performance of a portfolio based on historical data, a few parameters can be estimated:

$$\hat{\bar{r}}_i = \frac{1}{n}\sum_{k=1}^{n} r_i^k \qquad r_i^k = \text{return of asset } i \text{ in period } k$$

$$\hat{\sigma}_i^2 = \frac{1}{n-1}\sum_{k=1}^{n}\left(r_i^k - \hat{\bar{r}}_i\right)^2 \qquad \text{(standard estimators from}$$

$$\hat{\sigma}_{iM} = \frac{1}{n-1}\sum_{k=1}^{n}\left(r_i^k - \hat{\bar{r}}_i\right)\left(r_M^k - \hat{\bar{r}}_M\right) \text{ basic statistics courses)}$$

Another parameter is the **Jensen index** J, which measures how much the performance of an asset has deviated from the security market line:

$$\hat{\bar{r}} - r_f = J + \beta\left(\hat{\bar{r}}_M - r_f\right)$$

This index serves to measure two aspects:

a The true performance of an asset compared to CAPM;

b The validity of CAPM (all alphas would be zero if CAPM always holds).

Finally, the **Sharpe index** S measures how the fund performs relative to the capital market line:

$$\hat{\bar{r}} - r_f = S\hat{\sigma}$$

If the Sharpe index for a fund is smaller than that of the market portfolio, then the fund is probably inefficient.

5 Pricing with CAPM

Definition 5.1. Pricing with CAPM

The price P of an asset with a random payoff Q is

$$P = \frac{\bar{Q}}{1 + r_f + \beta(\bar{r}_M - r_f)},$$

where β is the beta of the asset and $\bar{Q}$ is the expected value of Q.

The pricing form follows directly from the definition of the expected rate of return and the CAPM:

$$\bar{r} = \frac{\bar{Q} - P}{P}$$
$$\Rightarrow \frac{\bar{Q} - P}{P} = r_f + \beta(\bar{r}_M - r_f)$$

Also, quantity $r_f + \beta(\bar{r}_M - r_f)$ can be regarded as the **risk adjusted interest rate** (in the context of CAPM). For assets with $\beta > 0$, if the expected market return increases, then the asset price decreases.

The CAPM exhibits **linear pricing**, which is an important property because there would be arbitrage opportunities without linear pricing. Remember that the price of a portfolio of assets is the sum of the prices of its constituent assets. This result is not obvious in the pricing form of the CAPM.

Considering the following theorem:

Theorem 5.2. Certainty equivalent pricing formula

The price P of an asset with payoff Q is:

$$P = \frac{1}{1 + r_f}\left[\bar{Q} - \frac{Cov[Q, r_M](\bar{r}_M - r_f)}{\sigma_M^2}\right]$$

The term in the brackets is called the **certainty equivalent** of random selling price Q.Expectation and covariance are linear operators; thus, pricing is also linear.

Proof: Substituting

$$\beta = \frac{Cov[r, r_M]}{\sigma_M^2} = \frac{Cov[Q/P - 1, r_M]}{\sigma_M^2} = \frac{Cov[Q, r_M]}{P\sigma_M^2}$$

into the pricing form of the CAPM yields

$$P = \frac{\bar{Q}}{1 + r_f + \frac{Cov[Q, r_M]}{P\sigma_M^2}(\bar{r}_M - r_f)}$$

$$\Rightarrow 1 = \frac{\bar{Q}}{P(1 + r_f) + \frac{Cov[Q, r_M]}{\sigma_M^2}(\bar{r}_M - r_f)}$$

$$\Rightarrow P = \frac{1}{1 + r_f}\left[\bar{Q} - \frac{Cov[Q, r_M]}{\sigma_M^2}(\bar{r}_M - r_f)\right],$$

which completes the proof. 😎

Also, linear pricing follows from the exclusion of arbitrage:

1. If the price of a portfolio of assets A and B is greater than prices of A and B separately $(P_A + P_B)$, then buy these assets separately and sell them together as a portfolio $P > P_A + P_B \Rightarrow$ Arbitrage!
2. If the price of the portfolio of assets A and B is less than the prices of A and B when sold separately, then buy portfolios and sell separately: $P < P_A + P_B \Rightarrow$ Arbitrage!

Thus, we must have:

$$P = P_A + P_B$$

Considering a project that requires an initial investment of C and gives an uncertain cash flow of Q in a year. Using the CAPM certainty equivalent pricing formula to determine the NPV of a project as:

$$\text{NPV} = -C + \frac{1}{1 + r_f}\left[\bar{Q} - \frac{Cov[Q, r_M](\bar{r}_M - r_f)}{\sigma_M^2}\right]$$

The project should be carried out if the NPV of cash flow using CAPM is greater than C. Yet the use of CAPM for choosing projects may be questionable:

1. Projects are lumpy, while CAPM assumes that weights are continuous $\Rightarrow$ A large project with a small beta may get a much higher weight in the portfolio than what CAPM requires;

2. CAPM assumes that projects are being priced relative to the market portfolio;

3. The beta-driven approach assumes that the investor (ultimately firm owners) invests in the market portfolio using capitalisation weights (possibly adjusted for the project being valued).

Many kinds of methods are employed in practice, e.g., risk-adjusted discount rates, real options, and decision trees.