Aalto University
School of Science

MS-E2114 Investment Science
Lecture VII: Factor models, parameter estimation, and utility

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Overview

Single factor model

Multifactor models

Arbitrage Pricing Theory (APT)

Parameter estimation

Utility theory and risk aversion

This lecture

- Factor models and arbitrage pricing theory
  - Seek to explain returns and correlations between assets
  - The random return of an asset is explained by other random variables which are common to all assets
  - Reduces the number of parameter estimates that are needed for mean-variance optimization
- Model parameter estimation
- von Neumann and Morgenstern expected utility theory visited briefly

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Single factor model

- Explain asset returns with common random variables
  - E.g. GDP (gross domestic product) or stock index
- Rate of return of asset i expressed as

$$r_i = a_i + b_i f + e_i,$$

where

  - a_i and b_i are constants
  - f is the random explanatory variable (i.e., the factor)
  - e_i is the random error term

Single factor model

- Single-factor model for rate of return of asset i

$$r_i = a_i + b_i f + e_i$$

- Assumptions:
  - $\mathbb{E}[e_i] = 0$ (not restrictive as a_i can be chosen freely)
  - e_i is not correlated with f

$$\Rightarrow \mathbb{E}\left[(f - \bar{f})(e_i - \bar{e}_i)\right] = \mathbb{E}\left[(f - \bar{f})e_i\right] = 0$$

  - Error terms of the assets are uncorrelated

$$\Rightarrow \mathbb{E}\left[(e_i - \bar{e}_i)(e_j - \bar{e}_j)\right] = \mathbb{E}\left[e_i e_j\right] = 0, i \neq j$$

  - Variances of error terms are known

$$\Rightarrow \mathbb{E}\left[e_i^2\right] = \sigma_{e_i}^2$$

Single factor model

- With these assumptions, the expected rate of return for asset i is

$$\bar{r}_i = \mathbb{E}[r_i] = a_i + b_i\,\mathbb{E}[f] + \mathbb{E}[e_i]$$
$$\Rightarrow \bar{r}_i = a_i + b_i\bar{f}$$

- Variance of the return r_i is

$$\begin{aligned}\sigma_i^2 &= \mathsf{Var}[r_i] = \mathbb{E}\left[(r_i - \bar{r}_i)^2\right] = \mathbb{E}\left[(a_i + b_i f + e_i - a_i - b_i\bar{f})^2\right] \\ &= \mathbb{E}\left[\left(b_i(f - \bar{f}) + e_i\right)^2\right] = \mathbb{E}\left[b_i^2(f - \bar{f})^2 + 2b_i(f - \bar{f})e_i + e_i^2\right]\end{aligned}$$
$$\Rightarrow \sigma_i^2 = b_i^2\sigma_f^2 + \sigma_{e_i}^2$$

Single factor model

- Based on stated assumptions, the covariance between assets i and j is:

$$
\begin{aligned}
\sigma_{ij} &= \mathsf{Cov}[r_i, r_j] = \mathbb{E}\left[(r_i - \bar{r}_i)(r_j - \bar{r}_j)\right] \\
&= \mathbb{E}\left[\left(b_i(f - \bar{f}) + e_i\right)\left(b_j(f - \bar{f}) + e_j\right)\right] \\
&= \mathbb{E}\left[b_i b_j (f - \bar{f})^2 + (b_j e_i + b_i e_j)(f - \bar{f}) + e_i e_j\right] \\
\Rightarrow \sigma_{ij} &= b_i b_j \sigma_f^2, i \neq j
\end{aligned}
$$

Single factor model

- Thus it follows that

$$\begin{aligned}\mathsf{Cov}[r_i, f] &= \mathbb{E}\left[(r_i - \bar{r}_i)(f - \bar{f})\right] \\ &= \mathbb{E}\left[\left(b_i(f - \bar{f}) + e_i\right)(f - \bar{f})\right] \\ \Rightarrow \mathsf{Cov}[r_i, f] &= b_i \sigma_f^2 \\ \Rightarrow b_i &= \frac{\mathsf{Cov}[r_i, f]}{\sigma_f^2}\end{aligned}$$

- A total of $3n + 2$ parameters to be estimated
  - $\bar{f}, \sigma_f^2, a_i, b_i$, and $\sigma_{e_i}^2$, for $i = 1, 2, \ldots, n$

Estimating a_i and b_i

- Parameters a_i and b_i can be estimated from the time series of r_i and f
  - Estimates differ depending on the selected time span
  - Averaging and other statistical methods can be used to improve accuracy
- Standard statistical estimators

$$\hat{\bar{r}}_i = \frac{1}{n} \sum_{k=1}^{n} r_i^k$$

$$\hat{\sigma}_i^2 = \frac{1}{n-1} \sum_{k=1}^{n} \left(r_i^k - \hat{\bar{r}}_i \right)^2$$

$$\widehat{\mathsf{Cov}}[r_i, f] = \frac{1}{n-1} \sum_{k=1}^{n} \left(r_i^k - \hat{\bar{r}}_i \right) \left(f^k - \hat{\bar{f}} \right),$$

where superscript k denotes the kth sample

Estimating a_i and b_i

- The model parameters can be calculated from the standard estimates

$$b_i = \frac{\widehat{\mathsf{Cov}}[r_i, f]}{\hat{\sigma}_f^2}$$
$$a_i = \hat{\bar{r}}_i - b_i \hat{\bar{f}}$$

- The variance of error terms become

$$\sigma_i^2 = b_i^2 \sigma_f^2 + \sigma_{e_i}^2$$
$$\Rightarrow \hat{\sigma}_{e_i}^2 = \hat{\sigma}_i^2 - b_i^2 \hat{\sigma}_f^2$$

Portfolios in the single factor model

- Form a portfolio of n assets
  - Asset i has weight w_i
- Returns of the assets follow the factor model

$$r_i = a_i + b_i f + e_i$$

- Return of the portfolio

$$r = \sum_{w=1}^{n} w_i r_i = \sum_{i=1}^{n} w_i a_i + \left(\sum_{i=1}^{n} w_i b_i \right) f + \sum_{i=1}^{n} w_i e_i = a + bf + e,$$

where

$$a = \sum_{i=1}^{n} w_i a_i, \quad b = \sum_{i=1}^{n} w_i b_i, \quad e = \sum_{i=1}^{n} w_i e_i$$

Portfolios in the single factor model

- For the error term of the portfolio return, we have

$$\mathbb{E}[e] = \mathbb{E}\left[\sum_{i=1}^{n} w_i e_i\right] = \sum_{i=1}^{n} w_i \, \mathbb{E}[e_i]$$

$$\Rightarrow \mathbb{E}[e] = 0$$

$$\mathrm{Cov}[f, e] = \mathbb{E}\left[(f - \bar{f}) \sum_{i=1}^{n} w_i e_i\right] = \sum_{i=1}^{n} w_i \, \mathbb{E}[(f - \bar{f}) e_i]$$

$$\Rightarrow \mathrm{Cov}[f, e] = 0$$

$$\mathrm{Var}[e] = \mathbb{E}\left[\left(\sum_{i=1}^{n} w_i e_i\right)\left(\sum_{j=1}^{n} w_j e_j\right)\right] = \sum_{i=1}^{n} w_i^2 \, \mathbb{E}[e_i^2]$$

$$\Rightarrow \mathrm{Var}[e] = \sum_{i=1}^{n} w_i^2 \sigma_{e_i}^2$$

Portfolios in the single factor model

- Assume that assets have equal weights and the variance of error terms is $\sigma_{e_i}^2 = s^2$. Then the variance of the error term of the portfolio is

$$\sigma_e^2 = \mathsf{Var}[e] = \sum_{i=1}^{n} w_i^2 \sigma_{e_i}^2 = \sum_{i=1}^{n} \frac{1}{n^2} s^2 = \frac{1}{n} s^2,$$

and the variance of the portfolio return is

$$\sigma^2 = \mathsf{Var}[r] = b^2 \sigma_f^2 + \sigma_e^2,$$

where $\sigma_e^2 \to 0$ as $n \to \infty$

- Variance related to the error terms e_i can be diversified
- Variance related to terms $b_i f$ cannot be diversified

Single factor model and CAPM

- Suppose you want to postulate a single-factor model (factor = r_M) similar to the CAPM as follows:

$$r_i - r_f = \alpha_i + \beta_i(r_M - r_f) + e_i$$

- Taking the expectation of this postulated factor model gives

$$\bar{r}_i - r_f = \alpha_i + \beta_i(\bar{r}_M - r_f)$$

- Covariance of $r_i - r_f$ with r_M is

$$\sigma_{iM} = \mathsf{Cov}[r_i - r_f, r_M] = \mathsf{Cov}[\alpha_i + \beta_i(r_M - r_f) + e_i, r_M] = \beta_i \sigma_M^2$$

$$\Rightarrow \beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

- This is in agreement with the CAPM

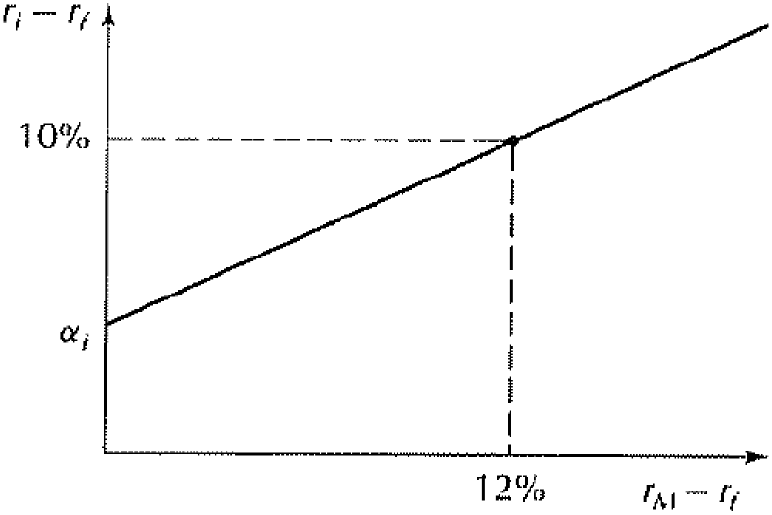
Characteristic line

- **Characteristic line** is drawn by plotting r_i, as given by equation

$$r_i - r_f = \alpha_i + \beta_i(r_M - r_f),$$

  as a function of its factor r_M in the $(r_M - r_f, r_i - r_f)$-space
  - e_i is assumed to be at its expectation, 0
  - Slope of the line is equal to β_i
  - Intercept of the line is equal to α_i
  - CAPM predicts that $\alpha_i = 0$
  - Measurements of r_i and its factor r_M can be plotted in a scatter diagram against this line
- Security market line is drawn in $(\beta_i, \bar{r}_i)$-space
- Capital market line is drawn in $(\sigma, \bar{r})$-space

Characteristic line



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Multifactor models

- The return can be explained with more than one factor
- With two factors

$$r_i = a_i + b_{i1}f_i + b_{i2}f_2 + e_i,$$

  where a_i is the intercept and b_{i1}, b_{i2} are factor loadings
- Assumptions
  - Expected error $\mathbb{E}[e_i] = 0$
  - Error terms are uncorrelated with factors and each other
  - **Factors can correlate with each other**

Multifactor models

- Expected return in the two factor model

$$\bar{r}_i = \mathbb{E}[r_i] = a_i + b_{i1}\bar{f} + b_{i2}\bar{f}_2$$

- Covariance

$$\begin{aligned}\mathsf{Cov}[r_i, r_j] &= \mathbb{E}\left[\,\left(b_{i1}(f_1 - \bar{f}_1) + b_{i2}(f_2 - \bar{f}_2) + e_i\right)\right.\\ &\qquad \left.\left(b_{j1}(f_1 - \bar{f}_1) + b_{j2}(f_2 - \bar{f}_2) + e_j\right)\,\right]\\ &= \begin{cases} b_{i1}b_{j1}\sigma_{f_1}^2 + (b_{i1}b_{j2} + b_{i2}b_{j1})\sigma_{f_1,f_2} + b_{i2}b_{j2}\sigma_{f_2}^2, & i \neq j \\ b_{i1}^2\sigma_{f_1}^2 + 2b_{i1}b_{i2}\sigma_{f_1,f_2} + b_{i2}^2\sigma_{f_2}^2 + \sigma_{e_i}^2, & i = j \end{cases}\end{aligned}$$

Estimating the parameters of factor models

- Loadings b_{i1}, b_{i2} can be estimated from the covariance matrix

$$\begin{aligned}
\mathsf{Cov}[r_i, f_1] &= \mathbb{E}\left[\left(b_{i1}(f_1 - \bar{f}_1) + b_{i2}(f_2 - \bar{f}_2) + e_i\right)(f_1 - \bar{f}_1)\right] \\
&= b_{i1}\sigma_{f_1}^2 + b_{i2}\sigma_{f_1,f_2} \\
\mathsf{Cov}[r_i, f_2] &= b_{i2}\sigma_{f_2}^2 + b_{i1}\sigma_{f_1,f_2}
\end{aligned}$$

  - Solve these equations for b_{i1} and b_{i2}

Estimating the parameters of factor models

- The use of multiple factors can be considered if a single factor model has a large error term variance
- If the error term variance is nearly as high as the variance of returns, the factor model does not explain much
- Too many factors leads to overfitting ⇒ Poor predictions

Selection of factors

- No unambiguous answer - depends on what the key factors are believed to be
- External factors, such as
  - Gross National Product (GNP)
  - Consumer price indices
  - Unemployment rate
- Factors extracted from the market, such as
  - Market portfolio return
  - Average return of companies in one industry
  - Days since the last market peak
- Firm characteristics, such as
  - Price-earnings ratio
  - Dividend payout ratio
  - Earnings forecast

Selection of factors

- Fama-French discuss factors such as:
  1. Market risk
  2. Firm size
  3. Book-to-market ratio
- Book-to-market ratio = Inverse of price (i.e., market capitalization) / book value (P/B) ratio
- For details see Fama & French (1993): Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics* 33 (optional reading, available at `https://doi.org/10.1016/0304-405X(93)90023-5`)

Overview

Arbitrage Pricing Theory (APT)

- In APT, the parameters of a factor model are chosen to exclude **arbitrage opportunities**
  - Under the no-arbitrage principle, not all parameter combinations are possible
- Key assumptions
  - There is a large number of assets
  - Instead of optimizing with respect to mean-variance (as in CAPM), the investors just prefer higher returns to lower returns
  - S. Ross (1976): The Arbitrage Theory of Capital Asset Pricing. *Journal of Economic Theory* 13, 341-360.

APT: Example with two assets

- A single factor with two assets and no error term

$$r_i = a_i + b_i f,$$
$$r_j = a_j + b_j f$$

- Invest in assets i (weight $w_i = w$) and j ($w_j = 1 - w$) that follow a single factor model
- Portfolio return

$$\begin{aligned} r &= w(a_i + b_i f) + (1 - w)(a_j + b_j f) \\ &= w a_i + (1 - w) a_j + (w b_i + (1 - w) b_j) f \end{aligned}$$

- Select the weight w so that the coefficient of factor f is 0

$$w b_i + (1 - w) b_j = 0$$
$$\Rightarrow w = \frac{b_j}{b_j - b_i}$$

APT: Example with two assets

- The portfolio with coefficient 0 for factor f is risk free (no variance), hence its return must be $r_f = \lambda_0$

$$
\begin{aligned}
r &= \frac{b_j}{b_j - b_i} a_i + \left(1 - \frac{b_j}{b_j - b_i}\right) a_j \\
&= \frac{b_j}{b_j - b_i} a_i - \frac{b_i}{b_j - b_i} a_j = \lambda_0
\end{aligned}
$$

- In this setup, λ_0 denotes the risk-free interest rate

APT: Example with two assets

- Given a risk-free interest rate λ_0, we find out that the factor model parameters of assets i and j must be proportional to each other to ensure absence of arbitrage:

$$
\begin{aligned}
r = \frac{b_j}{b_j - b_i} a_i - \frac{b_i}{b_j - b_i} a_j &= \lambda_0 \\
\Rightarrow b_j a_i - b_i a_j &= \lambda_0 (b_j - b_i) \\
\Rightarrow b_j (a_i - \lambda_0) &= b_i (a_j - \lambda_0) \\
\Rightarrow \frac{a_i - \lambda_0}{b_i} &= \frac{a_j - \lambda_0}{b_j}
\end{aligned}
$$

- Otherwise, the factor model would offer arbitrage opportunities
  - E.g., different riskless asset combinations would imply different risk-free interest rates

APT: Example with two assets

- Thus, for every asset i, ratio $(a_i - \lambda_0)/b_i$ must be equal to some constant c

$$\Rightarrow \quad \frac{a_i - \lambda_0}{b_i} = c$$
$$\Leftrightarrow \quad a_i = \lambda_0 + b_i c$$

- Thus

$$\begin{aligned} \bar{r}_i &= a_i + b_i\bar{f} = \lambda_0 + b_i c + b_i\bar{f} \\ &= \lambda_0 + b_i(c + \bar{f}) = \lambda_0 + b_i\lambda_1, \end{aligned}$$

where $\lambda_1 = c + \bar{f}$ is the **price of risk** associated with factor f, i.e. the **factor price**.

- This can be generalized to several factors

Simple APT (no error terms)

Definition

(**Simple APT**) Suppose that there are n assets whose rates of return are governed by $m < n$ factors according to the equation

$$r_i = a_i + \sum_{j=1}^{m} b_{ij} f_j$$

for all assets $i = 1, 2, \dots, n$. Then there are constants $\lambda_0, \lambda_1, \dots, \lambda_m$ such that expected rates of return are given by

$$\bar{r}_i = \lambda_0 + \sum_{j=1}^{m} b_{ij} \lambda_j$$

for all assets $i = 1, 2, \dots, n$.

- λ_j = **price of risk** of factor j (i.e., **factor price**)
- b_{ij} = **factor loading** of factor j of asset i

Factor model with error terms

- Suppose now that there is also an error term e_i in the factor model of return of asset i with m factors

$$r_i = a_i + \sum_{j=1}^{m} b_{ij} f_j + e_i$$

- Next, form a portfolio of n assets using weights w_i

$$r = \sum_{w=1}^{n} w_i r_i = \sum_{i=1}^{n} w_i a_i + \sum_{j=1}^{m} \sum_{i=1}^{n} w_i b_{ij} f_j + \sum_{i=1}^{n} w_i e_i$$

$$= a + \sum_{j=1}^{m} b_j f_j + e$$

where

$$a = \sum_{i=1}^{n} w_i a_i, \quad b_j = \sum_{i=1}^{n} w_i b_{ij}, \quad e = \sum_{i=1}^{n} w_i e_i$$

Well-diversified portfolios

- Variance of the error term of the portfolio is

$$\sigma_e^2 = \sum_{i=1}^{n} w_i^2 \sigma_{e_i}^2$$

- Assume that all asset error term variances $\sigma_{e_i}^2$ are bounded, that is,

$$\sigma_{e_i}^2 \leq s^2$$

  for some s, and assume that all assets have similar weights (i.e., we have $w_i \leq W/n$ for some $W \approx 1$)
- This means that the portfolio is **well-diversified**

Well-diversified portfolio

- With the assumptions of similar and bounded weights, we have

$$\sigma_e^2 = \sum_{i=1}^{n} w_i^2 \sigma_{e_i}^2 \leq \sum_{i=1}^{n} \frac{W^2}{n^2} s^2 = \frac{1}{n} W^2 s^2$$
$$\Rightarrow \lim_{n \to \infty} \sigma_e^2 = 0$$

- Hence, a well-diversified portfolio with a large number of assets has practically no non-diversifiable risk
- At limit, the rate of return of such a portfolio is fully explained by the factor model (because the error terms tend to go to zero, as n goes to infinity)

$$r = a + \sum_{j=1}^{m} b_j f_j,$$

General APT

- At limit, as the error terms have gone to zero, simple APT states the expected rate of return of a *well-diversified portfolio with a very large number of individual assets* is

$$\bar{r} = \lambda_0 + \sum_{j=1}^{m} b_j \lambda_j,$$

- If the above holds for a well-diversified portfolio with a very large n, then the same must also hold for an *individual asset* i, since different well-diversified portfolios may differ just by a small amount of the asset i. Thus, we have:

$$\bar{r}_i = \lambda_0 + \sum_{j=1}^{m} b_{ij} \lambda_j,$$

- This pricing equation is referred to as the **General APT**
- Note: The rigorous proofs are quite technical

APT and CAPM

- In the CAPM, the returns are essentially explained by a factor model
- Some insights can be gained if the assumptions of the general APT hold

APT and CAPM

- Let us assume that
  1. the CAPM holds,
  2. the general APT holds (the number of assets n is large and market portfolio is well-diversified), and
  3. the returns of individual assets are determined by the following two factor model:

$$r_i = a_i + b_{i1}f_1 + b_{i2}f_2 + e_i$$

- Covariance with the market portfolio is now

$$\begin{aligned}\mathrm{Cov}[r_M, r_i] &= \mathbb{E}\left[(r_M - \bar{r}_M)\left(b_{i1}(f_1 - \bar{f}_1) + b_{i2}(f_2 - \bar{f}_2) + e_i\right)\right] \\ &= b_{i1}\,\mathrm{Cov}[r_M, f_1] + b_{i2}\,\mathrm{Cov}[r_M, f_2] + \mathrm{Cov}[r_M, e_i]\end{aligned}$$

APT and CAPM

- Since the assumptions of APT hold, we have $\mathsf{Cov}[r_M, e_i] \approx 0$ and thus

$$\mathsf{Cov}[r_M, r_i] = b_{i1}\,\mathsf{Cov}[r_M, f_1] + b_{i2}\,\mathsf{Cov}[r_M, f_2]$$

- Dividing by σ_M^2 gives the beta of an asset

$$\begin{aligned}\beta_i &= b_{i1}\frac{\mathsf{Cov}[f_1, r_M]}{\sigma_M^2} + b_{i2}\frac{\mathsf{Cov}[f_2, r_M]}{\sigma_M^2}\\ &= b_{i1}\beta_{f_1} + b_{i2}\beta_{f_2}\end{aligned}$$

- The β_i of asset i is the factor-loading-weighted sum of the factors' betas

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Parameter estimation

- Annual return r_y formed from monthly returns $r_1, r_2, \ldots, r_{12}$

$$1 + r_y = (1 + r_1)(1 + r_2) \cdots (1 + r_{12})$$

- Assume that monthly returns are small:

$$\begin{aligned} 1 + r_y &\approx 1 + r_1 + r_2 + \cdots + r_{12} \\ \Rightarrow r_y &= r_1 + r_2 + \cdots + r_{12} \end{aligned}$$

- Monthly returns are equally distributed and uncorrelated

$$\begin{aligned} \bar{r}_y &= 12\bar{r} \\ \sigma_y^2 &= \mathbb{E}\left[\left(\sum_{i=1}^{12}(r_i - \bar{r})\right)^2\right] = \mathbb{E}\left[\sum_{i=1}^{12}(r_i - \bar{r})^2\right] = 12\sigma^2 \end{aligned}$$

Parameter estimation

- If there are p periods in a year, then

$$\bar{r}_p = \bar{r}_y / p$$
$$\sigma_p = \sigma_y / \sqrt{p}$$

- When the number of periods p becomes larger, the ratio between the standard deviation and expected return for each period increases
  - $\Rightarrow$ Finding short term estimators becomes more difficult
  - If the yearly parameters are $\mathbb{E}[r_y] = 12\%$ and $\sigma_y = 15\%$, the monthly parameters $p = 12$ are $\mathbb{E}[r_p] = 1\%$ and $\sigma_p = 1/\sqrt{12} \cdot 15\% = 4.33\%$
  - The one month return is within the interval $1 \pm 4.33\%$ with a 68% probability, which is a rather wide confidence interval
- Thus, single period expected returns are hard to estimate reliably even if the time series are long

Parameter estimation

- Let there be a time series of *n independent and identically distributed observations*, denoted by r_i, where each observation is drawn from a random variable with an expected value $\bar{r}$ and standard deviation σ
- **Unbiased estimator** of expected rate of return is

$$\hat{\bar{r}} = \frac{1}{n}\sum_{i=1}^{n} r_i$$

because expected value of the estimator is the true expected rate of return $\bar{r}$:

$$\mathbb{E}\left[\hat{\bar{r}}\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[r_i] = \frac{1}{n}\sum_{i=1}^{n}\bar{r} = \bar{r}$$

Parameter estimation

- Variance of the unbiased estimator of the expected rate of return is:

$$\sigma_{\hat{\bar{r}}}^2 = \mathsf{Var}\left[\hat{\bar{r}}\right] = \mathsf{Var}\left[\frac{1}{n}\sum_{i=1}^{n} r_i\right] = \frac{1}{n^2}\mathsf{Var}\left[\sum_{i=1}^{n} r_i\right]$$

Because the observations are independent, we have

$$\mathsf{Var}\left[\hat{\bar{r}}\right] = \frac{1}{n^2}\sum_{i=1}^{n}\mathsf{Var}\left[r_i\right] = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{n}{n^2}\sigma^2 = \frac{1}{n}\sigma^2$$

- Standard deviation of the unbiased estimator thus is:

$$\sigma_{\hat{\bar{r}}} = \frac{1}{\sqrt{n}}\sigma$$

Parameter estimation

- Standard deviation $\sigma_{\hat{\hat{r}}}$ of estimator $\hat{\hat{r}}$ decreases slowly with n, because $\sqrt{n}$ is in its denominator
- Let monthly $\mathbb{E}[r] = 1\%$ and $\sigma = 4.33\%$ and consider a time series of $n = 12$ months

$$\sigma_{\hat{\hat{r}}} = \frac{1}{\sqrt{12}} 4.33\% = 1.25\%$$

- Should we want to estimate the standard deviation which is within 10% of the expected returns ($0.1 \cdot 1\% = 0.10\%$), then we would need a time series of 156 years and 3 months

$$\sigma_{\hat{\hat{r}}} = \frac{1}{\sqrt{n}} 4.33\% = 0.10\%$$

$$\Rightarrow n = \left(\frac{4.33\%}{0.10\%} \right)^2 = 1875 = 12 \cdot 156.25$$

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Investor's risk preferences

- We have discussed the construction of efficient portfolios
  - Mean-variance portfolio theory
  - CAPM
- Which one out of the efficient portfolios should the investor select?
- Specifically, portfolios can be characterized by their expected return and risk (standard deviation), that is, by an ordered pair (r, σ)

$\Rightarrow$ Which combination of these parameters should be selected?

Expected utility theory (EUT)

(von Neumann & Morgenstern 1947)

- In EUT, investors' preferences under risk are consistent with a utility function $U : \mathbb{R} \to \mathbb{R}$
  - Wealth level x_1 preferred to wealth level x_2 if and only if

$$U(x_1) > U(x_2)$$

  - Random variable A is preferred to random variable B if and only if

$$\mathbb{E}[U(A)] > \mathbb{E}[U(B)]$$

- von Neumann-Morgenstern utility functions are unique up to positive affine transformations

  $\Rightarrow$ $U(x)$ and $V(x)$ represent the same preferences if and only if

$$U(x) = aV(x) + b,$$

  where $a > 0$ and $b \in \mathbb{R}$

Example on applying expected utility theory

- Investor invests in either
  - A: Bank account for a profit of 6 k€, or
  - B: Stock that yields a profit of
    - 10 k€ (probability 0.4)
    - 5 k€ (probability 0.4)
    - 1 k€ (probability 0.2)
- Investor's utility function is $U(x) = \sqrt{x}$ (unit of x is k€)

$$\mathbb{E}\left[U(A)\right] = U(6) = 2.45$$

$$\mathbb{E}\left[U(B)\right] = 0.4U(10) + 0.4U(5) + 0.2U(1) = 2.36$$

$\Rightarrow$ A is preferred to B, because $\mathbb{E}\left[U(A)\right] = 2.45 > 2.36 = \mathbb{E}\left[U(B)\right]$

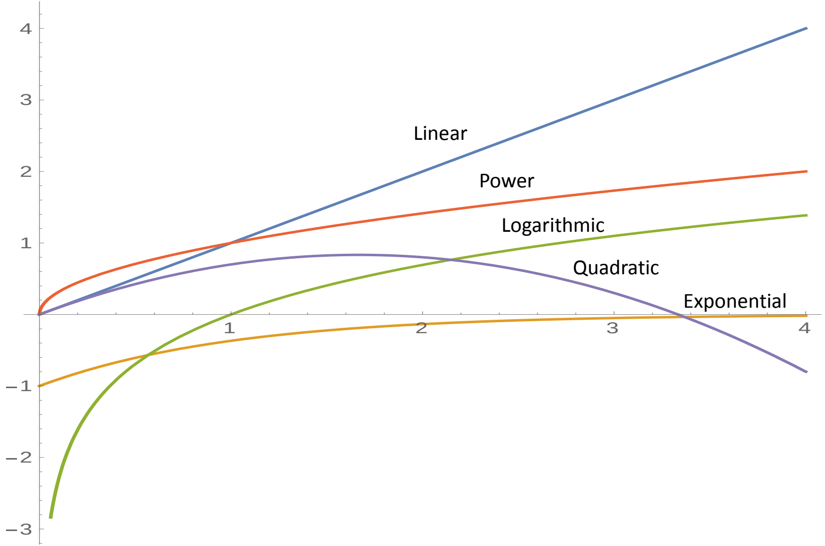
- Note that $\mathbb{E}[A] = 6 < 6.2 = \mathbb{E}[B]$

Widely used utility functions

- Linear
$$U(x) = x$$
- Exponential ($a > 0$)
$$U(x) = -e^{-ax}$$
- Logarithmic
$$U(x) = \ln x$$
- Power ($b \leq 1, b \neq 0$)
$$U(x) = bx^b$$
- Quadratic
$$U(x) = x - bx^2$$

  (increasing for $x < 1/(2b)$

Widely used utility functions



Certainty equivalent

- The **certainty equivalent** of a random variable X is the certain wealth c for which

$$\mathbb{E}\left[U(c)\right] = \mathbb{E}\left[U(X)\right]$$
$$\Leftrightarrow U(c) = \mathbb{E}\left[U(X)\right]$$

- E.g, for a 50% chance to win 100 € and 50% chance of winning nothing, the certainty equivalent could be $c = 40$€
- If U has an inverse function U^{-1}, certainty equivalent can be calculated as

$$c = U^{-1}\left(\mathbb{E}\left[U(X)\right]\right)$$

Risk aversion

- Investor is:
  - **Risk neutral** if for all random variables X, his or her certainty equivalent for X is $\mathbb{E}[X]$
  - **Risk averse** if for all non-constant random variables X, his or her certainty equivalent for X is less than $\mathbb{E}[X]$
  - **Risk seeking** if for all non-constant random variables X, his or her certainty equivalent for X is more than $\mathbb{E}[X]$
- In EUT, investor with utility function U is:
  - Risk neutral if U is linear
  - Risk averse if U is strictly concave, i.e.,

    $$U(\lambda x + (1-\lambda)y) > \lambda U(x) + (1-\lambda)U(y)$$

    for all $x \neq y$ and $0 < \lambda < 1$
  - Risk seeking if U is strictly convex, i.e.,

    $$U(\lambda x + (1-\lambda)y) < \lambda U(x) + (1-\lambda)U(y)$$

    for all $x \neq y$ and $0 < \lambda < 1$

Risk aversion coefficient

- Arrow-Pratt risk aversion coefficient

$$a(x) = -\frac{U''(x)}{U'(x)}$$

- Measures the degree of risk aversion (concavity) at point x
- Measures the **relative rate of change** of **slope** of U at x
  - Let $k(x) = U'(x)$ be the slope of U at x
  - Relative rate of change of $k(x)$ is

$$\frac{dk(x)/dx}{k(x)} = \frac{U''(x)}{U'(x)} = -a(x)$$

Risk aversion coefficient

- For the exponential utility function, risk aversion coefficient is constant

$$U(x) = -e^{-bx} \Rightarrow U'(x) = be^{-bx}, U''(x) = -b^2 e^{-bx}$$
$$\Rightarrow a(x) = -\frac{-b^2 e^{-bx}}{be^{-bx}} = b$$

- For logarithmic utility function, risk aversion decreases with wealth

$$U(x) = \ln x$$
$$\Rightarrow a(x) = \frac{1}{x}$$

Elicitation of utility functions

- The utility function may help the investor choose investments that suit him or her
- Elicitation methods
  - Ask for certainty equivalents to get the value of U for different random variables
  - Select the functional form of utility function, fix some parameters to 1, proceed by carrying out more utility assessments
  - Questionnaires (Luenberger p. 238)

Utility function and the mean-variance criterion

- Risk aversion is related to the mean-variance criterion
- Example: Assume quadratic utility

$$U(x) = ax - \frac{1}{2}bx^2, \quad \text{where } a > 0, b \geq 0$$

  - This is increasing for $x \leq a/b$
  - Assume that the initial wealth level is 0 (the result can be extended for positive wealth levels)
  - Because $\mathbb{E}[Y^2] = \mathsf{Var}[Y] + \mathbb{E}[Y]^2$, portfolio with random wealth Y has

$$\begin{aligned}\mathbb{E}\left[U(Y)\right] &= \mathbb{E}\left[aY - \frac{1}{2}bY^2\right] = a\,\mathbb{E}[Y] - \frac{1}{2}b\,\mathbb{E}[Y^2] \\ &= a\,\mathbb{E}[Y] - \frac{1}{2}b\,\mathbb{E}[Y]^2 - \frac{1}{2}b\,\mathsf{Var}[Y]\end{aligned}$$

$\Rightarrow$ Thus, for a quadratic utility functions, the optimal portfolio can be chosen based on expected return and variance

Overview

Single factor model

Multifactor models

Arbitrage Pricing Theory (APT)

Parameter estimation

Utility theory and risk aversion