MSE2114 - Investment Science Lecturer Notes VII

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Abstract

In this lectures, factor models and arbitrage pricing theory seek to explain returns and correlations between assets. For random returns, the models are based on random variables, which can be explained by the behaviour of other variables. By using the parameter estimation are required for mean-variance optimization.

Contents

1 Single Factor Model

A single-factor model explains asset returns with common random variables. The rate of return of asset i expressed as:

$$r_i = a_i + b_i f + e_i,$$

where,

- a_i and b_i are constants;
- f is the random explanatory variable (i.e., the factor);
- e_i is the random error term.

This formulation is backed up by the following assumptions:

- $\mathbb{E}[e_i] = 0$ (not restrictive as a_i can be chosen freely);
- e_i is not correlated with f;

$$\Rightarrow \mathbb{E}\left[(f - \bar{f})(e_i - \bar{e}_i)\right] = \mathbb{E}\left[(f - \bar{f})e_i\right] = 0$$

- Error terms of the assets are uncorrelated:

$$\Rightarrow \mathbb{E}\left[(e_i - \bar{e}_i)(e_j - \bar{e}_j)\right] = \mathbb{E}\left[e_i e_j\right] = 0, i \neq j$$

- Variances of error terms are known:

$$\Rightarrow \mathbb{E}\left[e_i^2\right] = \sigma_{e_i}^2$$

With such assumptions, the expected rate of return is:

$$\begin{aligned} \bar{r}_i &= \mathbb{E}[r_i] = a_i + b_i \mathbb{E}[f] + \mathbb{E}[e_i] \\ \Rightarrow \bar{r}_i &= a_i + b_i \bar{f} \end{aligned}$$

While the variance of the return r_i is:

$$\begin{aligned} \sigma_i^2 &= Var[r_i] = \mathbb{E}\left[(r_i - \bar{r}_i)^2\right] = \mathbb{E}\left[(a_i + b_i f + e_i - a_i - b_i \bar{f})^2\right] \\ &= \mathbb{E}\left[\left(b_i(f - \bar{f}) + e_i\right)^2\right] = \mathbb{E}\left[b_i^2 (f - \bar{f})^2 + 2 b_i (f - \bar{f}) e_i + e_i^2\right] \\ \Rightarrow \sigma_i^2 &= b_i^2 \sigma_f^2 + \sigma_{e_i}^2 \end{aligned}$$

and the covariance between different assets i and j:

$$\begin{aligned} \sigma_{ij} &= Cov[r_i, r_j] = \mathbb{E}\left[(r_i - \bar{r}_i)(r_j - \bar{r}_j)\right] \\ &= \mathbb{E}\left[\left(b_i(f - \bar{f}) + e_i\right)\left(b_j(f - \bar{f}) + e_j\right)\right] \\ &= \mathbb{E}\left[b_i b_j (f - \bar{f})^2 + (b_j e_i + b_i e_j)(f - \bar{f}) + e_i e_j\right] \\ \Rightarrow \sigma_{ij} &= b_i b_j \sigma_f^2, i \neq j \end{aligned}$$

With those statistical values in mind, the parameters from the single-factor model can be calculated. Thus it follows that:

$$\begin{aligned} Cov[r_i, f] &= \mathbb{E}\left[(r_i - \bar{r}_i)(f - \bar{f})\right] \\ &= \mathbb{E}\left[\left(b_i(f - \bar{f}) + e_i\right)(f - \bar{f})\right] \\ \Rightarrow Cov[r_i, f] &= b_i \sigma_f^2 \\ \Rightarrow b_i &= \frac{Cov[r_i, f]}{\sigma_f^2} \end{aligned}$$

A total of $3n+2$ parameters to be estimated, among them $\bar{f}, \sigma_f^2, a_i, b_i$, and $\sigma_{e_i}^2$, for $i = 1, 2, \dots, n$. Parameters a_i and b_i can be estimated from the time series of r_i and f. Therefore, estimates differ depending on the selected period and averaging and other statistical methods can be used to improve accuracy.

Standard statistical estimators:

$$\begin{aligned} \hat{\bar{r}}_i &= \frac{1}{n}\sum_{k=1}^{n} r_i^k \\ \hat{\sigma}_i^2 &= \frac{1}{n-1}\sum_{k=1}^{n}\left(r_i^k - \hat{\bar{r}}_i\right)^2 \\ \widehat{Cov}[r_i, f] &= \frac{1}{n-1}\sum_{k=1}^{n}\left(r_i^k - \hat{\bar{r}}_i\right)\left(f^k - \hat{\bar{f}}\right), \end{aligned}$$

where superscript k denotes the kth sample.

Finally, the model parameters can be calculated from the standard estimates:

$$\begin{aligned} b_i &= \frac{\widehat{Cov}[r_i, f]}{\hat{\sigma}_f^2} \\ a_i &= \hat{\bar{r}}_i - b_i \hat{\bar{f}} \end{aligned}$$

The variance of error terms becomes:

$$\begin{aligned} \sigma_i^2 &= b_i^2 \sigma_f^2 + \sigma_{e_i}^2 \\ \Rightarrow \hat{\sigma}_{e_i}^2 &= \hat{\sigma}_i^2 - b_i^2 \hat{\sigma}_f^2 \end{aligned}$$

For a portfolio, the same analysis can be done. The collective return is given by:

$$r = \sum_{w=1}^{n} w_i r_i = \sum_{i=1}^{n} w_i a_i + \left(\sum_{i=1}^{n} w_i b_i\right) f + \sum_{i=1}^{n} w_i e_i = a + bf + e,$$

where,

$$a = \sum_{i=1}^{n} w_i a_i, \quad b = \sum_{i=1}^{n} w_i b_i, \quad e = \sum_{i=1}^{n} w_i e_i$$

For the error term of the portfolio return, we have:

$$
\begin{aligned}
\mathbb{E}[e] &= \mathbb{E}\left[\sum_{i=1}^{n} w_i e_i\right] = \sum_{i=1}^{n} w_i \mathbb{E}[e_i] \\
\Rightarrow \mathbb{E}[e] &= 0 \\
Cov[f,e] &= \mathbb{E}\left[(f-\bar{f})\sum_{i=1}^{n} w_i e_i\right] = \sum_{i=1}^{n} w_i \mathbb{E}[(f-\bar{f})e_i] \\
\Rightarrow Cov[f,e] &= 0 \\
Var[e] &= \mathbb{E}\left[\left(\sum_{i=1}^{n} w_i e_i\right)\left(\sum_{j=1}^{n} w_j e_j\right)\right] = \sum_{i=1}^{n} w_i^2 \mathbb{E}[e_i^2] \\
\Rightarrow Var[e] &= \sum_{i=1}^{n} w_i^2 \sigma_{e_i}^2
\end{aligned}
$$

Assuming that assets have equal weights and the variance of error terms is $\sigma_{e_i}^2 = s^2$. Then, the variance of the error term of the portfolio is:

$$\sigma_e^2 = Var[e] = \sum_{i=1}^{n} w_i^2 \sigma_{e_i}^2 = \sum_{i=1}^{n} \frac{1}{n^2} s^2 = \frac{1}{n} s^2,$$

And the variance of the portfolio return is:

$$\sigma^2 = Var[r] = b^2 \sigma_f^2 + \sigma_e^2,$$

where $\sigma_e^2 \to 0$ as $n \to \infty$:

- Variance related to the error terms e_i can be diversified;
- Variance related to terms $b_i f$ cannot be diversified.

Compared to CAPM, the single factor is modelled as the **market**. Thus, the single-factor model is remodelled as:

$$r_i - r_f = \alpha_i + \beta_i (r_M - r_f) + e_i$$

Taking the expectation of this postulated factor model gives:

$$\bar{r}_i - r_f = \alpha_i + \beta_i (\bar{r}_M - r_f)$$

And the covariance of $r_i - r_f$ with r_M is:

$$
\begin{aligned}
\sigma_{iM} &= Cov[r_i - r_f, r_M] = Cov[\alpha_i + \beta_i (r_M - r_f) + e_i, r_M] = \beta_i \sigma_M^2 \\
\Rightarrow \beta_i &= \frac{\sigma_{iM}}{\sigma_M^2}
\end{aligned}
$$

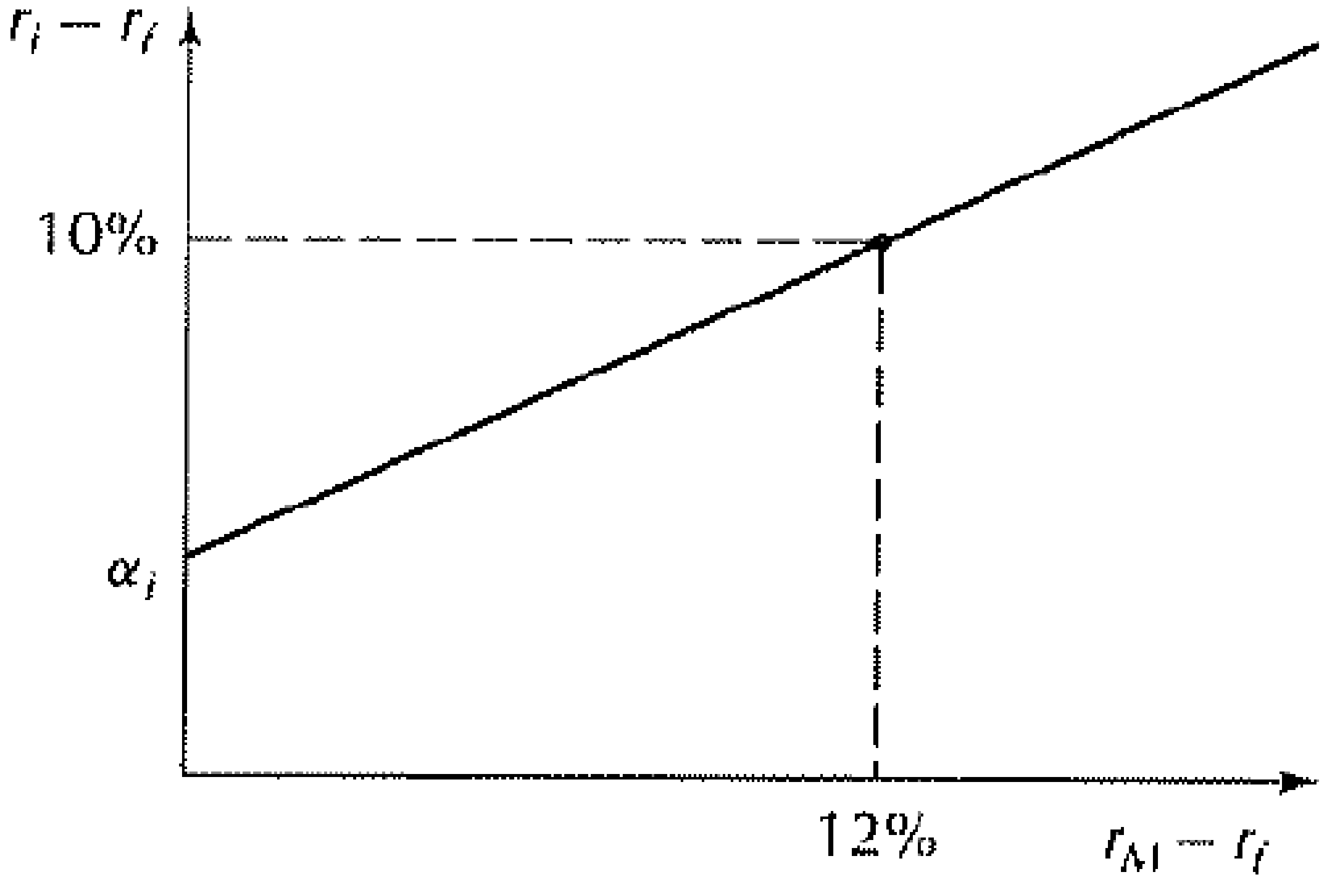
Another "line" can be calculated as a characteristic line, which is drawn by plotting r_i, as given by the equation:

$$r_i - r_f = \alpha_i + \beta_i (r_M - r_f),$$

as a function of its factor r_M in the $(r_M - r_f, r_i - r_f)$-space:

- e_i is assumed to be at its expectation, 0
- Slope of the line is equal to β_i
- Intercept of the line is equal to α_i
- CAPM predicts that $\alpha_i = 0$
- Measurements of r_i and its factor r_M can be plotted in a scatter diagram against this line

Security market line is drawn in $(\beta_i, \bar{r}_i)$-space and capital market line is drawn in $(\sigma, \bar{r})$-space.



2 Multifactor Model

The return can be calculated by extrapolating from a single factor to multiple factors. Starting with two factors:

$$r_i = a_i + b_{i1} f_i + b_{i2} f_2 + e_i,$$

where a_i is the intercept and b_{i1}, b_{i2} are factor loadings and the following assumptions:

- Expected error $\mathbb{E}[e_i] = 0$
- Error terms are uncorrelated with factors, and each other
- **Factors can correlate with each other**

For this model, the expected return in the two-factor model is:

$$\bar{r}_i = \mathbb{E}[r_i] = a_i + b_{i1} \bar{f} + b_{i2} \bar{f}_2$$

And the covariance:

$$
\begin{aligned}
Cov[r_i, r_j] &= \mathbb{E}\big[\, \big(b_{i1}(f_1 - \bar{f}_1) + b_{i2}(f_2 - \bar{f}_2) + e_i\big) \\
&\qquad \big(b_{j1}(f_1 - \bar{f}_1) + b_{j2}(f_2 - \bar{f}_2) + e_j\big)\, \big] \\
&= \begin{cases} b_{i1}b_{j1}\sigma^2_{f_1} + (b_{i1}b_{j2} + b_{i2}b_{j1})\sigma_{f_1,f_2} + b_{i2}b_{j2}\sigma^2_{f_2}, & i \neq j \\ b^2_{i1}\sigma^2_{f_1} + 2b_{i1}b_{i2}\sigma_{f_1,f_2} + b^2_{i2}\sigma^2_{f_2} + \sigma^2_{e_i}, & i = j \end{cases}
\end{aligned}
$$

By estimating the parameters as done previously for the single-factor model, the following calculation for covariance can be expressed:

$$
\begin{aligned}
Cov[r_i, f_1] &= \mathbb{E}\left[\left(b_{i1}(f_1 - \bar{f}_1) + b_{i2}(f_2 - \bar{f}_2) + e_i\right)(f_1 - \bar{f}_1)\right] \\
&= b_{i1}\sigma^2_{f_1} + b_{i2}\sigma_{f_1,f_2} \\
Cov[r_i, f_2] &= b_{i2}\sigma^2_{f_2} + b_{i1}\sigma_{f_1,f_2}
\end{aligned}
$$

Which is derived from the covariance matrix.

Multiple factors can be considered if a single-factor model has a large error term variance. If the error term variance is nearly as high as the variance of returns, the factor model does not explain much. Too many factors lead to **overfitting** $\Rightarrow$ Poor predictions.

Several factors can be used to estimate different factors:

- Gross National Product (GNP);
- Consumer price indices;
- Unemployment rate.

Factors extracted from the market, such as:

- Market portfolio return;
- Average return of companies in one industry;
- Days since the last market peak.

Firm characteristics, such as:

- Price-earnings ratio;
- Dividend payout ratio;
- Earnings forecast.

Fama-French discusses factors such as:

1. Market risk;
2. Firm size;
3. Book-to-market ratio.

Book-to-market ratio = Inverse of price (i.e., market capitalization) / book value (P/B) ratio. For details, see Fama & French (1993): Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics* 33 (optional reading, available at https://doi.org/10.1016/0304-405X(93)90023-5)/

3 Arbitrage Pricing Theory (ATP

A particular type of single-factor model is the arbitrage pricing theory model, where the parameters are chosen to exclude any arbitrage opportunities (which remove some combinations between all parameters).

In this model, such assumptions are required:

- There is a large number of assets;
- Instead of optimizing concerning mean-variance (as in CAPM), the investors just prefer higher returns to lower returns;
- S. Ross (1976): The Arbitrage Theory of Capital Asset Pricing. *Journal of Economic Theory* 13, 341-360.

Starting with a single factor with two assets and no error term:

$$\begin{aligned} r_i &= a_i + b_i f, \\ r_j &= a_j + b_j f \end{aligned}$$

Invest in assets i (weight $w_i = w$) and j ($w_j = 1 - w$) that follow a single factor model:

In a portfolio, the return is:

$$\begin{aligned} r &= w(a_i + b_i f) + (1 - w)(a_j + b_j f) \\ &= wa_i + (1 - w)a_j + (wb_i + (1 - w)b_j)\, f \end{aligned}$$

The selection of the weight w should be that the coefficient of factor f is 0:

$$\begin{aligned} & wb_i + (1 - w)b_j = 0 \\ \Rightarrow & w = \frac{b_j}{b_j - b_i} \end{aligned}$$

The portfolio with coefficient 0 for factor f is risk-free (no variance). Hence, its return must be $r_f = \lambda_0$:

$$\begin{aligned} r &= \frac{b_j}{b_j - b_i} a_i + \left(1 - \frac{b_j}{b_j - b_i}\right) a_j \\ &= \frac{b_j}{b_j - b_i} a_i - \frac{b_i}{b_j - b_i} a_j = \lambda_0 \end{aligned}$$

In this setup, λ_0 denotes the risk-free interest rate.

Given a risk-free interest rate λ_0, we find out that the factor model parameters of assets i and j must be proportional to each other to ensure the absence of arbitrage:

$$\begin{aligned} & r = \frac{b_j}{b_j - b_i} a_i - \frac{b_i}{b_j - b_i} a_j = \lambda_0 \\ \Rightarrow & b_j a_i - b_i a_j = \lambda_0 (b_j - b_i) \\ \Rightarrow & b_j (a_i - \lambda_0) = b_i (a_j - \lambda_0) \\ \Rightarrow & \frac{a_i - \lambda_0}{b_i} = \frac{a_j - \lambda_0}{b_j} \end{aligned}$$

Otherwise, the factor model would offer arbitrage opportunities. For, different riskless asset combinations would imply different risk-free interest rates.

Thus, for every asset i, ratio $(a_i - \lambda_0)/b_i$ must be equal to some constant c:

$$\Rightarrow \quad \frac{a_i - \lambda_0}{b_i} = c$$
$$\Leftrightarrow \quad a_i = \lambda_0 + b_i c$$

Thus:

$$\begin{aligned}\bar{r}_i &= a_i + b_i \bar{f} = \lambda_0 + b_i c + b_i \bar{f} \\ &= \lambda_0 + b_i(c + \bar{f}) = \lambda_0 + b_i \lambda_1,\end{aligned}$$

where $\lambda_1 = c + \bar{f}$ is the **price of risk** associated with factor f, i.e. the **factor price**. This can be generalized to several factors.

Another special version of APT is **without error terms** defined as simple APT.

Definition 3.1. Simple APT

Suppose that there are n assets whose rates of return are governed by $m < n$ factors according to the equation:

$$r_i = a_i + \sum_{j=1}^{m} b_{ij} f_j$$

for all assets $i = 1, 2, \ldots, n$. Then there are constants:
$\lambda_0, \lambda_1, \ldots, \lambda_m$ such that expected rates of return are given by:

$$\bar{r}_i = \lambda_0 + \sum_{j=1}^{m} b_{ij} \lambda_j$$

for all assets $i = 1, 2, \ldots, n$.

Suppose now that there is also an error term e_i in the factor model of return of asset i with m factors:

$$r_i = a_i + \sum_{j=1}^{m} b_{ij} f_j + e_i$$

Next, form a portfolio of n assets using weights w_i:

$$\begin{aligned}r = \sum_{w=1}^{n} w_i r_i &= \sum_{i=1}^{n} w_i a_i + \sum_{j=1}^{m} \sum_{i=1}^{n} w_i b_{ij} f_j + \sum_{i=1}^{n} w_i e_i \\ &= a + \sum_{j=1}^{m} b_j f_j + e\end{aligned}$$

where,

$$a = \sum_{i=1}^{n} w_i a_i, \quad b_j = \sum_{i=1}^{n} w_i b_{ij}, \quad e = \sum_{i=1}^{n} w_i e_i$$

The variance of the error term of the portfolio is:

$$\sigma_e^2 = \sum_{i=1}^{n} w_i^2 \sigma_{e_i}^2$$

Assume that all asset error term variances $\sigma_{e_i}^2$ are bounded, that is:

$$\sigma_{e_i}^2 \leq s^2$$

For some s, assume that all assets have similar weights (i.e., we have $w_i \leq W/n$ for some $W \approx 1$). This means that the portfolio is **well-diversified**.

With the assumptions of similar and bounded weights, we have:

$$\sigma_e^2 = \sum_{i=1}^{n} w_i^2 \sigma_{e_i}^2 \leq \sum_{i=1}^{n} \frac{W^2}{n^2} s^2 = \frac{1}{n} W^2 s^2$$
$$\Rightarrow \lim_{n \to \infty} \sigma_e^2 = 0$$

Hence, a well-diversified portfolio with many assets has practically no non-diversifiable risk. At the limit, the rate of return of such a portfolio is fully explained by the factor model (because the error terms tend to go to zero, as n goes to infinity):

$$r = a + \sum_{j=1}^{m} b_j f_j,$$

//

At limit, as the error terms have gone to zero, simple APT states the expected rate of return of a *well-diversified portfolio with a very large number of individual assets* is:

$$\bar{r} = \lambda_0 + \sum_{j=1}^{m} b_j \lambda_j,$$

//

Suppose the above holds for a well-diversified portfolio with a very large n. In that case, the same must also hold for an *individual asset* i, since different well-diversified portfolios may differ just by a small amount of the asset i. Thus, we have:

$$\bar{r}_i = \lambda_0 + \sum_{j=1}^{m} b_{ij} \lambda_j,$$

This pricing equation is referred to as the **General APT**.

In the CAPM, the returns are essentially explained by a factor model. Some insights can be gained if the assumptions of the general APT hold. Assuming that:

1. the CAPM holds,
2. the general APT holds (the number of assets n is large and market portfolio is well-diversified), and
3. the returns of individual assets are determined by the following two-factor model:
$$r_i = a_i + b_{i1} f_1 + b_{i2} f_2 + e_i$$

Covariance with the market portfolio is now:

$$Cov[r_M, r_i] = \mathbb{E}\left[(r_M - \bar{r}_M)\left(b_{i1}(f_1 - \bar{f}_1) + b_{i2}(f_2 - \bar{f}_2) + e_i\right)\right]$$
$$= b_{i1}Cov[r_M, f_1] + b_{i2}Cov[r_M, f_2] + Cov[r_M, e_i]$$

Since the assumptions of APT hold, we have $Cov[r_M, e_i] \approx 0$ and thus:

$$Cov[r_M, r_i] = b_{i1}Cov[r_M, f_1] + b_{i2}Cov[r_M, f_2]$$

Dividing by σ_M^2 gives the beta of an asset:

$$\beta_i = b_{i1}\frac{Cov[f_1, r_M]}{\sigma_M^2} + b_{i2}\frac{Cov[f_2, r_M]}{\sigma_M^2}$$
$$= b_{i1}\beta_{f_1} + b_{i2}\beta_{f_2}$$

The β_i of asset i is the factor-loading-weighted sum of the factors' betas.

4 Parameter Estimation

Assuming the following return rate:

$$1 + r_y = (1 + r_1)(1 + r_2)\cdots(1 + r_{12})$$

Assume that monthly returns are small:

$$1 + r_y \approx 1 + r_1 + r_2 + \cdots + r_{12}$$
$$\Rightarrow r_y = r_1 + r_2 + \cdots + r_{12}$$

Monthly returns are equally distributed and uncorrelated:

$$\bar{r}_y = 12\bar{r}$$
$$\sigma_y^2 = \mathbb{E}\left[\left(\sum_{i=1}^{12}(r_i - \bar{r})\right)^2\right] = \mathbb{E}\left[\sum_{i=1}^{12}(r_i - \bar{r})^2\right] = 12\sigma^2$$

When the number of periods p becomes larger, the ratio between the standard deviation and expected return for each period increases:

⇒ Finding short term estimators becomes more difficult;

- If the yearly parameters are $\mathbb{E}[r_y] = 12\%$ and $\sigma_y = 15\%$, the monthly parameters $p = 12$ are $\mathbb{E}[r_p] = 1\%$ and $\sigma_p = 1/\sqrt{12} \cdot 15\% = 4.33\%$;

- The one-month return is within the interval $1 \pm 4.33\%$ with a 68% probability, a rather wide confidence interval.

Thus, single-period expected returns are hard to estimate reliably even if the time series are long.

Let there be a time series of n **independent and identically distributed observations**, denoted by r_i, where each observation is drawn from a random variable with an expected value $\bar{r}$ and standard deviation σ:

Unbiased estimator of expected rate of return is:

$$\hat{\bar{r}} = \frac{1}{n}\sum_{i=1}^{n} r_i$$

Because the expected value of the estimator is the true expected rate of return $\bar{r}$:

$$\mathbb{E}\left[\hat{\bar{r}}\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[r_i] = \frac{1}{n}\sum_{i=1}^{n}\bar{r} = \bar{r}$$

The variance of the unbiased estimator of the expected rate of return is:

$$\sigma^2_{\hat{\bar{r}}} = Var\left[\hat{\bar{r}}\right] = Var\left[\frac{1}{n}\sum_{i=1}^{n} r_i\right] = \frac{1}{n^2}Var\left[\sum_{i=1}^{n} r_i\right]$$

Because the observations are independent, we have:

$$Var\left[\hat{\bar{r}}\right] = \frac{1}{n^2}\sum_{i=1}^{n}Var\left[r_i\right] = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{n}{n^2}\sigma^2 = \frac{1}{n}\sigma^2$$

The standard deviation of the unbiased estimator thus is:

$$\sigma_{\hat{\bar{r}}} = \frac{1}{\sqrt{n}}\sigma$$

Standard deviation $\sigma_{\hat{\bar{r}}}$ of estimator $\hat{\bar{r}}$ decreases slowly with n, because $\sqrt{n}$ is in its denominator. As an example, let monthly $\mathbb{E}[r] = 1\%$ and $\sigma = 4.33\%$ and consider a time series of $n = 12$ months:

$$\sigma_{\hat{\bar{r}}} = \frac{1}{\sqrt{12}}4.33\% = 1.25\%$$

Should we want to estimate the standard deviation, which is within 10% of the expected returns ($0.1 \cdot 1\% = 0.10\%$), then we would need a time series of 156 years and 3 months:

$$\sigma_{\hat{\bar{r}}} = \frac{1}{\sqrt{n}}4.33\% = 0.10\%$$
$$\Rightarrow n = \left(\frac{4.33\%}{0.10\%}\right)^2 = 1875 = 12 \cdot 156.25$$

5 Utility theory and risk aversion

In EUT (Expected Utility Theory), investors' preferences under risk are consistent with a utility function $U : \mathbb{R} \to \mathbb{R}$:

- Wealth level x_1 preferred to wealth level x_2 if and only if
$$U(x_1) > U(x_2)$$
- Random variable A is preferred to random variable B if and only if
$$\mathbb{E}\left[U(A)\right] > \mathbb{E}\left[U(B)\right]$$

von Neumann-Morgenstern utility functions are unique up to positive affine transformations. $U(x)$ and $V(x)$ represent the same preferences if and only if:

$$U(x) = aV(x) + b,$$

where $a > 0$ and $b \in \mathbb{R}$.

Different types of utility functions:

- Linear

$$U(x) = x$$

- Exponential $(a > 0)$

$$U(x) = -e^{-ax}$$

- Logarithmic

$$U(x) = \ln x$$

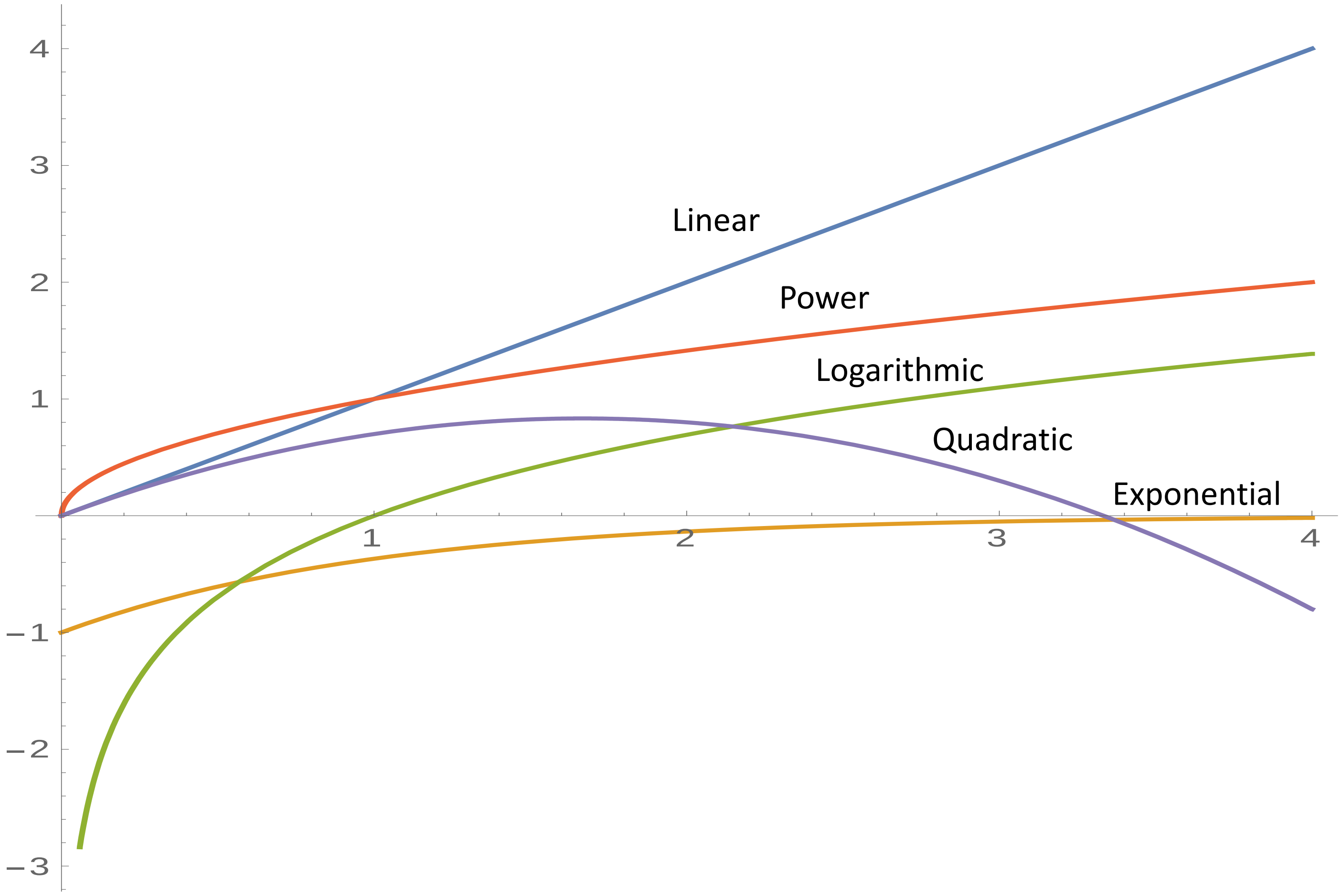
- Power $(b \leq 1, b \neq 0)$

$$U(x) = bx^b$$

- Quadratic

$$U(x) = x - bx^2$$

(increasing for $x < 1/(2b)$)



The **certainty equivalent** of a random variable X is the certain wealth c for which:

$$\mathbb{E}[U(c)] = \mathbb{E}[U(X)]$$
$$\Leftrightarrow U(c) = \mathbb{E}[U(X)]$$

For a 50% chance to win 100 € and a 50% chance of winning nothing, the certainty equivalent could be $c = 40$€. If U has an inverse function U^{-1}, certainty equivalent can be calculated as:

$$c = U^{-1}(\mathbb{E}[U(X)])$$

An investor is:

- **Risk neutral** if for all random variables X, his or her certainty equivalent for X is $\mathbb{E}[X]$
- **Risk averse** if for all non-constant random variables X, his or her certainty equivalent for X is less than $\mathbb{E}[X]$

- **Risk seeking** if for all non-constant random variables X, his or her certainty equivalent for X is more than $\mathbb{E}[X]$

In EUT, the investor with utility function U is:

- Risk neutral if U is linear,
- Risk averse if U is strictly concave, i.e.,

$$U(\lambda x + (1-\lambda)y) > \lambda U(x) + (1-\lambda)U(y)$$

  for all $x \neq y$ and $0 < \lambda < 1$.

- Risk seeking if U is strictly convex, i.e.,

$$U(\lambda x + (1-\lambda)y) < \lambda U(x) + (1-\lambda)U(y)$$

  for all $x \neq y$ and $0 < \lambda < 1$.

Arrow-Pratt risk aversion coefficient:

$$a(x) = -\frac{U''(x)}{U'(x)}$$

It measures the degree of risk aversion (concavity) at point x and also measures the **relative rate of change of slope** of U at x:

- Let $k(x) = U'(x)$ be the slope of U at x;
- Relative rate of change of $k(x)$ is:

$$\frac{dk(x)/dx}{k(x)} = \frac{U''(x)}{U'(x)} = -a(x)$$

For the exponential utility function, the risk aversion coefficient is constant:

$$U(x) = -e^{-bx} \Rightarrow U'(x) = be^{-bx}, U''(x) = -b^2e^{-bx}$$
$$\Rightarrow a(x) = -\frac{-b^2e^{-bx}}{be^{-bx}} = b$$

//

For the logarithmic utility function, risk aversion decreases with wealth:

$$U(x) = \ln x$$
$$\Rightarrow a(x) = \frac{1}{x}$$

The utility function may help the investor choose investments that suit them:

Via elicitation methods:

- Ask for certainty equivalents to get the value of U for different random variables;
- Select the functional form of a utility function, fix some parameters to 1, and proceed by carrying out more utility assessments;

Risk aversion is related to the mean-variance criterion. Assume quadratic utility:

$$U(x) = ax - \frac{1}{2}bx^2, \quad \text{where } a > 0, b \geq 0$$

- This is increasing for $x \leq a/b$;
- Assume that the initial wealth level is 0 (the result can be extended for positive wealth levels);
- Because $\mathbb{E}[Y^2] = Var[Y] + \mathbb{E}[Y]^2$, portfolio with random wealth Y has:

$$\begin{aligned}\mathbb{E}\left[U(Y)\right] &= \mathbb{E}\left[aY - \frac{1}{2}bY^2\right] = a\mathbb{E}[Y] - \frac{1}{2}b\mathbb{E}[Y^2] \\ &= a\mathbb{E}[Y] - \frac{1}{2}b\mathbb{E}[Y]^2 - \frac{1}{2}bVar[Y]\end{aligned}$$

⇒ Thus, the optimal portfolio can be chosen based on expected return and variance for quadratic utility functions.