Aalto University
School of Science

MS-E2114 Investment Science
Lecture VI: Capital asset pricing model

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Overview

Capital asset pricing model

Systematic and non-systematic risk

Assessing historical performance with CAPM

Pricing with CAPM

This lecture

- Last week we constructed efficient portfolios
  - One and two fund theorems
- Today's topic is the Capital Asset Pricing Model (CAPM)
  - Seminal papers (click authors' names to access)

    Sharpe W. (1964) Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk, *Journal of Finance* 19, 425-442.

    Lintner J. (1965) The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets, *Review of Economics and Statistics*, Vol. XLVII, 13-37.

    Mossin J. (1968) Optimal Multiperiod Portfolio Policies, *Journal of Business* 41, 215-229.
- CAPM is an extension of the mean-variance portfolio framework and the one-fund theorem, in particular

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One-fund theorem

- Recall that when there is a risk-free asset, a mean-variance investor invests in a combination of a one fund F and the risk-free asset
  - All other combinations of assets are less efficient
- The expected return $\bar{r}$ of any efficient portfolio can thus be expressed as a function of its standard deviation σ as

$$\bar{r} = r_f + \frac{\bar{r}_F - r_f}{\sigma_F}\sigma$$

where r_f is the risk-free interest rate; and $\bar{r}_F$ and σ_F are the expected return and standard deviation of the one fund F, respectively

CAPM: Market is the one fund

- The assumptions of the Capital Asset Pricing Model (CAPM) imply that all investors invest in the *same* fund and that
- This one fund F is therefore the whole market M
- This fund M is called the **market**, the **market portfolio** or the **market fund**
- If the following assumptions hold, all investors in the market build the same one-fund
  1. All investors are one-period mean-variance investors
  2. All investors have the same estimates of means, variances and covariances for each asset
  3. All investors can borrow and lend at the risk-free interest rate without limit

Market equilibrium

- What if the estimates are not identical?
- We can argue that markets tend towards an equilibrium
  - Better informed investors have different 'one funds'
    - $\Rightarrow$ Supply and demand for assets change
    - $\Rightarrow$ Prices change
    - $\Rightarrow$ Estimates of expected return and variance change
    - $\Rightarrow$ The other investors' estimates change
- Large institutional investors are key players
  - Their investments drive estimates to the equilibrium
  - Consequently not all investors need to optimize
    - It suffices to have many enough big investors who calculate the equilibrium
- All the same, we assume that $F = M$

Market as the one fund

- Recognizing that the one fund is the market has several implications
- First, any efficient portfolio is a combination of the risk-free asset and the market
- This is called the **capital market line**

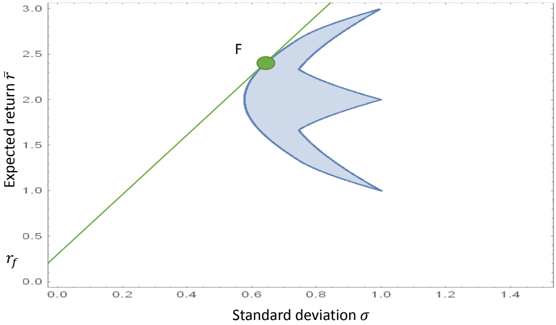
$$\bar{r} = r_f + \frac{\bar{r}_M - r_f}{\sigma_M}\sigma$$

  where $\bar{r}_M$ is the expected return of the market.
- The expression $\frac{\bar{r}_M - r_f}{\sigma_M}$ is called the (market) **price of risk**.

Capital market line

- Starts from $(0, r_f)$ and passes through $(\sigma_M, \mathbb{E}[r_M])$
- Includes all efficient portfolios

$$\bar{r} = r_f + \overbrace{\frac{\bar{r}_M - r_f}{\sigma_M}}^{\text{Price of risk}} \sigma$$



Market as the one fund

- We can observe the assets' **capitalization weights**: How large a share is the market capitalization of some company of the total market portfolio?
- In the one-fund theorem, we knew the expected returns, variances and covariances and then solved for the optimal porfolio weights
- But if we know the optimal weights, we can solve for the other parameters, such as expected returns or covariances
- Specifically, when we solve for the expected returns based on capitalization weights, variances and covariances, we get the **Capital Asset Pricing Model** (**CAPM**)
- We are effectively using the one-fund theorem in the reverse direction

Capital asset pricing model

Theorem

*(**Capital asset pricing model**) If the market portfolio M is efficient, the expected return $\bar{r}_i$ of any asset i satisfies*

$$\bar{r}_i - r_f = \beta_i(\bar{r}_M - r_f),$$

where

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}.$$

Note that the capital asset pricing model tells what returns the assets must have **within** the (optimal) one fund. It is the optimality relationship between expected returns, variances, covariances, and weights.

Terminology in CAPM

- The term $\bar{r}_i - r_f$ is the **expected excess rate of return** of asset i
- Likewise, the term $\bar{r}_M - r_f$ is the expected excess rate of return of the market portfolio
- The multiplier

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

is called the asset's **beta**

Capital asset pricing model

Proof in Luenberger: For any α, consider investing a portion α in asset i and $1-\alpha$ in the market portfolio M. The expected return and volatility of this portfolio are

$$\bar{r}_\alpha = \alpha \bar{r}_i + (1-\alpha)\bar{r}_M$$

$$\sigma_\alpha = \sqrt{\alpha^2\sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha)^2\sigma_M^2}$$

Curve depicted by the pairs $(\sigma_\alpha, \bar{r}_\alpha), \alpha \geq 0$ that parallel the capital market line at $\alpha = 0$. The tangent of this line can be computed from the following differentials

$$\frac{d\bar{r}_\alpha}{d\alpha} = \frac{d}{d\alpha}\left(\alpha\bar{r}_i + (1-\alpha)\bar{r}_M\right) = \bar{r}_i - \bar{r}_M$$

$$\frac{d\sigma_\alpha}{d\alpha} = \frac{1}{2\sigma_\alpha}\left(2\alpha\sigma_i^2 + 2(1-2\alpha)\sigma_{iM} - 2(1-\alpha)\sigma_M^2\right)$$

$$\left.\frac{d\sigma_\alpha}{d\alpha}\right|_{\alpha=0} = \frac{1}{2\sigma_\alpha}\left(2\sigma_{iM} - 2\sigma_M^2\right) = \frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}$$

Capital asset pricing model

When $\alpha = 0$, the curve must be parallel to the capital market line

$$\left.\frac{d\bar{r}_\alpha}{d\sigma_\alpha}\right|_{\alpha=0} = \left.\frac{d\bar{r}_\alpha}{d\alpha}\frac{d\alpha}{d\sigma_\alpha}\right|_{\alpha=0} = (\bar{r}_i - \bar{r}_M)\left(\frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}\right)^{-1} = \frac{\bar{r}_M - r_f}{\sigma_M}$$

$$\Rightarrow \bar{r}_i - \bar{r}_M = \frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}\frac{\bar{r}_M - r_f}{\sigma_M} = \left(\frac{\sigma_{iM}}{\sigma_M^2} - 1\right)(\bar{r}_M - r_f)$$

$$= \frac{\sigma_{iM}}{\sigma_M^2}(\bar{r}_M - r_f) - \bar{r}_M + r_f$$

$$\Rightarrow \bar{r}_i - r_f = \frac{\sigma_{iM}}{\sigma_M^2}(\bar{r}_M - r_f),$$

which completes the proof. □

Capital asset pricing model

Alternative proof:

- Recall from the one-fund theorem that the optimality conditions for the one fund were:

$$0 = \frac{\partial}{\partial w_k} \frac{\sum_{i=1}^{n} w_i(\bar{r}_i - r_f)}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}}}, \quad k = 1, 2, \ldots, n$$

$$\Rightarrow \bar{r}_k - r_f = \frac{\sum_{i=1}^{n} w_i(\bar{r}_i - r_f)}{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}} \sum_{i=1}^{n} w_i \sigma_{ik}, \quad k = 1, 2, \ldots, n$$

Capital asset pricing model

- Since the market fund is

$$r_M = \sum_{i=1}^{n} w_i r_i$$

we have

$$\bar{r}_M = \sum_{i=1}^{n} w_i \bar{r}_i, \quad \bar{r}_M - r_f = \sum_{i=1}^{n} w_i (\bar{r}_i - r_f)$$

$$\sigma_M^2 = \mathsf{Var}\left[r_M\right] = \mathsf{Var}\left[\sum_{i=1}^{n} w_i r_i\right] = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

$$\sigma_{kM} = \mathsf{Cov}\left[r_k, r_M\right] = \mathsf{Cov}\left[r_k, \sum_{i=1}^{n} w_i r_i\right] = \sum_{i=1}^{n} w_i \sigma_{ik}$$

Capital asset pricing model

- Thus, we have

$$\bar{r}_k - r_f = \frac{\sum_{i=1}^{n} w_i(\bar{r}_i - r_f)}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}} \sum_{i=1}^{n} w_i \sigma_{ik}, \quad k = 1, 2, \ldots, n$$

$$\Rightarrow \bar{r}_k - r_f = \frac{\bar{r}_M - r_f}{\sigma_M^2} \sigma_{kM}, \quad k = 1, 2, \ldots, n$$

- By rearranging the terms and denoting

$$\beta_k = \frac{\sigma_{kM}}{\sigma_M^2},$$

we get

$$\bar{r}_k - r_f = \beta_k \left(\bar{r}_M - r_f\right), \quad k = 1, 2, \ldots, n \quad \square$$

Capital asset pricing model (CAPM)

- Summary of key assumptions
  1. Investors consider only the expected return and the variance of returns
  2. All investors have the same estimates concerning asset parameters (expected returns, covariances)
  3. Unlimited lending/borrowing at the risk-free rate
  4. There are no transaction costs
  5. There are no taxes
  6. Any quantity of an asset can be purchased
  7. Individual investors cannot influence prices
  8. Unlimited shorting of risky assets is possible
  9. All assets have efficient markets
- These are the assumptions of the one-fund theorem + those needed to ensure that $F = M$

Interpretations of beta

- The beta can be interpreted as standard-deviation-scaled correlation to market:

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2} = \rho_{iM}\frac{\sigma_i}{\sigma_M}$$

  where $\rho_{iM} = \sigma_{iM}/(\sigma_i\sigma_M)$ is the correlation coefficient between asset i and the market
- If the return of the market portfolio r_M is interpreted as a **factor** influencing the return on the asset r_i, then the beta can be also interpreted as a **factor loading** of r_M (i.e, the multiplier used with factor r_M)

$$r_i = r_f + \beta_i(r_M - r_f) + \varepsilon_i = r_f(1 - \beta_i) + \beta_i r_M + \varepsilon_i$$

- We will get back to this later in this lecture
- Also, Lecture 7 will discuss this interpretation more

Beta of assets

- The beta β_i has a seemingly surprising impact on price
  - If β_i is large, there is a large risk premium
    - The asset is not suitable for diversification
    - Possibly a large weight in the market
    - If $\beta_i > 1$, we know that $\sigma_i > \sigma_M$, because correlation is limited to 1
  - If $\beta_i = 0$, there is no risk premium
    - All asset risk can be eliminated, including the fact that the asset is part of the market
  - If $\beta_i < 0$, the risk premium is negative
    - Correlation with the market is negative: The value of the asset tends to increase when the markets go down
    - The asset has insurance value, even if variance high
    - If $\beta_i < -1$, we know that $\sigma_i > \sigma_M$, because correlation is limited to -1

Beta and optimal portfolio choice

- Note that the beta of an asset or beta of a fund alone is **not** a guide for investment selection
- The theory implies that you should always invest in a combination of the market fund and the risk-free asset; this will be optimal
- If you buy some fund or asset (other than the market portfolio), the theory assumes that you will make additional investments so that you will have the market portfolio in the end, which ultimately justifies the expected returns of assets based on their betas
- If you do not make these additional investments, then your portfolio will not be efficient and will not fall on the capital market line

What can you use the CAPM for?

- If you know the prices and capitalization weights of all assets in the market, knowing their expected returns is not particularly helpful, because in any case you will just invest in the market portfolio
- However, if you do not know a price of an asset, e.g., because it is a new asset (such as an initial public offering), but you know its random cash flow in the next period (along with its correlations with other assets), then you can use the CAPM to estimate its market price

What can you use the CAPM for?

1. CAPM provides a framework for the reasonable pricing of individual assets
   - May help identify mispriced assets in practice
   - Pricing of new assets which are introduced to the market
2. As a factor model, CAPM explains the relation between of asset prices and their correlation with the market
   - Many stocks (but not all) tend to correlate with the market
   - Historically, this was one of the results that made CAPM interesting

Example: Beta

- Let $r_f = 0.08, \bar{r}_M = 0.12, \sigma_M = 0.15$
- Consider an asset A whose covariance with the market portfolio is $\sigma_{AM} = 0.045$
- The required expected return for this asset is

$$\begin{aligned} \bar{r}_A &= r_f + \frac{\sigma_{AM}}{\sigma_M^2}(\bar{r}_M - r_f) \\ &= 0.08 + \frac{0.045}{0.15^2}(0.12 - 0.08) \\ \Rightarrow \bar{r}_A &= 0.16 = 16\% \end{aligned}$$

- This required return is higher than that of the market portfolio 12%, because the covariance is high
- E.g., if we know that $\rho_{AM} = 0.5$, then $\sigma_A = \sigma_{AM}/(\rho_{AM}\sigma_M) = 0.045/(0.5 \cdot 0.15) = 0.6 = 60\%$

Quoted betas of stocks

KONE Oyj (KNEBV.HE)

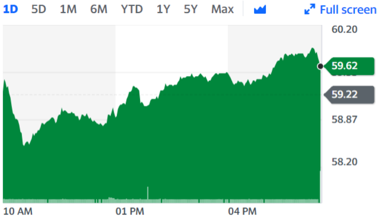
Helsinki - Helsinki Real Time Price. Currency in EUR

59.62 +0.40 (+0.68%)

At close: October 15 6:29PM EEST

Summary Chart Conversations Statistics Historical Data Profile Financials Analysis Options Holders Sustainability

Previous Close	**59.22**	Market Cap	**30.926B**
Open	**59.38**	Beta (5Y Monthly)	**0.55**
Bid	**59.74 x 0**	PE Ratio (TTM)	**30.25**
Ask	**59.78 x 0**	EPS (TTM)	**1.97**
Day's Range	**58.50 - 59.90**	Earnings Date	**Oct 28, 2021**
52 Week Range	**58.20 - 75.86**	Forward Dividend & Yield	**1.75 (2.94%)**
Volume	**1,102,520**	Ex-Dividend Date	**Mar 03, 2021**
Avg. Volume	**630,516**	1y Target Est	**67.23**



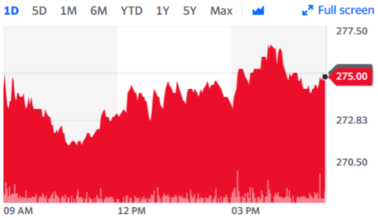
Quoted betas of stocks

Volkswagen AG (VOW.DE)
XETRA - XETRA Delayed Price. Currency in EUR

274.80 -0.40 (-0.15%)
At close: October 15 5:35PM CEST

Summary | Chart | Conversations | Statistics | Historical Data | Profile | Financials | Analysis | Options | Holders | Sustainability

Previous Close	275.20	Market Cap	121.819B
Open	275.60	Beta (5Y Monthly)	1.40
Bid	274.40 x N/A	PE Ratio (TTM)	7.81
Ask	274.80 x N/A	EPS (TTM)	35.19
Day's Range	271.60 - 276.60	Earnings Date	Oct 29, 2020
52 Week Range	131.80 - 357.40	Forward Dividend & Yield	4.80 (1.75%)
Volume	35,353	Ex-Dividend Date	Jul 23, 2021
Avg. Volume	45,900	1y Target Est	288.25



Beta of a portfolio

- Beta is linear in portfolio weights

$$\mathrm{Cov}\left[\sum_{i=1}^{n} w_i r_i, r_M\right] = \mathbb{E}\left[\left(\sum_{i=1}^{n} w_i r_i - \sum_{i=1}^{n} w_i \bar{r}_i\right)(r_M - \bar{r}_M)\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{n} w_i (r_i - \bar{r}_i)(r_M - \bar{r}_M)\right] = \sum_{i=1}^{n} w_i \,\mathbb{E}\left[(r_i - \bar{r}_i)(r_M - \bar{r}_M)\right]$$

$$\Rightarrow \mathrm{Cov}\left[\sum_{i=1}^{n} w_i r_i, r_M\right] = \sum_{i=1}^{n} w_i \,\mathrm{Cov}[r_i, r_M]$$

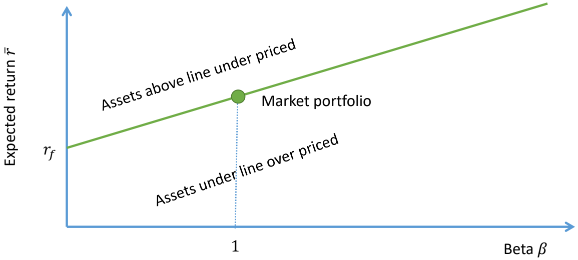
$$\Rightarrow \beta = \frac{\mathrm{Cov}\left[\sum_{i=1}^{n} w_i r_i, r_M\right]}{\sigma_M^2} = \sum_{i=1}^{n} w_i \frac{\mathrm{Cov}\left[r_i, r_M\right]}{\sigma_M^2}$$

$$\Rightarrow \beta = \sum_{i=1}^{n} w_i \beta_i$$

Security market line

- The CAPM formula as a linear pricing relationship for **individual assets** i is called the **security market line**

$$\bar{r}_i = r_f + \beta_i(\bar{r}_M - r_f), \quad \text{where } \beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$



- Compare this relationship with the capital market line, that is, the pricing relationship for any **efficient fund**

$$\bar{r} = r_f + \frac{\sigma}{\sigma_M}(\bar{r}_M - r_f)$$

Overview

Capital asset pricing model

Systematic and non-systematic risk

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Systematic risk

- Let us consider the random return r_i of asset i in the following "additive error model" form

$$r_i = r_f + \beta_i(r_M - r_f) + \varepsilon_i,$$

where r_M is the random rate of return of the market and ε_i is a new random variable that makes the equation true

- This is the **factor model** form we discussed briefly earlier
- This is, we simply define a new random variable ε_i by adding and deducting $r_f + \beta_i(r_M - r_f)$ from r_i:

$$\begin{aligned} r_i &= r_i + (r_f + \beta_i(r_M - r_f)) - (r_f + \beta_i(r_M - r_f)) \\ \Rightarrow r_i &= r_f + \beta_i(r_M - r_f) + \underbrace{(r_i - r_f - \beta_i(r_M - r_f))}_{\varepsilon_i} \end{aligned}$$

- What properties must the variable ε_i have?

Systematic risk

- The new random variable ε_i has the following properties under CAPM

A Expected return of the asset i must follow the security market line so that

$$
\begin{aligned}
\mathbb{E}[r_i] &= \mathbb{E}[r_f + \beta_i(r_M - r_f) + \varepsilon_i] \\
&= r_f + \beta_i\left(\mathbb{E}[r_M] - r_f\right) + \mathbb{E}[\varepsilon_i] \\
\Rightarrow \mathbb{E}[\varepsilon_i] &= 0
\end{aligned}
$$

Equivalently

$$
\begin{aligned}
\mathbb{E}[\varepsilon_i] &= \mathbb{E}[r_i - r_f - \beta_i(r_M - r_f)] \\
&= \mathbb{E}[r_i] - r_f - \beta_i(\mathbb{E}[r_M] - r_f) \\
&= r_f + \beta_i(\bar{r}_M - r_f) - r_f - \beta_i(\bar{r}_M - r_f) = 0
\end{aligned}
$$

Systematic risk

B Linearity of covariance with respect to one variate yields

$$\begin{aligned}\mathsf{Cov}[r_i, r_M] &= \mathsf{Cov}[r_f + \beta_i(r_M - r_f) + \varepsilon_i, r_M] \\ &= \mathsf{Cov}[r_f + \beta_i(r_M - r_f), r_M] + \mathsf{Cov}[\varepsilon_i, r_M] \\ &= \beta_i\sigma_M^2 + \mathsf{Cov}[\varepsilon_i, r_M] = \mathsf{Cov}[r_i, r_M] + \mathsf{Cov}[\varepsilon_i, r_M]\end{aligned}$$

$$\Rightarrow \mathsf{Cov}[\varepsilon_i, r_M] = 0$$

Equivalently:

$$\begin{aligned}\varepsilon_i &= r_i - r_f - \beta_i(r_M - r_f) = (\beta_i - 1)r_f + r_i - \beta_i r_M \\ \mathsf{Cov}[\varepsilon_i, r_M] &= \mathsf{Cov}[(\beta_i - 1)r_f + r_i - \beta_i r_M, r_M] \\ &= 0 + \underbrace{\mathsf{Cov}[r_i, r_M]}_{\beta_i \mathsf{Var}[r_M]} - \beta_i \mathsf{Var}[r_M] = 0\end{aligned}$$

Systematic risk

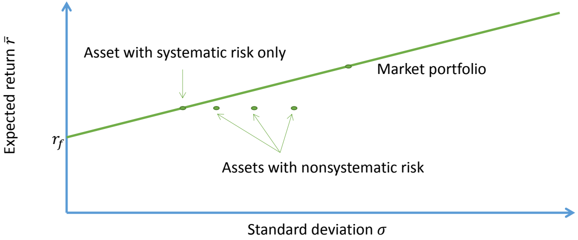
- Using the formula for the variance of a sum, the variance of the return r_i can be written as

$$\begin{aligned}\mathsf{Var}[r_i] &= \mathsf{Var}[r_f + \beta_i(r_M - r_f) + \varepsilon_i] = \mathsf{Var}[\beta_i r_M + \varepsilon_i] \\ &= \beta_i^2\,\mathsf{Var}[r_M] + 2\beta_i \underbrace{\mathsf{Cov}[r_M, \varepsilon_i]}_{=0} + \mathsf{Var}[\varepsilon_i]\end{aligned}$$

$$\Rightarrow \sigma_i^2 = \underbrace{\beta_i^2 \sigma_M^2}_{\substack{\text{Systematic} \\ \text{(market) risk}}} + \underbrace{\mathsf{Var}[\varepsilon_i]}_{\substack{\text{Non-systematic} \\ \text{risk}}}$$

Systematic risk

- Investment risk
  - Systematic risk $\beta_i^2 \, \mathrm{Var}[r_M]$
    - This is the risk that correlates with the market portfolio and cannot be diversified
  - Non-systematic risk $\mathrm{Var}[\varepsilon_i]$
    - This does not correlate with the market portfolio and can therefore be diversified



Example: Diversification of non-systematic risk

- Select n assets such that

$$\mathbb{E}[r_i] = \bar{r}, \quad i = 1, 2, \ldots, n$$
$$\mathsf{Cov}[\varepsilon_i, \varepsilon_j] = \begin{cases} \sigma_\varepsilon^2, & i = j \\ 0, & i \neq j \end{cases}$$

- Because the expected returns are the same, CAPM implies that the assets have the same beta

$$\begin{aligned} \mathbb{E}[r_i] = \bar{r} = r_f + \beta_i(\bar{r}_M - r_f) \quad & \forall i = 1, 2, \ldots, n \\ \Rightarrow \beta_i = \beta \quad & \forall i = 1, 2, \ldots, n \end{aligned}$$

Example: Diversification of non-systematic risk

- Invest an equal portion $w_i = \frac{1}{n}$ into every asset
- The variance of the resulting portfolio is

$$\mathsf{Var}\left[\sum_{i=1}^{n} w_i r_i\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \, \mathsf{Cov}\left[r_i, r_j\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \, \mathsf{Cov}\left[r_f + \beta_i(r_M - r_f) + \varepsilon_i, r_f + \beta_j(r_M - r_f) + \varepsilon_j\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \left(\mathsf{Cov}[\beta_i r_M, \beta_j r_M] + 2\,\mathsf{Cov}[\beta_i r_M, \varepsilon_j] + \mathsf{Cov}[\varepsilon_i, \varepsilon_j]\right)$$

$$\Rightarrow \mathsf{Var}\left[\sum_{i=1}^{n} w_i r_i\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{n^2}\beta^2\sigma_M^2 + \sum_{i=1}^{n} \frac{1}{n^2}\sigma_\varepsilon^2 = \beta^2\sigma_M^2 + \frac{1}{n}\sigma_\varepsilon^2$$

- Non-systematic risk is diversified.

Overview

Capital asset pricing model

Systematic and non-systematic risk

Assessing historical performance with CAPM

Pricing with CAPM

Assessing historical performance with CAPM

- How well <u>has</u> the portfolio performed based on historical data?
  - How well has an index fund succeeded in diversifying risk?
  - Estimate CAPM model parameters

$$\hat{\bar{r}}_i = \frac{1}{n}\sum_{k=1}^{n} r_i^k \qquad r_i^k = \text{return of asset } i \text{ in period } k$$

$$\hat{\sigma}_i^2 = \frac{1}{n-1}\sum_{k=1}^{n}\left(r_i^k - \hat{\bar{r}}_i\right)^2 \qquad \text{(standard estimators from}$$

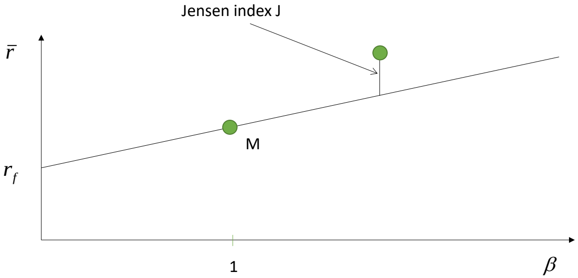
$$\hat{\sigma}_{iM} = \frac{1}{n-1}\sum_{k=1}^{n}\left(r_i^k - \hat{\bar{r}}_i\right)\left(r_M^k - \hat{\bar{r}}_M\right) \text{ basic statistics courses)}$$

Jensen's alpha index

- **Jensen index** J measures how much the performance of an asset has deviated from the security market line

$$\hat{\bar{r}} - r_f = J + \beta\left(\hat{\bar{r}}_M - r_f\right)$$

- This index serves to measure two aspects:
  a The true performance of an asset compared to CAPM
  b The validity of CAPM (all alphas would be zero, if CAPM always holds)

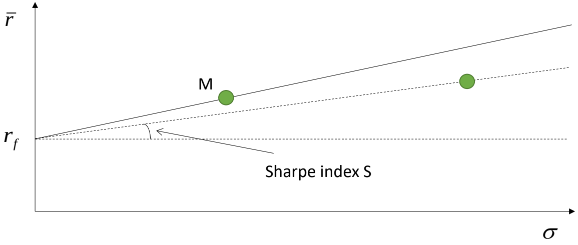


Sharpe index

- **Sharpe index** S measures how the fund performs relative to the capital market line

$$\hat{\bar{r}} - r_f = S\hat{\sigma}$$

- If the Sharpe index for a fund is smaller than that of the market portfolio, then the fund is probably not efficient



Assessing historical performance with CAPM

	Return (%)		
Year	ABC	S&P (market)	T-bills (no risk)
1	14	12	7.0
2	10	7	7.5
3	19	20	7.7
4	-8	-2	7.5
5	23	12	8.5
6	28	23	8.0
7	20	17	7.3
8	14	20	7.0
9	-9	-5	7.5
10	19	16	8.0
Mean	13	12	7.6
St.dev.	12.4	9.4	0.5
Cov(ABC,S&P)	0.011		
Beta	1.204	1	
Jensen	0.001	0	
Sharpe	0.436	0.467	

Overview

Pricing form of the CAPM

Definition

(**Pricing with CAPM**) The price P of an asset with a random payoff Q is

$$P = \frac{\bar{Q}}{1 + r_f + \beta(\bar{r}_M - r_f)},$$

where β is the beta of the asset and $\bar{Q}$ is the expected value of Q.

Pricing form of the CAPM

- The pricing form follows directly from the definition of the expected rate of return and the CAPM:

$$\bar{r} = \frac{\bar{Q} - P}{P}$$
$$\Rightarrow \frac{\bar{Q} - P}{P} = r_f + \beta(\bar{r}_M - r_f)$$

- Quantity $r_f + \beta(\bar{r}_M - r_f)$ can be regarded as the **risk adjusted interest rate** (in the context of CAPM)
- For assets with $\beta > 0$, if the expected market return increases, then the asset price decreases

Example: Pricing with CAPM

- Consider investing in a mutual fund which invests
  1. a portion $\alpha = 0.1$ in a risk-free asset at the rate $r_f = 0.07$
  2. a portion $1 - \alpha = 0.9$ into market portfolio with expected return $\bar{r}_M = 0.15$
- How much to pay for a nominal 100 € share of this fund?
  - Beta of the fund is $\beta = 0.9$
  - The expected return of this 100 € share is

$$\begin{aligned}\bar{Q} &= \alpha(1 + r_f) \cdot 100 + (1 - \alpha)(1 + \bar{r}_M) \cdot 100 \\ &= 0.1 \cdot 1.07 \cdot 100 + 0.9 \cdot 1.15 \cdot 100 = 114.20\end{aligned}$$

  - The price of the 100 € nominal share is

$$P = \frac{\bar{Q}}{1 + r_f + \beta(\bar{r}_M - r_f)} = \frac{114.20}{1 + 0.07 + 0.9 \cdot (0.15 - 0.07)} = 100$$

Linear pricing

- The CAPM exhibits **linear pricing**
- The price of a portfolio of assets = The sum of prices of its constituent assets
- This result is not obvious in the pricing form of the CAPM
- Important property, because without linear pricing there would be arbitrage opportunities

Linear pricing

Theorem

*(**Certainty equivalent pricing formula**) The price P of an asset with payoff Q is*

$$P = \frac{1}{1 + r_f} \left[\bar{Q} - \frac{\mathsf{Cov}[Q, r_M](\bar{r}_M - r_f)}{\sigma_M^2} \right]$$

- The term in the brackets is called the **certainty equivalent** of random selling price Q
- Expectation and covariance are linear operators; thus pricing is also linear

Linear pricing

Proof: Substituting

$$\beta = \frac{\text{Cov}[r, r_M]}{\sigma_M^2} = \frac{\text{Cov}[Q/P - 1, r_M]}{\sigma_M^2} = \frac{\text{Cov}[Q, r_M]}{P\sigma_M^2}$$

into the pricing form of the CAPM yields

$$P = \frac{\bar{Q}}{1 + r_f + \frac{\text{Cov}[Q, r_M]}{P\sigma_M^2}(\bar{r}_M - r_f)}$$

$$\Rightarrow 1 = \frac{\bar{Q}}{P(1 + r_f) + \frac{\text{Cov}[Q, r_M]}{\sigma_M^2}(\bar{r}_M - r_f)}$$

$$\Rightarrow P = \frac{1}{1 + r_f}\left[\bar{Q} - \frac{\text{Cov}[Q, r_M]}{\sigma_M^2}(\bar{r}_M - r_f)\right],$$

which completes the proof. □

Linear pricing

- Linear pricing also follows from the exclusion of arbitrage
  1. If the price of portfolio of assets A and B is greater than prices of A and B separately $(P_A + P_B)$, then buy these assets separately and sell them together as a portfolio $P > P_A + P_B \Rightarrow$ Arbitrage!
  2. If the price of the portfolio of assets A and B is less then the prices of A and B when sold separately, then buy portfolios and sell separately: $P < P_A + P_B \Rightarrow$ Arbitrage!
- Thus, we must have

$$P = P_A + P_B$$

Project choice

- Consider a project requires an initial investment C and gives an uncertain cash flow Q in a year
- Use the CAPM certainty equivalent pricing formula to determine the NPV of project as

$$\text{NPV} = -C + \frac{1}{1+r_f}\left[\bar{Q} - \frac{\text{Cov}[Q, r_M](\bar{r}_M - r_f)}{\sigma_M^2}\right]$$

- The project should be carried out if NPV of cash flow using CAPM is greater than C

Project choice

- Yet the use of CAPM for choosing projects may be questionable
  1. Projects are lumpy, while CAPM assumes that weights are continuous ⇒ A large project with a small beta may be get a much higher weight in the portfolio that what CAPM requires
  2. CAPM assumes that projects are being priced relative to the market portfolio
  3. The beta-driven approach assumes that the investor (ultimately firm owners) invest in the market portfolio using capitalization weights (possibly adjusted for the project being valued)
- Many kinds of methods are employed in practice
  - E.g., risk-adjusted discount rates, real options, decision trees

Overview

Capital asset pricing model

Systematic and non-systematic risk

Assessing historical performance with CAPM

Pricing with CAPM