MSE2114 - Investment Science Lecturer Notes VIII

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Abstract

In this lecture, we are introducing the concept of derivative securities which are A financial instrument whose payoff is determined by (derived from) the value of another variable, typically the price of another ("underlying") asset.

Contents

1 Forward Contract

First, a definition:

Forward contract is commitment to buy or sell a predetermined amount of the underlying asset at a specific price and time. For example, a commitment to sell 1000 barrels of oil after 9 months for 100 €/barrel. For those, the **forward price** F is the price paid at the time of delivery, which is usually F determined so that the value of the contract is zero at the time of making the contract. Also, the value of the underlying asset is determined in the **spot market**.

Buyer of the underlying asset is said to have a **long position**. For instance, if the oil price were to rise to $P > 100$ €/barrel, the buyer has secured the purchase at the lower predetermined price of 100 €. Seller of the underlying asset is said to have a **short position**. Similarly, if the oil price were to fall to $P < 100$ €/barrel, the seller has secured the sale at the higher predetermined price of 100 €/barrel.

The forward price formula is given by the following definition:

Definition 1.1. Forward price value

(**Forward price formula**) Suppose that the asset whose spot price at time $t = 0$ is S can be stored at zero cost and that short selling is possible. Then the theoretical forward price F for delivery at T is:

$$F = \frac{S}{d(0,T)} = (1 + s_T)S,$$

where $d(0,T) = \frac{1}{1+s_T}$ is the risk-free discount factor for the time period $[0,T]$, derived from the risk free interest rate s_T .

Proof: If there are no storage costs, the forward price is implied by forward rates through the following arbitrage arguments:

1. At time 0, take a short position in the forward contract of the asset at the forward price F at time T (i.e., you are committed to sell the asset);
2. At time 0, borrow money equal to asset price S at the risk-free interest rate $s_T = 1/(d(0,T) - 1)$;
3. At time 0, buy the asset immediately from the market at price S and store it until time T (for free);
4. At time T, sell the asset at the forward price F;
5. At time T, pay back the loan principal S and the accrued interest Ss_T;
6. You had zero cash flow in the beginning and your cash flow at time T is $F - S(1 + s_T) = F - S/d(0,T)$;
7. In order to avoid arbitrage opportunities, we must have:

$$F - S/d(0,T) = 0$$
$$\Rightarrow F = \frac{S}{d(0,T)}$$ 😎

In practice, storage cost also exist and it should be taking into account.

Definition 1.2. Forward price value with carrying costs

(**Forward price formula with carrying costs**) Suppose that the holding costs for an asset are $c(k)$ per unit at time k, and that it can be sold short. Suppose the initial price is S. Then the theoretical forward price is:

$$F = \frac{S}{d(0,T)} + \sum_{k=0}^{T-1} \frac{c(k)}{d(k,T)},$$

where $d(k,T)$ is the risk-free discount factor from time k to time T.

Proof: The forward price can again be derived from the no arbitrage assumption using the following logic:

1. At time 0, take a short position in the forward contract of the asset at the forward price F at time T (i.e., you are committed to sell the asset);

2. At time 0, borrow money equal to asset price S at the risk-free interest rate $s_T = 1/(d(0,T) - 1)$;

3. At time 0, buy the asset immediately from the market at price S and store it until time T;

4. At each time k, borrow money equal to the storage cost $c(k)$ at the risk-free interest rate $f_{k,T} = 1/(d(k,T) - 1)$;

5. At each time k, pay the storage cost $c(k)$;

6. At time T, sell the asset at the forward price F;

7. At time T, pay back the loan principal and the accrued interest, which in total is:

$$-S(1+s_T) - \sum_{k=0}^{T-1} c(k)(1+f_{k,T})$$

8. You had zero cash flow in the beginning and at each time k, and your cash flow at time T is:

$$\begin{aligned} &F - S(1+s_T) - \sum_{k=0}^{T-1} c(k)(1+f_{k,T}) \\ =&F - S/d(0,T) - \sum_{k=0}^{T-1} c(k)/d(k,T) \end{aligned}$$

9. In order to avoid arbitrage opportunities, we must have:

$$\begin{aligned} &F - S/d(0,T) - \sum_{k=0}^{T-1} c(k)/d(k,T) = 0 \\ \Rightarrow &F = S/d(0,T) + \sum_{k=0}^{T-1} c(k)/d(k,T) \end{aligned}$$ 😎

The value of a forward contract can be also explained using this theory:

Definition 1.3. The value of a forward

The value of a forward: Suppose a forward contract for delivery at time T in the future has a delivery price F_0 and that at time $t > 0$ the forward price is F_t. The value of the initial contract at time t is:

$$f_t = (F_t - F_0)d(t,T),$$

where $d(t,T)$ is the discount factor over the period from t to T.

Proof: Consider the following:

1. At time 0, the investor makes a forward contract to take a long position in the asset at forward price F_0 at time 0;

2. At time t, the investor makes a new forward contract to take a short position in the asset at forward price F_t at time t. This contract does not cost anything and can be closed through the long position at time T;

3. At time T, the investor receives a cash flow $F_t - F_0$ for certain, i.e., the asset is delivered from the long position to the short position regardless of the price, fulfilling the obligations for the two positions.

The time-t-present value of this certain cash flow at time T is by definition:

$$(F_t - F_0)d(t, T)$$

To eliminate arbitrage, this certain value must be equal to the price of the long forward contract f_t at time t, i.e.,

$$f_t = (F_t - F_0)d(t, T)$$ 😎

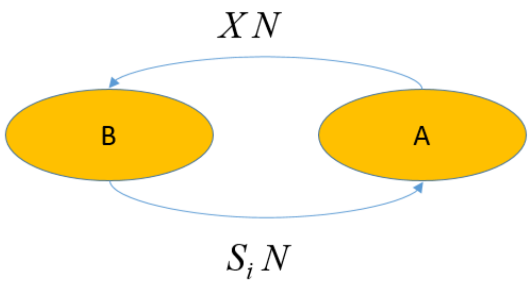
2 Swaps

In financial terms, a **swap** is an agreement to transform one cash flow stream into another. It typically one cash flow stream is random (risky) and the other one is fixed (therefore, riskless). For example, a **swap** a cash flow stream that depends on a variable interest rate (such as EURIBOR) to a fixed cash flow stream. In this transaction, the random cash flow stream is called **variable leg** and he fixed cash flow stream is called **fixed leg**.

Just for curiosity, the size of EUR interest rate swap markets is hundreds of billions of euros and a simple swap is commonly referred as **plain vanilla swap**.

For a swap, there is also a value of commodity. Considering a swap for N units of a commodity with M periods. In this swap, in each period, party A (e.g., manufacturer) which receives a payment which equals the spot price S_i of the commodity at time i times N and pays a fixed price X for each commodity unit N.

At time i, party A has the cash flow of $(S_i - X)N$ while party B has the opposite cash flows. The swap allows party A to buy the commodity effectively for a fixed price.



Let F_i be the unit forward price for delivery at time i. Taking a long forward position at this price (at time zero) allows party A to eliminate the risk associated with the spot price while the fixed payment X involves no uncertainties. The present value of this certain cash flow at time i must therefore be:

$$f_i = d(0, i)(F_i - X)N$$

This yields the total value of the swap for party A as:

$$V = \sum_{i=1}^{M} f_i = \sum_{i=1}^{M} d(0, i)(F_i - X)N$$

where X is often selected so that the initial value of the swap is zero.

3 Future Contract

Private contracts such as forwards can be closed only with the original counterparty, where there is no competition and the counterparty can decline to give a reasonable price.

Another possibility is to make another, opposing forward with another counterparty to remove the price risk of the swap. However, in this case, you have the risk of default of two counterparties:

1. the original swap counterparty and;

2. the opposing swap counterparty;

3. closing the swap with the original counterparty would have removed all risk (and they know it), but they may not give a good price (just for that reason).

This leads to a need for a better system **futures contracts**.

A **futures contract** is a derivative instrument on an asset whose delivery price ("futures price") is changed continuously to the last traded futures price. Futures contracts are traded on an exchange where every investor has an account, called a **margin account**.

The difference between the earlier delivery price and the new delivery price is credited to or debited from the investor's margin account by the exchange daily. If there is only relatively little money left on a margin account, the exchange can issue a **margin call** to force the investor to put more money on the account.

Value of the futures contract is always zero following the adjustment of the delivery price. With futures contracts, every investor in the market is holding either a long or short position in exactly the same kind of a derivative instrument (for each maturity).

A standardized product is easy to trade and can attract many investors. Counterparty for the futures contract is the exchange it can be closed out with the exchange, leaving no counterparty risk.

A curiosity of futures contracts is that the margin account is debited and credited by the difference in delivery price (which takes place at maturity) directly, not the *present value* of the difference, as one might expect. However, it might be explained by the ease in practice.

If the futures price increases:

- Those in the long position receive to their margin account a deposit which corresponds to the increase in the futures price;

- This deposit is taken from those who have a short position.

If the futures price decreases:

- A deposit is taken from those in long position;

- This deposit is received by those in short position.

The margin account needs an initial deposit for possible compensations (initial margin):

- Typically about 5-10% of the price of the contract;

- If the margin becomes too small, a margin call is issued to force investor to put more money on the account;

- The account (and the futures position) will be closed if the owner does not make an additional deposit.

The main difference between **forward** and **future** contracts relies on the fact that futures are marked-to-market on a daily basis and with forwards, cash flows occur only when the contract is terminated.

However, there is a equivalence between then, given by the following theorem:

Theorem 3.1. Futures-forward equivalence

uppose that interest rates are known to follow expectations dynamics. Then the theoretical futures and forward prices of respective contracts are identical.

It is in fact enough that, in each period, the investor can predict the next period's discount factor to maturity. If the margin account were debited and credited by the present value of the change in delivery price, the equivalence would always hold.

Proof: Let F_0 = futures price and G_0 = forward price at time $t = 0$. Let there be T periods and let the discount rate from period j to $k > j$ be $d(j,k)$.

Consider the following two strategies:

A : Take the following positions in futures

0: Go long in $d(1,T)$ units of futures

1: Increase position to $d(2,T)$

$\vdots$

k: Increase position to $d(k+1,T)$

$\vdots$

$T-1$: Increase position to 1

In period $k+1$, the profit from previous period is $(F_{k+1}-F_k)d(k+1,T)$. Then, invest these at the risk-free rate until T.

A : At time T, the value of the investment made at the end of period k is:

$$V_k = \frac{(F_{k+1}-F_k)d(k+1,T)}{d(k+1,T)} = F_{k+1} - F_k$$

This no cost strategy yields the profit

$$\sum_{k=0}^{T-1}(F_{k+1}-F_k) = F_T - F_0 = S_T - F_0$$

B : Take a long position with a single forward contract:

- No initial cash flow
- At the end the profit is $S_T - G_0$

- If the strategy is to buy **A** and sell **B** then
  - There is no commitment but the profit will be $G_0 - F_0$
  - It follows that $G_0 = F_0$ (or else there would be arbitrage) 😎

An alternative proof is:

Considering, F_0 = futures price at time 0 and G_0 = forward price at time 0. There are T periods.

At time t, discount factor from time i to j is $d_t(i,j)$. Also, At time t, the amount credited to the margin account (after the futures price has moved from F_{t-1} to F_t) is:

$$\Delta_t = F_t - F_{t-1}$$

.

No discounting or present value is applied to the cash posted to the margin account.

At time $t-1$, choose your futures position to be long $x_t = d_t(t,T)$ futures contracts which requires that this amount is known at time $t-1$ already, which would not be possible if the interest rates are random, hence the assumption that expectations dynamics hold.

At time t, the exchange credits us for:

$$x_t\Delta_t = d_t(t,T)(F_t - F_{t-1})$$

At time t, we can deposit $x_t\Delta_t$ at the risk-free interest rate until time T (this is $1/[d_t(t,T)-1)]$, so that at time T we will have:

$$x_t\Delta_t/d_t(t,T) = d_t(t,T)(F_t - F_{t-1})/d_t(t,T) = F_t - F_{t-1}$$

Thus, in total, we have at time T:

$$\sum_{t=1}^{T} x_t\Delta_t/d_t(t,T) = \sum_{t=1}^{T}(F_t - F_{t-1}) = F_T - F_0$$

If we go short in a respective forward, then we will make $-(S_T - G_0)$ at time T. Also, the delivery price of the futures contract matches the spot price at maturity, i.e., $F_T = S_T$.

To have no arbitrage, we must have at time T:

$$
\begin{aligned}
&F_T - F_0 - (S_T - G_0) = 0 \\
\Leftrightarrow &S_T - F_0 - S_T + G_0 = 0 \\
\Leftrightarrow &F_0 = G_0
\end{aligned}
$$

😎

Note that if the margin account were debited and credited by the present value of the change in delivery price, $(F_t - F_{t-1})d_t(t,T)$, x_t could be set to 1 and no advance knowledge of the interest rates would be needed, and the futures-forward equivalence would always hold.

Final definitions related to this equivalence:
Backwardation occurs when the current spot price of an underlying asset is higher than the prices in the futures market.
Contango occurs if the forward curve is upward sloping with higher futures contract price for each successive maturity date.

Expected spot price at maturity (an unknown quantity) and futures price may be different:

- Cannot be really compared, because the expected spot price at maturity cannot be observed from market data

There are slightly different interpretations of these terms, some employ the use of expected spot price.

4 Hedging

In a **perfect hedge**, one takes a position that is **equal and opposite** to the investment that one wishes to hedge and the price risk is eliminated (cf. earlier examples on commodity swaps). In this scenario, perfect hedging may not be possible if futures with suitable terms are not available.

Perfect hedging is not always possible, due to a few reasons:

- Commodity may not have a suitable futures market;
- The available contracts may not have suitable terms (e.g. delivery date);
- The supply of futures contracts is insufficient;
- Markets are not liquid enough.

Minimum variance hedge: Use a hedging instrument (e.g., a futures contract) such that the variance of a portfolio consisting of (i) the investment and (ii) the hedge is minimized.
Considering a commitment to sell W units of commodity at time T: at the spot price S_T, the delivery is worth $x = WS_T$ and hedge this position with h units worth of futures contracts.

At time T, cash flow is:

$$y = x + (F_T - F_0)h$$

Variance of this cash flow is:

$$
\begin{aligned}
Var[y] &= \mathbb{E}\left[\left(x - \bar{x} + (F_T - \bar{F}_T)h\right)^2\right] \\
&= Var[x] + 2hCov[x, F_T] + h^2 Var[F_T]
\end{aligned}
$$

The goal is to choose $h = h^*$ to minimize variance $Var[y]$ and set the derivative with respect to h at $h = h^*$ to 0:

$$\begin{aligned}\frac{d}{dh}Var[y]\Big|_{h=h^*} &= 2Cov[x, F_T] + 2h^* Var[F_T] = 0\\ \Rightarrow h^* &= -\frac{Cov[x, F_T]}{Var[F_T]}\end{aligned}$$

Inserting this into the expression for $Var[y]$ gives the minimum variance:

$$Var[y]\big|_{h=h^*} = Var[x] - \frac{Cov[x, F_T]^2}{Var[F_T]}$$

When $x = WS_T$, the hedge is:

$$h^* = -\frac{Cov[S_T, F_T]}{Var[F_T]}W \equiv -\beta W$$

If the markets for commodities are identical so that $F_T = S_T$, we have $h^* = -W$ and thus:

$$\begin{aligned}Var[y]\big|_{h=h^*} &= Var[WS_T] - \frac{Cov[WS_T, F_T]^2}{Var[F_T]}\\ &= W^2 Var[S_T] - \frac{W^2 Cov[F_T, F_T]^2}{Var[F_T]}\\ \Rightarrow Var[y]\big|_{h=h^*} &= 0\end{aligned}$$

4.1 Optimal hedging in expected utility theory

Minimum variance hedge does not account for the investor's risk preferences and he investor might prefer another hedging strategy.

Risk preferences can be modeled using, for example, Expected Utility Theory (EUT, see Lecture 7).

In optimal hedging, an EUT investor maximizes the expected utility of a portfolio containing the investment and its hedge:

$$\max_h \mathbb{E}\left[U(y)\right] = \max_h \mathbb{E}\left[U(x + (F_T - F_0)h)\right]$$