MSE2114 - Investment Science
Lecturer Notes IX

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Abstract

In this lecture, we cover price processes and options. The payoff from an option typically depends on *timing*, i.e., if and when the option is exercised and the options pricing theory calls for the modelling of asset prices.

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1 Price processes

A definition to start:

Price processes are decisions about multiperiod investments can be analyzed by modelling asset prices as stochastic processes, It can be divided into discrete processes and continuous process.

Discrete processes are binomial lattices. These are simple and adequate for the analysis of many types of investments. It behaves as an additive model:

$$S(k+1) = aS(k) + u(k)$$

or multiplicative model:

$$S(k+1) = u(k)S(k)$$

Continuous processes are known as **itô-processes**, where the price can change by any amount within a given period interval. Some Itô-processes have analytical solutions:

$$dx(t) = a(x,t)dt + b(x,t)dz$$

In the additive model, considering the following price process:

$$S(k+1) = aS(k) + u(k), \quad k = 0, 1, \ldots, N-1,$$

where $u(k)$ is random and a is constant (usually $a > 0$) and $S(k)$ is therefore $(k = 1, 2, \ldots, N)$.

$$\begin{aligned}
S(1) &= aS(0) + u(0) \\
S(2) &= aS(1) + u(1) = a^2S(0) + au(0) + u(1) \\
S(3) &= aS(2) + u(2) = a^3S(0) + a^2u(0) + au(1) + u(2) \\
&\vdots \\
S(k) &= a^kS(0) + \sum_{i=0}^{k-1} a^{k-1-i}u(i)
\end{aligned}$$

Considering another additive price process:

$$S(k) = a^kS(0) + \sum_{i=0}^{k-1} a^{k-1-i}u(i)$$

If $u(k)$ is normally distributed, the price process is a sum of normal random variables and hence normally distributed. If $u(k)$ has zero expectation so that $\mathbb{E}[u(k)] = 0$, the expected value of the additive price process is:

$$\mathbb{E}[S(k)] = a^kS(0)$$

The additive model is partly unrealistic because $u(i)$'s can be negative, then $S(k)$ can become negative, too. The volatility of $S(k+1)$ given $S(k)$ is not proportional to $S(k)$, contrary to what is suggested by empirical studies of actual asset prices.

In the multiplicative model:

$$S(k+1) = u(k)S(k), \quad k = 0, 1, \ldots, N-1$$

Inpendent random variables $u(k)$ model the relative change of the price in one period. The multiplicative model is additive in terms of the logarithms of price, because:

$$\begin{aligned} S(k+1) &= u(k)S(k) \\ \Rightarrow \ln S(k+1) &= \ln S(k) + \ln u(k) \end{aligned}$$

If $w(k) = \ln u(k)$ are normally distributed, then $u(k)$ are "lognormally" distributed and:

$$\ln S(k) = \ln S(0) + \sum_{i=0}^{k-1} w(i)$$

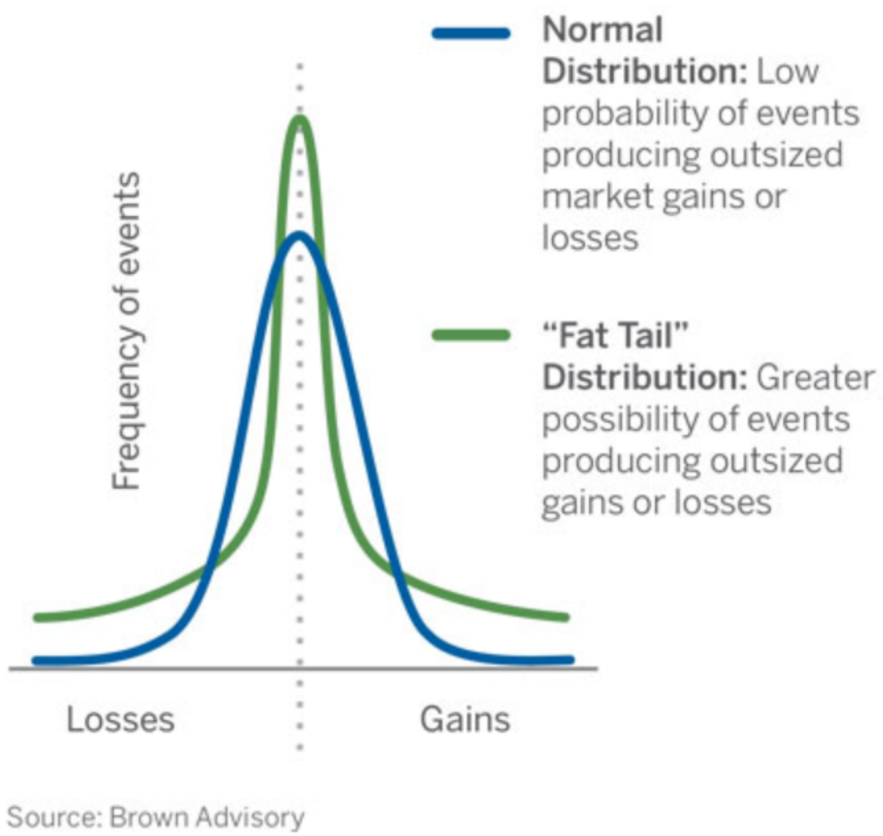
Considering the following multiplicative price process:

$$\ln S(k) = \ln S(0) + \sum_{i=0}^{k-1} w(i)$$

If $\mathbb{E}[w(k)] = \nu_p$ and $Var[w(k)] = \sigma_p^2$, then:

$$\begin{aligned} \mathbb{E}[\ln S(k)] &= \ln S(0) + k\nu_p \\ Var[\ln S(k)] &= k\sigma_p^2 \end{aligned}$$

Real stock prices are approximately lognormal. However, empirical distributions tend to have fatter tails than those of the lognormal distribution. Extreme price changes are more frequent than predicted by the lognormal distribution.



Binomial lattice are a generalizable numerical method for evaluating options. Essentially, the model uses lattice-based models of the varying price over time of the underlying financial instrument.

- Model parameters S_0, Δt, d, u and p:

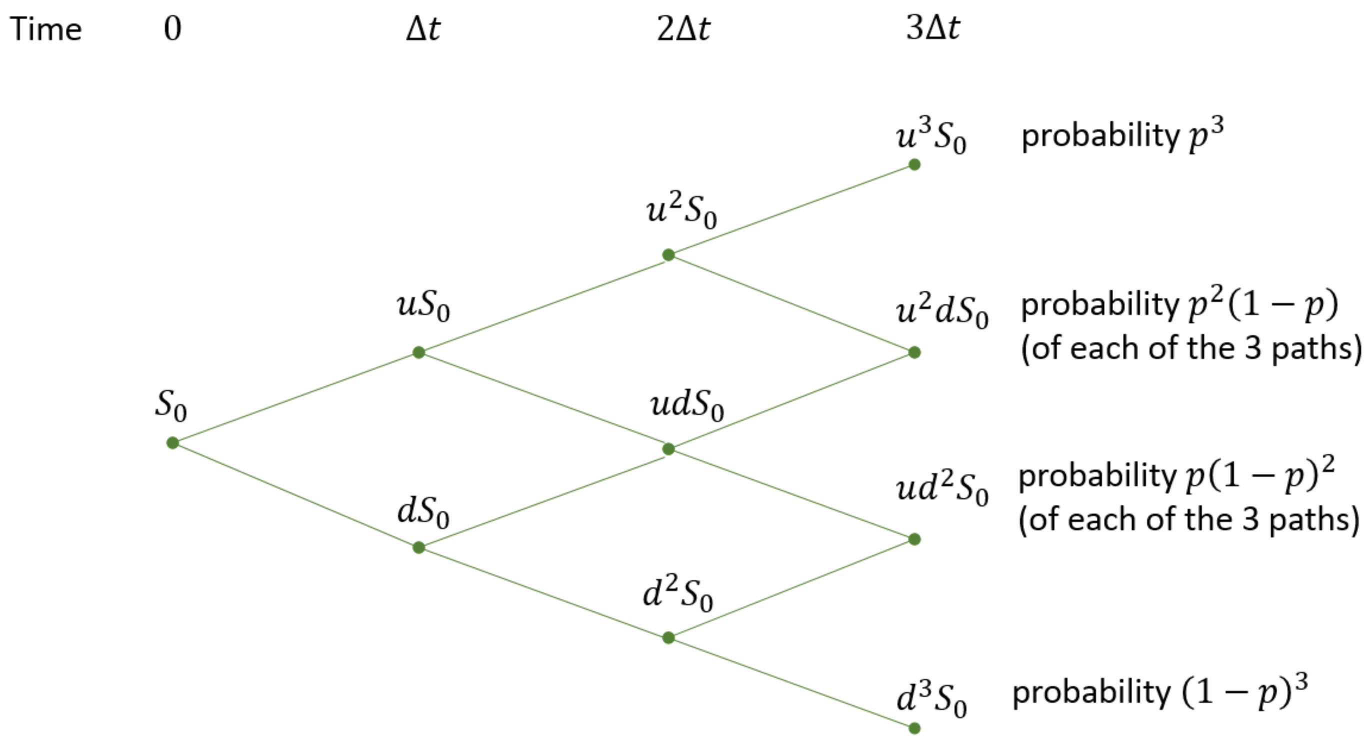
  1. Initial price S_0;
  2. Period length Δt (e.g., one week);
  3. Relative price changes down d and up u;
  4. Probability of price going up p.

- Expected relative shift of price is:

$$\mathbb{E}[u(k)] = pu = (1-p)d$$

  and expected shift of price logarithm is:

$$\mathbb{E}[w(k)] = p \ln u + (1-p) \ln d$$



The lattice can be constructed to conform to the desired expected growth rate and variance. Let the ν and σ^2 be the **yearly** expectation and variance of the logarithmic price process, respectively. These are defined as:

$$\nu T = \mathbb{E}[\ln(S_T/S_0)]$$
$$\sigma^2 T = Var[\ln(S_T/S_0)]$$

Note that there is an error in the course book as formulas are missing T (time in years).

Period-specific parameters can be estimated from data as follows:

$$
\begin{aligned}
\hat{\nu}_p &= \frac{1}{N}\sum_{k=0}^{N-1} \ln u(k) = \frac{1}{N}\sum_{k=0}^{N-1} \ln \frac{S(k+1)}{S(k)} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} [\ln S(k+1) - \ln S(k)] = \frac{1}{N} \ln \frac{S(N)}{S(0)} \\
\hat{\sigma}_p^2 &= \widehat{Var}[w(k)] = \widehat{Var}[\ln u(k)] \\
&= \frac{1}{N-1}\sum_{k=0}^{N-1} \left[\ln \frac{S(k+1)}{S(k)} - \hat{\nu}_p\right]^2
\end{aligned}
$$

Yearly parameters are linked to periodic estimates. Both expectation and variance are additive in terms of time. If p is the length of the period, then annual (no subscripts) and periodic parameters (sub-scripted by p) are related by:

$$
\begin{aligned}
\hat{\nu}_p = \hat{\nu} p \Leftrightarrow \hat{\nu} = \frac{1}{p}\hat{\nu}_p \\
\hat{\sigma}_p^2 = \hat{\sigma}^2 p \Leftrightarrow \hat{\sigma}^2 = \frac{1}{p}\hat{\sigma}_p^2
\end{aligned}
$$

For a general time difference Δt, we have:

$$
\begin{aligned}
\hat{\nu}_{\Delta t} &= \hat{\nu}\Delta t \\
\hat{\sigma}_{\Delta t}^2 &= \hat{\sigma}^2 \Delta t
\end{aligned}
$$

Let $S(0) = 1$ so that:

$$
\mathbb{E}[\ln S(1)] = \mathbb{E}[\ln S(0) + w(0)] = p \ln u + (1-p) \ln d
$$

Then the variance of the logarithmic price is:

$$
\begin{aligned}
Var[\ln S(1)] =& p[\ln u - p\ln u - (1-p)\ln d]^2 \\
&+ (1-p)[\ln d - p \ln u - (1-p)\ln d]^2 \\
=& p(1-p)\left(\ln u - \ln d\right)^2
\end{aligned}
$$

Denote $U = \ln u$ and $D = \ln d$ to obtain:

$$
\begin{aligned}
\mathbb{E}[\ln S(1)] &= pU + (1-p)D \\
Var[\ln S(1)] &= p(1-p)(U-D)^2
\end{aligned}
$$

We now require that the expectation and variance match the annual desired values (here, Δt is period length):

$$
\begin{aligned}
pU + (1-p)D &= \nu \Delta t \\
p(1-p)(U-D)^2 &= \sigma^2 \Delta t
\end{aligned}
$$

The quantities ν and σ^2 are annual parameters. There are three unknown parameters and only two equations. This extra degree of freedom can be exploited to set $d = 1/u$ so that $D = \ln d = \ln 1 - \ln u = -U$ and hence.

$$(2p-1)U = \nu \Delta t$$
$$4p(1-p)U^2 = \sigma^2 \Delta t$$

These equations give:

$$p = \frac{1}{2} + \frac{1/2}{\sqrt{\sigma^2 / (\nu^2 \Delta t) + 1}}, \quad U = \ln u = \sqrt{\sigma^2 \Delta t + (\nu \Delta t)^2}$$

For small Δt, these are approximately equal to:

$$p = \frac{1}{2}\left(1 + \frac{\nu}{\sigma}\sqrt{\Delta t}\right), \quad u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}$$

We have now fitted the parameters of the binomial lattice to match the desired parameters. E.g, the observed expectation and variance of the price process.

2 Options

Option is a contract which gives its owner the right, but not obligation, to sell or buy an asset at prespecified terms. For example:

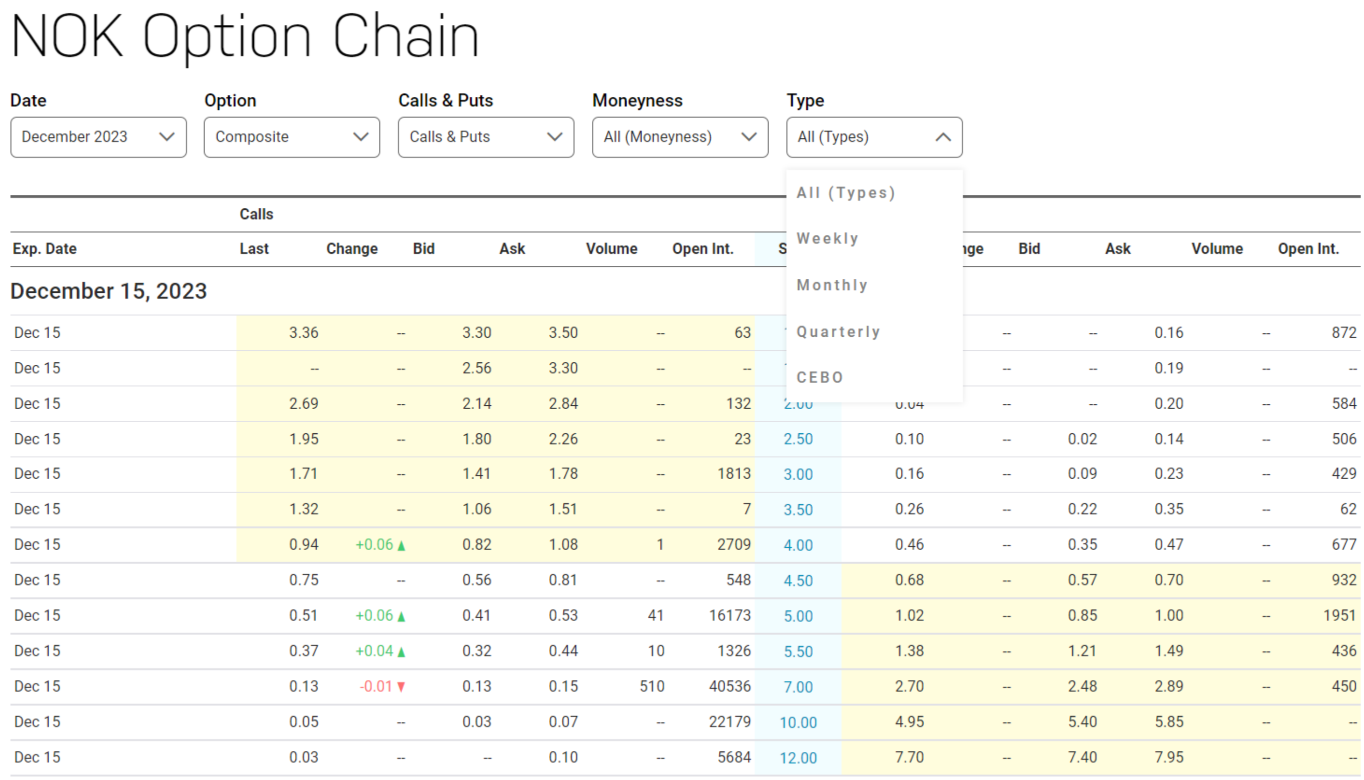
- Right to buy 1000 shares of company A for 20 € per share on 30 May 2024;
- Right to sell 10 tons of oil for 100 € / barrel in March 2025.

Some important definitions:

- **Underlying asset** = the asset which the option gives the right to buy or sell;
- **Call option** = option to buy the asset;
- **Put option** = right to sell the asset;
- **Expiration date** = date by/upon which the option must be exercised (and after which the option expires);
- **Exercise/strike price** = price paid for the asset when the option is exercised;
- **Premium** = price of the option.

textbfAmerican option can be exercised at any time before expiration while **european option** can be exercised only on the expiration date. Note that this classification refers to contract type, not location!

Upon expiry, the value of an option depends on the price of the asset and the strike price. If the price of company A stock is 20 € /share, the value of an expiring put option for selling 1000 shares at 25 € /share is $1000 \times (25-20)$ € = 5000 €.



NOK Option Chain

	Calls												
Exp. Date	Last	Change	Bid	Ask	Volume	Open Int.	S	[illegible]	ge	Bid	Ask	Volume	Open Int.
December 15, 2023													
Dec 15	3.36	--	3.30	3.50	--	63	[illegible]	[illegible]	--	--	0.16	--	872
Dec 15	--	--	2.56	3.30	--	--	[illegible]	[illegible]	--	--	0.19	--	--
Dec 15	2.69	--	2.14	2.84	--	132	2.00	0.04	--	--	0.20	--	584
Dec 15	1.95	--	1.80	2.26	--	23	2.50	0.10	--	0.02	0.14	--	506
Dec 15	1.71	--	1.41	1.78	--	1813	3.00	0.16	--	0.09	0.23	--	429
Dec 15	1.32	--	1.06	1.51	--	7	3.50	0.26	--	0.22	0.35	--	62
Dec 15	0.94	+0.06 ▲	0.82	1.08	1	2709	4.00	0.46	--	0.35	0.47	--	677
Dec 15	0.75	--	0.56	0.81	--	548	4.50	0.68	--	0.57	0.70	--	932
Dec 15	0.51	+0.06 ▲	0.41	0.53	41	16173	5.00	1.02	--	0.85	1.00	--	1951
Dec 15	0.37	+0.04 ▲	0.32	0.44	10	1326	5.50	1.38	--	1.21	1.49	--	436
Dec 15	0.13	-0.01 ▼	0.13	0.15	510	40536	7.00	2.70	--	2.48	2.89	--	450
Dec 15	0.05	--	0.03	0.07	--	22179	10.00	4.95	--	5.40	5.85	--	--
Dec 15	0.03	--	--	0.10	--	5684	12.00	7.70	--	7.40	7.95	--	--

The buyer of an option must pay a premium (the purchase price) to the seller of the option. The premiums for options traded in exchanges are determined in the market. Asset quantities, expiration dates and strike prices are all standardized.

The risk associated with an option is asymmetric for the seller and the buyer. The buyer has the right - but no the obligation - to exercise the option and possible loss is limited to the size of the premium.

The seller of the options must fulfil his or her obligation if the buyer chooses to exercise the option and the seller may incur losses (e.g., when selling call options and the price of the asset increases considerably above the strike price) and sellers are required to have margin accounts.

Options are often purchased in order to hedge one's position against risks.

3 Options pricing theory

The value of an option depends on:

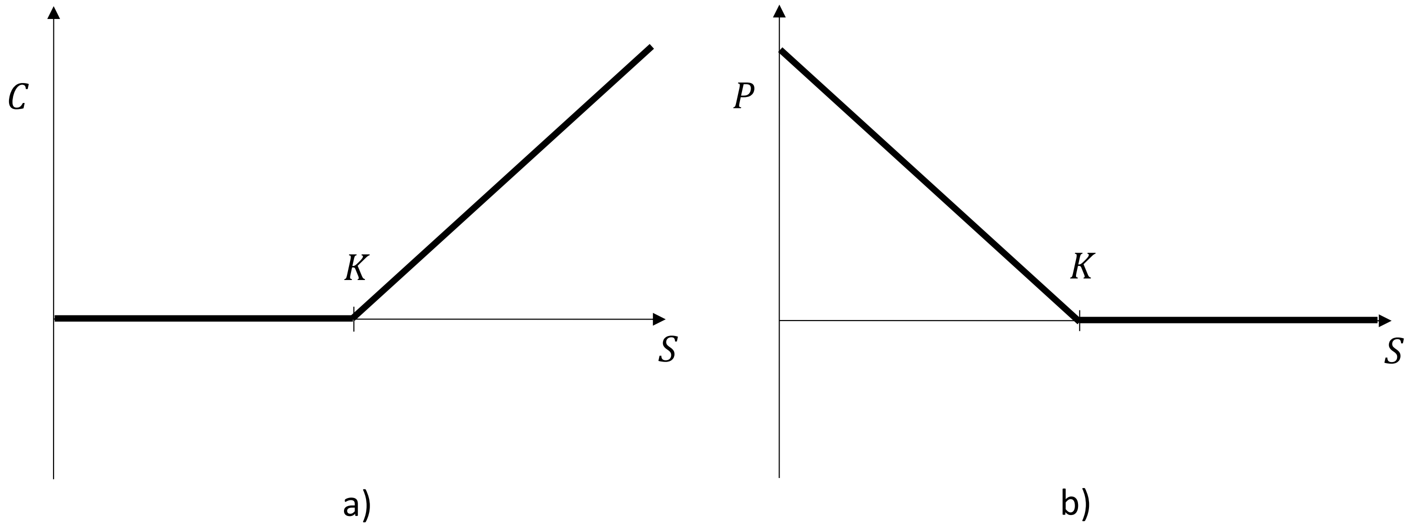
1. Price of underlying asset;
2. Strike price;
3. Time to expiry;
4. Volatility of the price of the underlying asset;
5. Interest rates;
6. Dividends of the asset.

Consider a call with strike price K, if at the time of expiry T, the price of underlying asset S is higher than K, then the value of the call is $S-K$. If S is less than K, then the option is worthless, meaning that upon expiry, the value of the call is:

$$C = \max\{0, S - K\}$$

Consider a put with strike price K, if at the time of expiry T, the price of underlying asset S is lower than K, then the value of the put is $K - S$. If S is greater than K, then the option is worthless, meaning upon expiry, the value of the put is:

$$P = \max\{0, K - S\}$$



a) The value $C = \max\{0, S - K\}$ of a call at expiration;

b) The value $P = \max\{0, K - S\}$ of a put at expiration;

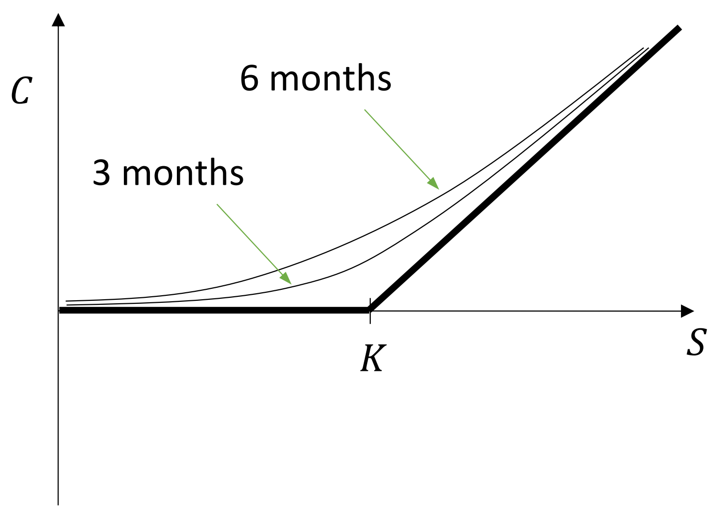
Let S_t be the price of the underlying asset at time $t < T$. A call option is said to be:

- **In the money** if $S_t > K$;
- **At the money** if $S_t = K$;
- **Out of the money** if $S_t < K$.

A put option is said to be:

- **In the money** if $S_t < K$;
- **At the money** if $S_t = K$;
- **Out of the money** if $S_t > K$.

Even if the call option is out of the money, the option still has value, because the price of the underlying asset may become higher before expiry.



Consider a call option which is out of the money, where the more volatile the asset, the greater the chance that its price will exceed the strike price. Higher interest rates make call options more valuable.

Those are the alternatives: Buy 1 000 shares at \$10 each for a total investment of \$10 000 or buy call options with \$10 strike price at \$1 for \$1 x 1 000 = \$1 000 and invest the rest \$9 000 at the risk free interest rate. With higher rates, the return on this \$9 000 is higher, making the call option more valuable.

The effect of different factors on options prices can be summarized with the following table.

	Impact when factor increases	
Factor	Call	Put
Price of underlying asset	+	-
Strike price	-	+
Time to expiry	+	+
Price volatility of underlying asset	+	+
Prevailing interest rate	+	-
Dividends	-	+

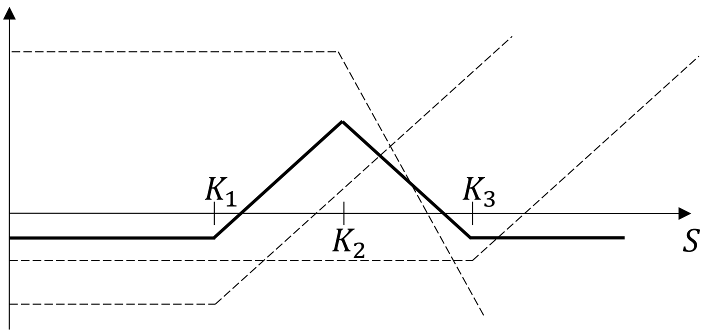
Options are often combined to construct a given desired financial position. Example: Butterfly spread:

- Buy two calls with strike prices K_1 and K_3 such that $K_3 > K_1$;
- Sell two calls with strike price K_2 such that $K_1 < K_2 < K_3$;
- Usually K_2 chosen so that it is close to the price of the underlying asset.

Such portfolio has the following properties:

A: It yields a profit if the price of the underlying asset does not change much;

B: It has a low risk even if the price of the underlying asset would change significantly.



Considering the following theorem:

Theorem 3.1. Put-call parity

et C and P be the prices of a European call and a European put, both with a strike price of K and defined on the same stock with price S. The put-call parity states that:

$$C - P + dK = S,$$

where d is the risk-free discount factor to the expiration date.

Proof: Consider the following position at time $t < T$:

1. Buy a call at C_t;
2. Sell a put option at P_t;
3. Deposit $d(t,T)K$ at the risk-free rate $(= 1/d(t,T) - 1)$.

Then, consider the following at time T:

A: If $S_T \geq K$, then the call yields a profit $S_T - K$, the put is worthless, and the deposit yields the cash flow $d(t,T)K/d(t,T) = K$;

$\Rightarrow$ Total cash flow is $(S_T - K) + K = S_T$;

B: If $S_T < K$, then the call is worthless, the short position on put yields a loss of (i.e., you have to pay) $K - S_T$ and the deposit yields the cash flow K;

$\Rightarrow$ Total cash flow is $K - (K - S_T) = S_T$.

Thus, the position has the same value as the underlying asset at time T:

The position and the asset must have the same value at the preceding time t, too. Hence, at time t, it must hold that $C_t - P_t + d(t,T)K = S_t$.

😎