Aalto University
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MS-E2114 Investment Science
Lecture IX: Basic options theory

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Overview

Price processes

Options

Options pricing theory

This lecture

- Last week, we covered contracts for forwards and futures
  - For simple derivative securities, the theoretical price is straightforward to calculate using no arbitrage assumption
- In this lecture, we cover price processes and options
  - The payoff from an option typically depends on *timing*, i.e., if and <u>when</u> the option is exercised

  ⇒ Options pricing theory calls for the modelling of asset prices

Overview

Price processes

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Price processes

- Decisions about multiperiod investments can be analyzed by modelling asset prices as stochastic processes
- Discrete processes $\Rightarrow$ binomial lattices
  - These are simple and adequate for the analysis of many types of investments
  - Additive model

$$S(k+1) = aS(k) + u(k)$$

  - Multiplicative model

$$S(k+1) = u(k)S(k)$$

- Continuous processes $\Rightarrow$ Itô-processes
  - Price can change by any amount within a given period interval
  - Some Itô-processes have analytical solutions
  - Itô-process

$$dx(t) = a(x,t)dt + b(x,t)dz$$

Additive model

- Consider the price process

$$S(k+1) = aS(k) + u(k), \quad k = 0, 1, \ldots, N-1,$$

where $u(k)$ is random and a is constant (usually $a > 0$)

- $S(k)$ is therefore ($k = 1, 2, \ldots, N$)

$$\begin{aligned}
S(1) &= aS(0) + u(0) \\
S(2) &= aS(1) + u(1) = a^2 S(0) + au(0) + u(1) \\
S(3) &= aS(2) + u(2) = a^3 S(0) + a^2 u(0) + au(1) + u(2) \\
&\vdots \\
S(k) &= a^k S(0) + \sum_{i=0}^{k-1} a^{k-1-i} u(i)
\end{aligned}$$

Additive model

- Additive price process

$$S(k) = a^k S(0) + \sum_{i=0}^{k-1} a^{k-1-i} u(i)$$

- If $u(k)$ is normally distributed, the price process is a sum of normal random variables and hence normally distributed
- If $u(k)$ has zero expectation so that $\mathbb{E}[u(k)] = 0$, the expected value of the additive price process is

$$\mathbb{E}[S(k)] = a^k S(0)$$

- The additive model is partly unrealistic
  - $u(i)$'s can be negative $\Rightarrow S(k)$ can become negative, too
  - The volatility of $S(k+1)$ given $S(k)$ is not proportional to $S(k)$, contrary to what is suggested by empirical studies of actual asset prices

Multiplicative model

- In the multiplicative model

$$S(k+1) = u(k)S(k), \quad k = 0, 1, \ldots, N-1$$

- Independent random variables $u(k)$ model the relative change of the price in one period
- The multiplicative model is additive in terms of the logarithms of price, because

$$\begin{aligned} S(k+1) &= u(k)S(k) & \quad | \text{ take } \ln \\ \Rightarrow \ln S(k+1) &= \ln S(k) + \ln u(k) \end{aligned}$$

- If $w(k) = \ln u(k)$ are normally distributed, then $u(k)$ are lognormally distributed and

$$\ln S(k) = \ln S(0) + \sum_{i=0}^{k-1} w(i)$$

Multiplicative model

- Multiplicative price process

$$\ln S(k) = \ln S(0) + \sum_{i=0}^{k-1} w(i)$$

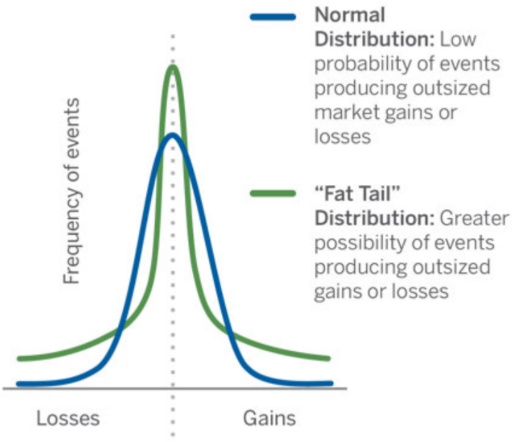
- If $\mathbb{E}[w(k)] = \nu_p$ and $\mathrm{Var}[w(k)] = \sigma_p^2$, then

$$\mathbb{E}[\ln S(k)] = \ln S(0) + k\nu_p$$
$$\mathrm{Var}[\ln S(k)] = k\sigma_p^2$$

- Real stock prices are approximately lognormal
- However, empirical distributions tend to have fatter tails than those of the lognormal distribution

$\Rightarrow$ Extreme price changes are more frequent than predicted by the lognormal distribution

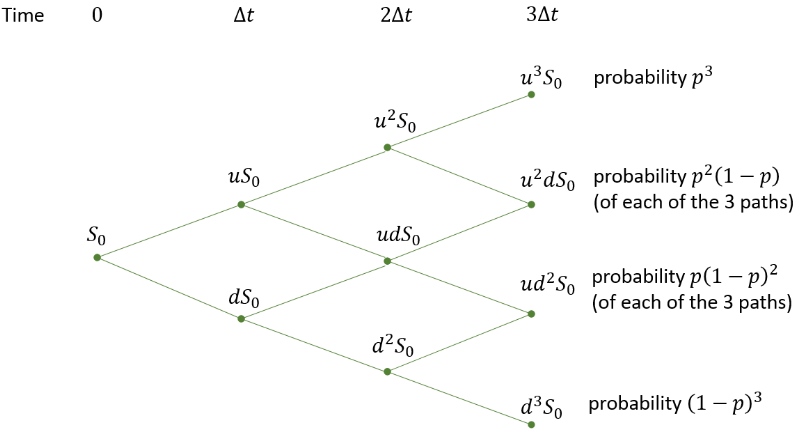
Comparison of fat tails with normal distribution



Source: Brown Advisory

Binomial lattice

- Model parameters S_0, Δt, d, u and p
  1. Initial price S_0
  2. Period length Δt (e.g., one week)
  3. Relative price changes down d and up u
  4. Probability of price going up p



Binomial lattice

- The lattice can be constructed to conform to the desired expected growth rate and variance
- Let the ν and σ^2 be the **yearly** expectation and variance of the logarithmic price process, respectively
- These are defined as

$$\nu T = \mathbb{E}[\ln(S_T/S_0)]$$
$$\sigma^2 T = \mathrm{Var}[\ln(S_T/S_0)]$$

- Note that there is an error in the course book as formulas are missing T (time in years)

Binomial lattice

- Period-specific parameters can be estimated from data as follows:

$$\hat{\nu}_p = \frac{1}{N}\sum_{k=0}^{N-1} \ln u(k) = \frac{1}{N}\sum_{k=0}^{N-1} \ln \frac{S(k+1)}{S(k)}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} [\ln S(k+1) - \ln S(k)] = \frac{1}{N} \ln \frac{S(N)}{S(0)}$$

$$\hat{\sigma}_p^2 = \widehat{\mathrm{Var}}[w(k)] = \widehat{\mathrm{Var}}[\ln u(k)]$$

$$= \frac{1}{N-1}\sum_{k=0}^{N-1} \left[\ln \frac{S(k+1)}{S(k)} - \hat{\nu}_p\right]^2$$

Binomial lattice

- Yearly parameters are linked to periodic estimates
- Both expectation and variance are additive in terms of time
- If p is the length of the period, then annual (no subscripts) and periodic parameters (subscripted by p) are related by

$$\hat{\nu}_p = \hat{\nu} p \Leftrightarrow \hat{\nu} = \frac{1}{p}\hat{\nu}_p$$

$$\hat{\sigma}_p^2 = \hat{\sigma}^2 p \Leftrightarrow \hat{\sigma}^2 = \frac{1}{p}\hat{\sigma}_p^2$$

- For a general time difference Δt, we have

$$\hat{\nu}_{\Delta t} = \hat{\nu}\Delta t$$

$$\hat{\sigma}_{\Delta t}^2 = \hat{\sigma}^2 \Delta t$$

Fitting the lattice parameters

- Let $S(0) = 1$ so that

$$\mathbb{E}[\ln S(1)] = \mathbb{E}[\ln S(0) + w(0)] = p \ln u + (1-p) \ln d$$

- Then the variance of the logarithmic price is

$$\begin{aligned}\mathrm{Var}[\ln S(1)] =& p[\ln u - p \ln u - (1-p) \ln d]^2 \\ &+ (1-p)[\ln d - p \ln u - (1-p) \ln d]^2 \\ =& p(1-p) (\ln u - \ln d)^2\end{aligned}$$

- Denote $U = \ln u$ and $D = \ln d$ to obtain

$$\mathbb{E}[\ln S(1)] = pU + (1-p)D$$
$$\mathrm{Var}[\ln S(1)] = p(1-p)(U-D)^2$$

Fitting the lattice parameters

- We now require that the expectation and variance match the annual desired values (here, Δt is period length):

$$pU + (1-p)D = \nu \Delta t$$
$$p(1-p)(U-D)^2 = \sigma^2 \Delta t$$

- The quantities ν and σ^2 are annual parameters
- There are three unknown parameters and only two equations
- This extra degree of freedom can be exploited to set $d = 1/u$ so that $D = \ln d = \ln 1 - \ln u = -U$ and hence

$$(2p-1)U = \nu \Delta t$$
$$4p(1-p)U^2 = \sigma^2 \Delta t$$

Fitting the lattice parameters

- These equations give

$$p = \frac{1}{2} + \frac{1/2}{\sqrt{\sigma^2/(\nu^2 \Delta t) + 1}}, \quad U = \ln u = \sqrt{\sigma^2 \Delta t + (\nu \Delta t)^2}$$

- For small Δt, these are approximately equal to

$$p = \frac{1}{2}\left(1 + \frac{\nu}{\sigma}\sqrt{\Delta t}\right), \quad u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}$$

- We have now fitted the parameters of the binomial lattice to match the desired parameters
- E.g, the observed expectation and variance of the price process

Overview

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Options pricing theory

Options

- **Option** = A contract which gives its owner the right, but not obligation, to sell or buy an asset at prespecified terms
  - Right to buy 1000 shares of company A for 20 € per share on 30 May 2024
  - Right to sell 10 tons of oil for 100 € / barrel in March 2025
- Terminology
  - **Underlying asset** = the asset which the option gives the right to buy or sell
  - **Call option** = option to buy the asset
  - **Put option** = right to sell the asset
  - **Expiration date** = date by/upon which the option must be exercised (and after which the option expires)
  - **Exercise/strike price** = price paid for the asset when the option is exercised
  - **Premium** = price of the option

Options

- **American option** can be exercised at any time before expiration
- **European option** can be exercised only on the expiration date
  - Classification refers to contract type, not location!
- Upon expiry, the value of an option depends on the price of the asset and the strike price
  - If the price of company A stock is 20 € /share, the value of an expiring put option for selling 1000 shares at 25 € /share is $1000 \times (25 - 20)$ € $=$ 5000 €

Example: Nokia Call and Put Options

NOK Option Chain

Date: December 2023 | Option: Composite | Calls & Puts: Calls & Puts | Moneyness: All (Moneyness) | Type: All (Types)

All (Types)
Weekly
Monthly
Quarterly
CEBO

	Calls												
Exp. Date	**Last**	**Change**	**Bid**	**Ask**	**Volume**	**Open Int.**	**S**		**nge**	**Bid**	**Ask**	**Volume**	**Open Int.**
December 15, 2023													
Dec 15	3.36	--	3.30	3.50	--	63	[illegible]	[illegible]	--	--	0.16	--	872
Dec 15	--	--	2.56	3.30	--	--	[illegible]	[illegible]	--	--	0.19	--	--
Dec 15	2.69	--	2.14	2.84	--	132	2.00	0.04	--	--	0.20	--	584
Dec 15	1.95	--	1.80	2.26	--	23	2.50	0.10	--	0.02	0.14	--	506
Dec 15	1.71	--	1.41	1.78	--	1813	3.00	0.16	--	0.09	0.23	--	429
Dec 15	1.32	--	1.06	1.51	--	7	3.50	0.26	--	0.22	0.35	--	62
Dec 15	0.94	+0.06 ▲	0.82	1.08	1	2709	4.00	0.46	--	0.35	0.47	--	677
Dec 15	0.75	--	0.56	0.81	--	548	4.50	0.68	--	0.57	0.70	--	932
Dec 15	0.51	+0.06 ▲	0.41	0.53	41	16173	5.00	1.02	--	0.85	1.00	--	1951
Dec 15	0.37	+0.04 ▲	0.32	0.44	10	1326	5.50	1.38	--	1.21	1.49	--	436
Dec 15	0.13	-0.01 ▼	0.13	0.15	510	40536	7.00	2.70	--	2.48	2.89	--	450
Dec 15	0.05	--	0.03	0.07	--	22179	10.00	4.95	--	5.40	5.85	--	--
Dec 15	0.03	--	--	0.10	--	5684	12.00	7.70	--	7.40	7.95	--	--

Determining the premium

- The buyer of an option must pay a premium (the purchase price) to the seller of the option
- The premiums for options traded in exchanges are determined in the market
  - Asset quantities, expiration dates and strike prices are all standardized

Risks of options

- The risk associated with an option is asymmetric for the seller and the buyer
  - The buyer has the right - but no the obligation - to exercise the option
    - ⇒ Possible loss is limited to the size of the premium
  - The seller of the options must fulfil his or her obligation if the buyer chooses to exercise the option
    - ⇒ The seller may incur losses (e.g., when selling call options and the price of the asset increases considerably above the strike price)
    - ⇒ Sellers are required to have margin accounts
- Options are often purchased in order to hedge one's position against risks

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Value of an option

- The value of an option depends on:
  1. Price of underlying asset
  2. Strike price
  3. Time to expiry
  4. Volatility of the price of the underlying asset
  5. Interest rates
  6. Dividends of the asset

Option value at expiration

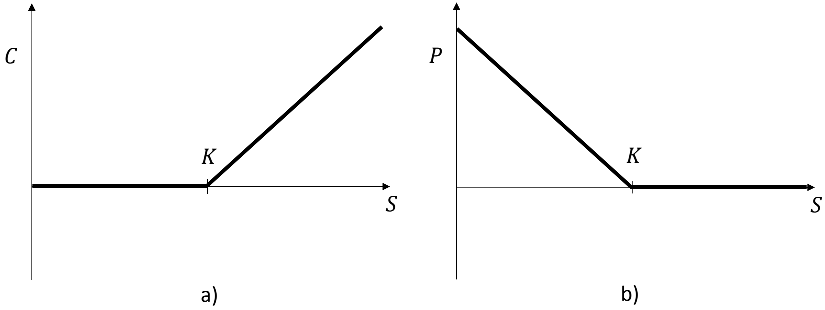
- Consider a call with strike price K
  - If at the time of expiry T, the price of underlying asset S is higher than K, then the value of the call is $S - K$
  - If S is less than K, then the option is worthless
  - ⇒ Upon expiry, the value of the call is

$$C = \max\{0, S - K\}$$

- Consider a put with strike price K
  - If at the time of expiry T, the price of underlying asset S is lower than K, then the value of the put is $K - S$
  - If S is greater than K, then the option is worthless
  - ⇒ Upon expiry, the value of the put is

$$P = \max\{0, K - S\}$$

Option value at expiration



a) The value $C = \max\{0, S - K\}$ of a call at expiration

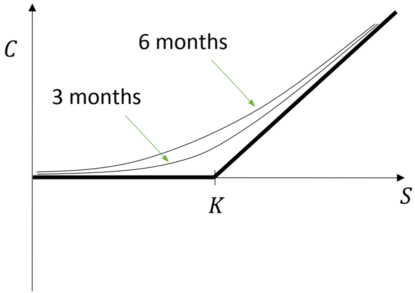
b) The value $P = \max\{0, K - S\}$ of a put at expiration

Time value of an option

- Let S_t be the price of the underlying asset at time $t < T$
- A call option is said to be
  - **In the money** if $S_t > K$
  - **At the money** if $S_t = K$
  - **Out of the money** if $S_t < K$
- A put option is said to be
  - **In the money** if $S_t < K$
  - **At the money** if $S_t = K$
  - **Out of the money** if $S_t > K$

Time value of an option

- Even if the call option is out of the money, the option still has value, because the price of the underlying asset may become higher before expiry



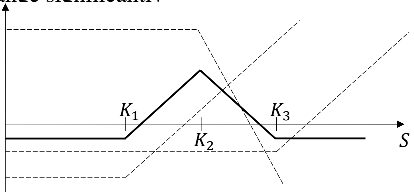
Other factors affecting the value of an option

- Consider a call option which is out of the money
  - The more volatile the asset, the greater the chance that its price will exceed the strike price
- Higher interest rates make call options more valuable
  - Alternative 1: Buy 1 000 shares at $10 each for a total investment of $10 000
  - Alternative 2: Buy call options with $10 strike price at $1 for $1 x 1 000 = $1 000 and invest the rest $9 000 at the risk free interest rate. With higher rates, the return on this $9 000 is higher, making the call option more valuable

	Impact when factor increases	
Factor	Call	Put
Price of underlying asset	+	-
Strike price	-	+
Time to expiry	+	+
Price volatility of underlying asset	+	+
Prevailing interest rate	+	-
Dividends	-	+

Combining options

- Options are often combined to construct a given desired financial position
- Example: Butterfly spread
  - Buy two calls with strike prices K_1 and K_3 such that $K_3 > K_1$
  - Sell two calls with strike price K_2 such that $K_1 < K_2 < K_3$
    - Usually K_2 chosen so that it is close to the price of the underlying asset
  - This portfolio has the following properties:
    - A: It yields a profit if the price of the underlying asset does not change much
    - B: It has a low risk even if the price of the underlying asset would change significantly



Put-call parity

Theorem

*(**Put-call parity**) Let C and P be the prices of a European call and a European put, both with a strike price of K and defined on the same stock with price S. The put-call parity states that*

$$C - P + dK = S,$$

where d is the risk-free discount factor to the expiration date.

Put-call parity

Proof: Consider the following position at time $t < T$:

1. Buy a call at C_t
2. Sell a put option at P_t
3. Deposit $d(t,T)K$ at the risk-free rate $(= 1/d(t,T) - 1)$

Then, consider the following at time T:

A: If $S_T \geq K$, then the call yields a profit $S_T - K$, the put is worthless, and the deposit yields the cash flow $d(t,T)K/d(t,T) = K$

⇒ Total cash flow is $(S_T - K) + K = S_T$

B: If $S_T < K$, then the call is worthless, the short position on put yields a loss of (i.e., you have to pay) $K - S_T$ and the deposit yields the cash flow K

⇒ Total cash flow is $K - (K - S_T) = S_T$

Put-call parity

Thus, the position has the same value as the underlying asset at time T

⇒ The position and the asset must have the same value at the preceding time t, too

Hence, at time t, it must hold that $C_t - P_t + d(t, T)K = S_t$. $\square$

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