10 Finite element methods for three-dimensional elasticity
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2. Energy methods and basic 1D finite element methods
   - bars/rods, beams, heat diffusion, seepage, electrostatics
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5. Abstract formulation and accuracy of finite element methods
6. Finite element methods for Euler–Bernoulli beams
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10 Finite element methods for three-dimensional linear elasticity

Contents

1. Strong and weak forms for linear elasticity problems
2. Finite element methods for linear elasticity problems

Learning outcome

A. Understanding the basic features of two and three-dimensional elasticity problems and ability to derive the basic formulations related to the problems
B. Basic knowledge and tools for solving two and three-dimensional elasticity problems by finite element methods – with locking free methods, in particular

References

Lecture notes: chapters 5, 6, 12
Text book: chapters 2.7–10, 3.5–6
10.0 Utilizing or avoiding 3D finite elements in structural analysis

Solid (3D) finite elements offer a flexible set of tools for analyzing various real life structural geometries – although the computational cost is often very high.
10.1 Strong and weak forms for linear elasticity problems

Let us consider a three-dimensional body (or structure) \( \Omega \subset R^3 \) subject to a distributed body force

\[
f(x, y, z) = f(x) = f_x(x)e_x + f_y(x)e_y + f_z(x)e_z \quad \text{in } \Omega
\]

and a distributed (surface) traction

\[
t(x, y, z; n) = t(x; n) = t_x(x; n)e_x + t_y(x; n)e_y + t_z(x; n)e_z \quad \text{on } \Gamma_t \subset \partial \Omega.
\]
10.1 Strong and weak forms for linear elasticity problems

Let us consider a three-dimensional body (or structure) \( \Omega \subset R^3 \) subject to a distributed \textit{body force}

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t(x, y, z; n) = t(x; n) = t_x(x; n)e_x + t_y(x; n)e_y + t_z(x; n)e_z \quad \text{on } \Gamma_t \subset \partial \Omega.
\]

In what follows, we will utilize the \textit{Cauchy’s lemma} and \textit{Cauchy’s law}, respectively,

\[
t(x; n) = -t(x; -n),
\]

\[
t(x; n) = \sigma(x)n,
\]

with \( \sigma = \sigma(x) \) (or \( \sigma = \sigma(x, t) \) in a time dependent case) denoting the \textit{Cauchy stress tensor} (measuring the force per unit area acting on a surface in the \textit{current configuration}).
10.1 Strong and weak forms for linear elasticity problems

The static equilibrium (or force balance) for the body follows from the Euler’s laws (or momentum principles, equivalent generalizations of Newton’s laws): Principle of linear momentum can be written in the form

$$\int_\Omega b \, d\Omega + \int_{s_b} t \, dS = \int_\Omega \rho \vec{\dot{v}} \, d\Omega$$

with the spatial (current configuration) mass density $\rho = \rho(\mathbf{x},t)$ and spatial velocity field $\mathbf{v} = \mathbf{v}(\mathbf{x},t)$. 
10.1 Strong and weak forms for linear elasticity problems

The static equilibrium (or force balance) for the body follows from the *Euler’s laws* (or *momentum principles*, equivalent generalizations of *Newton’s laws*): *Principle of linear momentum* can be written in the form

$$\int_{\Omega} \mathbf{b} \, d\Omega + \int_{S_t} \mathbf{t} \, dS = \int_{\Omega} \rho \mathbf{v} \cdot d\Omega$$

with the *spatial* (current configuration) *mass density* $\rho = \rho(x,t)$ and *spatial velocity field* $\mathbf{v} = \mathbf{v}(x,t)$. In the *static case*, the inertial (or kinetic) term vanishes:

$$\int_{\Omega} \mathbf{b} \, d\Omega + \int_{S_t} \mathbf{t} \, dS = 0.$$
The static equilibrium (or force balance) for the body follows from the *Euler’s laws* (or *momentum principles*, equivalent generalizations of *Newton’s laws*): *Principle of linear momentum* can be written in the form

\[ \int_{\Omega} b \, d\Omega + \int_{S} t \, dS = \int_{\Omega} \rho \dot{v} \, d\Omega \]

with the *spatial* (current configuration) *mass density* \( \rho = \rho(x,t) \) and *spatial velocity field* \( v = v(x,t) \). In the static case, the inertial (or kinetic) term vanishes:

\[ \int_{\Omega} b \, d\Omega + \int_{S} t \, dS = 0. \]

Cauchy’s law and *Gauss divergence theorem* (based on *Green’s formula*) imply the form

\[ 0 = \int_{\Omega} b \, d\Omega + \int_{S} t \, dS = \int_{\Omega} b \, d\Omega + \int_{S} \sigma n \, dS \]

\[ = \int_{\Omega} b \, d\Omega + \int_{\Omega} \text{div} \sigma \, d\Omega \]

from which one obtains the equilibrium equation:
10.1 Strong and weak forms for linear elasticity problems

\[ -\text{div } \sigma = b \quad \text{in } \Omega. \]

In case of dynamics, the inertial term (vanishing earlier above) would be added to the left hand side implying the \textit{equation of motion} in the form

\[ -\text{div } \sigma + \rho \frac{dv}{dt} = b \quad \text{in } \Omega. \]
10.1 Strong and weak forms for linear elasticity problems

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Principle of angular momentum is written in the form of equation

\[\int_{\Omega} r \times b \, d\Omega + \int_{S_t} r \times t \, dS = \frac{d}{dt} \int_{\Omega} r \times \rho v \, d\Omega\]
10.1 Strong and weak forms for linear elasticity problems

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\[- \text{div } \sigma + \rho \frac{dv}{dt} = b \quad \text{in } \Omega.\]

Principle of angular momentum is written in the form of equation

\[
\int_{\Omega} \mathbf{r} \times \mathbf{b} \, d\Omega + \int_{S_t} \mathbf{r} \times \mathbf{t} \, dS = \frac{d}{dt} \int_{\Omega} \mathbf{r} \times \rho \mathbf{v} \, d\Omega
\]

which implies, by using once again Cauchy’s law and Gauss divergence theorem, the moment equilibrium equation, or the moment symmetry condition for the Cauchy stress tensor,

\[\sigma = \sigma^T \quad \text{in } \Omega,\]

which holds true for both the static and dynamic case.
Let us consider the displacement fields defined by the original and deformed positions \( x \in \Omega \) and \( y(x) \in \tilde{\Omega} \), respectively, when the body \( \Omega \) deforms to \( \tilde{\Omega} \):

\[
u(x) = y(x) - x
\]
10.1 Strong and weak forms for linear elasticity problems

Let us consider the displacement fields defined by the original and deformed positions \( x \in \Omega \) and \( y(x) \in \widetilde{\Omega} \), respectively, when the body \( \Omega \) deforms to \( \widetilde{\Omega} \):

\[
\mathbf{u}(x) = y(x) - x
\]

Assuming small, or linear, deformations the displacement field implies the strains

\[
\varepsilon(\mathbf{u}) = \nabla \mathbf{u} = \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) / 2;
\]

\[
\varepsilon_x(x, y, z) = \frac{\partial u(x, y, z)}{\partial x}, \quad \varepsilon_y(x, y, z) = \frac{\partial v(x, y, z)}{\partial y}, \quad \varepsilon_z(x, y, z) = \frac{\partial w(x, y, z)}{\partial z},
\]

\[
\gamma_{xy}(x, y, z) = \frac{\partial u(x, y, z)}{\partial y} + \frac{\partial v(x, y, z)}{\partial x} = 2\varepsilon_{xy}(x, y, z)
\]

\[
\gamma_{xz}(x, y, z) = \frac{\partial u(x, y, z)}{\partial z} + \frac{\partial w(x, y, z)}{\partial x} = 2\varepsilon_{xz}(x, y, z),
\]

\[
\gamma_{yz}(x, y, z) = \frac{\partial v(x, y, z)}{\partial z} + \frac{\partial w(x, y, z)}{\partial y} = 2\varepsilon_{yz}(x, y, z).
\]
10.1 Strong and weak forms for linear elasticity problems

The principle of virtual work (or principle of virtual displacements) can be written in its usual form

\[ 0 = \delta W_{\text{int}} + \delta W_{\text{ext}} = -\int_{\Omega} \sigma : \delta \varepsilon \, d\Omega + \int_{\Omega} b \cdot \delta u \, d\Omega + \int_{S_t} t \cdot \delta u \, dS \]

\[ = -\int_{\Omega} \sigma : \bar{\nabla} (\delta u) \, d\Omega + \int_{\Omega} b \cdot \delta u \, d\Omega + \int_{S_t} t \cdot \delta u \, dS \]
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\[ = -\int_{\Omega} \sigma : \nabla (\delta u) \, d\Omega + \int_{\Omega} b \cdot \delta u \, d\Omega + \int_{S_t} t \cdot \delta u \, dS \]

where the first term is integrated by parts leading to equality

\[ 0 = -\int_{\Omega} \sigma : \nabla (\delta u) \, d\Omega + \int_{\Omega} b \cdot \delta u \, d\Omega + \int_{S_t} t \cdot \delta u \, dS \]

\[ = -\int_{\partial \Omega} \sigma n \cdot \delta u \, d\Omega + \int_{\Omega} \text{div} \sigma \cdot \delta u \, d\Omega + \int_{\Omega} b \cdot \delta u \, d\Omega + \int_{S_t} t \cdot \delta u \, dS \]
10.1 Strong and weak forms for linear elasticity problems

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\[ = -\int_{\partial \Omega} \sigma n \cdot \delta u \, d\Omega + \int_{\Omega} \text{div} \, \sigma \cdot \delta u \, d\Omega + \int_{\Omega} b \cdot \delta u \, d\Omega + \int_{S_t} t \cdot \delta u \, dS \]

\[ = -\int_{\partial \Omega \setminus S_t} \sigma n \cdot \delta u \, d\Omega + \int_{\Omega} (\text{div} \, \sigma + b) \cdot \delta u \, d\Omega + \int_{S_t} (t - \sigma n) \cdot \delta u \, dS \]

This finally implies the strong form of the elasticity problem:
10.1 Strong and weak forms for linear elasticity problems

For given loadings \(b\) and \(t\), and displacement \(u_0\), find \(u\) such that

\[
\begin{align*}
- \text{div } \sigma(u) &= b \quad \text{in } \Omega, \\
u &= u_0 \quad \text{on } S_u \subset \partial\Omega, \\
\sigma n &= t \quad \text{on } S_t \subset \partial\Omega,
\end{align*}
\]
10.1 Strong and weak forms for linear elasticity problems

For given loadings \( b \) and \( t \), and displacement \( u_0 \), find \( u \) such that

\[
- \text{div} \, \sigma(u) = b \quad \text{in} \quad \Omega, \\
\begin{aligned}
\quad u &= u_0 \quad \text{on} \quad S_u \subset \partial \Omega, \\
\quad \sigma n &= t \quad \text{on} \quad S_t \subset \partial \Omega,
\end{aligned}
\]

where \( S_u = \partial \Omega \setminus S_t \) and the linear strain-displacement relation is given as

\[
\varepsilon(u) = \tilde{\nabla} u = \left( \nabla u + (\nabla u)^T \right) / 2,
\]

while the strain-stress relation follows a given constitutive law. The generalized Hooke’s law of the theory of linear elasticity for isotropic materials implies

\[
\sigma(u) = D\varepsilon(u) = 2 \mu \varepsilon(u) + \lambda \text{tr}(\varepsilon(u)) I \\
= 2 \mu \varepsilon(u) + \lambda \text{div} u I
\]
10.1 Strong and weak forms for linear elasticity problems

For given loadings \( \mathbf{b} \) and \( \mathbf{t} \), and displacement \( \mathbf{u}_0 \), find \( \mathbf{u} \) such that

\[
- \text{div} \, \mathbf{\sigma}(\mathbf{u}) = \mathbf{b} \quad \text{in} \quad \Omega,
\]

\[
\mathbf{u} = \mathbf{u}_0 \quad \text{on} \quad S_u \subset \partial \Omega,
\]

\[
\mathbf{\sigma} \mathbf{n} = \mathbf{t} \quad \text{on} \quad S_t \subset \partial \Omega,
\]

where \( S_u = \partial \Omega \setminus S_t \) and the linear strain-displacement relation is given as

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\mathbf{\varepsilon}(\mathbf{u}) = \mathbf{\nabla} \mathbf{u} = \left( \mathbf{\nabla} \mathbf{u} + (\mathbf{\nabla} \mathbf{u})^T \right) / 2,
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\]

\[
= 2\mu \mathbf{\varepsilon}(\mathbf{u}) + \lambda \text{div} \mathbf{u} \mathbf{I}
\]

with Lamé parameters \( \lambda = E\nu / ((1 + \nu)(1 - 2\nu)) \), \( \mu = E / (2(1 + \nu)) = G \), where \( E \) and \( G \), respectively, denote the \textit{Young’s modulus} and \textit{shear modulus} usually applied in engineering applications. In principle, these material values can depend on coordinates \( x, y \) and \( z \), i.e., the material can be \textit{nonhomogeneous}. 

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10.1 Strong and weak forms for linear elasticity problems

The corresponding weak form is obtained from the virtual work expression above or, as usual, by multiplying the strong form by a test function, integrating over the domain and finally integrating by parts:

\[ \int_{\Omega} b \cdot \hat{u} \, d\Omega = -\int_{\Omega} \text{div} \sigma(u) \cdot \hat{u} \, d\Omega \]
The corresponding weak form is obtained from the virtual work expression above or, as usual, by multiplying the strong form by a test function, integrating over the domain and finally integrating by parts:

\[
\begin{align*}
\int_{\Omega} \mathbf{b} \cdot \hat{\mathbf{u}} \, d\Omega &= -\int_{\Omega} \text{div} \, \mathbf{\sigma}(u) \cdot \hat{\mathbf{u}} \, d\Omega \\
&= -\int_{\partial\Omega} \mathbf{\sigma}(u) n \cdot \hat{\mathbf{u}} \, d\Omega + \int_{\Omega} \mathbf{\sigma}(u) : \nabla \hat{\mathbf{u}} \, d\Omega
\end{align*}
\]
The corresponding weak form is obtained from the virtual work expression above or, as usual, by multiplying the strong form by a test function, integrating over the domain and finally integrating by parts:

$$\int_{\Omega} \mathbf{b} \cdot \mathbf{\hat{u}} \, d\Omega = -\int_{\Omega} \text{div} \, \sigma(\mathbf{u}) \cdot \mathbf{\hat{u}} \, d\Omega$$

$$= -\int_{\partial \Omega} (\sigma(\mathbf{u}) \mathbf{n}) \cdot \mathbf{\hat{u}} \, d\Omega + \int_{\Omega} \sigma(\mathbf{u}) : \nabla \mathbf{\hat{u}} \, d\Omega$$

$$= \int_{\partial \Omega \setminus \Omega_t} (\sigma(\mathbf{u}) \mathbf{n}) \cdot \mathbf{\hat{u}} \, dS - \int_{\Omega_t} (\sigma(\mathbf{u}) \mathbf{n}) \cdot \mathbf{\hat{u}} \, dS + \int_{\Omega} (D\varepsilon(\mathbf{u})) : \varepsilon(\mathbf{u}) \, d\Omega$$
10.1 Strong and weak forms for linear elasticity problems

The corresponding weak form is obtained from the virtual work expression above or, as usual, by multiplying the strong form by a test function, integrating over the domain and finally integrating by parts:

\[
\int_{\Omega} b \cdot \hat{u} \, d\Omega = - \int_{\Omega} \text{div} \sigma(u) \cdot \hat{u} \, d\Omega \\
= - \int_{\partial \Omega} (\sigma(u)n) \cdot \hat{u} \, d\Omega + \int_{\Omega} \sigma(u) : \nabla \hat{u} \, d\Omega \\
= \int_{\partial \Omega \setminus S_t} (\sigma(u)n) \cdot \hat{u} \, dS - \int_{S_t} (\sigma(u)n) \cdot \hat{u} \, dS + \int_{\Omega} (D\varepsilon(u)) : \varepsilon(u) \, d\Omega
\]

For simplicity, let us assume that surface traction is given on a part of the boundary and the rest of the boundary is homogeneously clamped:

\[
u = u_0 = 0 \quad \text{on} \quad S_u = \partial \Omega \setminus S_t \subset \partial \Omega
\]

\[
\sigma n = t \quad \text{on} \quad S_t \subset \partial \Omega.
\]

Hence, the weak form of the problem can be written in a short abstract form:
10.1 Strong and weak forms for linear elasticity problems

Weak form of the linear elasticity problem: Let a three-dimensional body be under the loading \( b \in [L^2(\Omega)]^3, \quad \Omega \subset R^3 \). Find \( u \in U \) such that

\[
\alpha(u, \hat{u}) = l(\hat{u}) \quad \forall \hat{u} \in U,
\]
10.1 Strong and weak forms for linear elasticity problems

**Weak form of the linear elasticity problem:** Let a three-dimensional body be under the loading $b \in [L^2(\Omega)]^3$, $\Omega \subset \mathbb{R}^3$. Find $u \in U$ such that

$$a(u, \hat{u}) = l(\hat{u}) \quad \forall \hat{u} \in U,$$

with the bilinear form, load functional and variational space

$$a(u, \hat{u}) = \int_\Omega (D\varepsilon(u)) : \varepsilon(\hat{u}) \, d\Omega,$$

$$l(\hat{u}) = \int_\Omega b \cdot \hat{u} \, d\Omega + \int_{S_t} t \cdot \hat{u} \, dS,$$

$$U = \{ \, v \in [H^1(\Omega)]^3 \mid v|_{S_u} = 0 \, \}.$$
10.1 Strong and weak forms for linear elasticity problems

**Weak form of the linear elasticity problem:** Let a three-dimensional body be under the loading \( b \in [L^2(\Omega)]^3 \), \( \Omega \subset \mathbb{R}^3 \). Find \( u \in U \) such that

\[
a(u, \hat{u}) = l(\hat{u}) \quad \forall \hat{u} \in U,
\]

with the bilinear form, load functional and variational space

\[
a(u, \hat{u}) = \int_\Omega (D\varepsilon(u)) : \varepsilon(\hat{u}) \, d\Omega,
\]

\[
l(\hat{u}) = \int_\Omega b \cdot \hat{u} \, d\Omega + \int_{S_t} t \cdot \hat{u} \, dS,
\]

\[
U = \{ \, v \in [H^1(\Omega)]^3 \mid v|_{S_u} = 0 \, \}.
\]

**Remark.** The bilinear form can be written in the form

\[
a(u, \hat{u}) = \int_\Omega (2\mu \varepsilon(u) : \varepsilon(\hat{u}) + \lambda \text{div} u \, \text{div} \hat{u}) \, d\Omega,
\]

\[
\tau : \theta = \sum_{i,j=1}^n \tau_{ij} \theta_{ij} = \sum_{i=1}^n \sum_{j=1}^n \tau_{ij} \theta_{ij}.
\]
10.1 Strong and weak forms for linear elasticity problems

Show that the bilinear form of the linear elasticity problem is elliptic and continuous with respect to the $H^1(\Omega)$ norm for $\mu > 0$, $0 < \lambda < \infty$ – with constants depending on the Lamé parameters $\mu$ and $\lambda$, however:

(i) $a(v, v) = \int_\Omega (D\varepsilon(v)) : \varepsilon(v) d\Omega$

\[ \geq ... \geq \alpha(\mu, \lambda) \| v \|_1^2 \quad \forall v \in U, \]

(ii) $a(v, \hat{v}) = \int_\Omega (D\varepsilon(v)) : \varepsilon(\hat{v}) d\Omega$

\[ \leq ... \leq C(\mu, \lambda) \| v \|_1 \| \hat{v} \|_1 \quad \forall v, \hat{v} \in U. \]

**Hint:** So called Korn’s inequality can be utilized in proving (i).
10.2 Finite element methods for linear elasticity problems

*Standard form* finite element method for the linear elasticity problem: Let a three-dimensional body be under the loading $b \in [L^2(\Omega)]^3$, $\Omega \subset R^3$. Find $u_h \in U_h \subset U$ such that

$$a(u_h, \hat{u}) = l(\hat{u}) \quad \forall \hat{u} \in U_h,$$
10.2 Finite element methods for linear elasticity problems

**Standard form finite element method for the linear elasticity problem:** Let a three-dimensional body be under the loading \( b \in [L^2(\Omega)]^3 \), \( \Omega \subset R^3 \). Find \( u_h \in U_h \subset U \) such that

\[
a(u_h, \hat{u}) = l(\hat{u}) \quad \forall \hat{u} \in U_h,
\]

with the bilinear form, load functional and variational space

\[
a(u_h, \hat{u}) = \int_{\Omega} (D\varepsilon(u_h)) : \varepsilon(\hat{u}) \, d\Omega = \int_{\Omega} (2\mu \varepsilon(u_h) : \varepsilon(\hat{u}) + \lambda \text{div} u_h \text{div} \hat{u}) \, d\Omega,
\]

\[
l(\hat{u}) = \int_{\Omega} b \cdot \hat{u} \, d\Omega + \int_{S_t} t \cdot \hat{u} \, dS,
\]

\[
U_h = \{ \, v \in [C(\Omega)]^3 \mid v|_{S_u} = 0, \ v|_K \in [P_k(K)]^3 \}.
\]
10.2 Finite element methods for linear elasticity problems

Standard form finite element method for the linear elasticity problem: Let a three-dimensional body be under the loading $b \in [L^2(\Omega)]^3$, $\Omega \subset R^3$. Find $u_h \in U_h \subset U$ such that

$$a(u_h, \hat{u}) = l(\hat{u}) \quad \forall \hat{u} \in U_h,$$

with the bilinear form, load functional and variational space

$$a(u_h, \hat{u}) = \int_\Omega (D\varepsilon(u_h)) : \varepsilon(\hat{u}) \, d\Omega = \int_\Omega (2\mu \varepsilon(u_h) : \varepsilon(\hat{u}) + \lambda \text{div} u_h \text{div} \hat{u}) \, d\Omega,$$

$$l(\hat{u}) = \int_\Omega b \cdot \hat{u} \, d\Omega + \int_{S_t} t \cdot \hat{u} \, dS,$$

$$U_h = \{ v \in [C(\Omega)]^3 \mid v_{|S_u} = 0, v_{|K} \in [P_k(K)]^3 \}.$$

Remark. This standard form finite element method is not optimal: For nearly incompressible materials, i.e, for $\nu \to 1/2$, the approximation converges to the exact solution extremely slowly due to so called volumetric locking phenomena which will be clarified in detail below.
Volumetric locking can be revealed by deriving a standard error estimate in the form

\[ \| u - u_h \|_1 \leq \frac{C(\mu, \lambda)}{\alpha(\mu, \lambda)} \| u - v \|_1 \quad \forall v \in U_h \]

\[ \leq \frac{C(\mu, \lambda)}{\alpha(\mu, \lambda)} c h^k \| u \|_{k+1}, \]
10.2 Finite element methods for linear elasticity problems

**Volumetric locking** can be revealed by deriving a standard error estimate in the form

\[
\| \mathbf{u} - \mathbf{u}_h \|_1 \leq \frac{C(\mu, \lambda)}{\alpha(\mu, \lambda)} \| \mathbf{u} - \mathbf{v} \|_1 \quad \forall \mathbf{v} \in \mathbf{U}_h
\]

\[
\leq \frac{C(\mu, \lambda)}{\alpha(\mu, \lambda)} c h^k \| \mathbf{u} \|_{k+1},
\]

Let us consider the ratio of the material constants in detail (cf. Break exercise 10):

\[
\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{2(1+\nu)(1-2\nu)} = \mu \frac{\nu}{(1-2\nu)} \rightarrow \infty, \text{ for } \nu \rightarrow 1/2
\]
10.2 Finite element methods for linear elasticity problems

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\[ \leq \frac{C(\mu, \lambda)}{\alpha(\mu, \lambda)} c h^k \left| u \right|_{k+1}, \]

Let us consider the ratio of the material constants in detail (cf. Break exercise 10):

\[ \mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{2(1+\nu)(1-2\nu)} = \mu \frac{\nu}{(1-2\nu)} \rightarrow \infty, \text{ for } \nu \rightarrow 1/2 \]

\[ \Rightarrow \ C \sim \max(2\mu, \lambda) \sim \lambda, \quad \alpha \sim \min(2\mu, \lambda) \sim 2\mu, \text{ for } \nu \rightarrow 1/2 \]
10.2 Finite element methods for linear elasticity problems

Volumetric locking can be revealed by deriving a standard error estimate in the form

\[ \| u - u_h \|_1 \leq \frac{C(\mu, \lambda)}{\alpha(\mu, \lambda)} \| u - v \|_1 \quad \forall v \in U_h \]

\[ \leq \frac{C(\mu, \lambda)}{\alpha(\mu, \lambda)} c h^k \| u \|_{k+1}, \]

Let us consider the ratio of the material constants in detail (cf. Break exercise 10):

\[ \mu = \frac{E}{2(1+\nu)} , \quad \lambda = \frac{E\nu}{2(1+\nu)(1-2\nu)} = \mu \frac{\nu}{(1-2\nu)} \to \infty, \quad \text{for } \nu \to 1/2 \]

\[ \Rightarrow C \sim \max(2\mu, \lambda) \sim \lambda, \quad \alpha \sim \min(2\mu, \lambda) \sim 2\mu, \quad \text{for } \nu \to 1/2 \]

\[ \Rightarrow \frac{C(\mu, \lambda)}{\alpha(\mu, \lambda)} \sim \frac{\lambda}{2\mu} \to \infty, \quad \text{for } \nu \to 1/2. \]

This shows that the error estimate is not uniform with respect to the Poisson’s ratio, and hence the method is far from optimal.
10.2 Finite element methods for linear elasticity problems

The nature of the problem, and hence the error estimate as well, depends on the Poisson’s ratio of the material, which can be clearly seen by dividing the weak form of the problem by $2\mu$ giving the form (assuming $t = 0$, for simplicity)

$$
\int_{\Omega} (\varepsilon(u) : \varepsilon(\hat{u}) + \frac{\lambda}{2\mu} \text{div } u \text{ div } \hat{u}) d\Omega = \int_{\Omega} \frac{b}{2\mu} \cdot \hat{u} d\Omega =: \int_{\Omega} \tilde{b} \cdot \hat{u} d\Omega.
$$
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\]

This shows that in the incompressibility limit (in practice, for all nearly incompressible materials) the coefficient of the second term in the scaled formulation above blows up,

\[
\frac{\lambda}{2\mu} \to \infty, \text{ for } \nu \to 1/2,
\]

which implies that the whole term blows up unless the rest of the integrand is able to balance the behaviour by converging to zero (with the same speed):

\[
\text{div} u \ \text{div} \hat{u} \to 0 \quad \forall \hat{u} \in U \quad \Rightarrow \quad \text{div} u \to 0.
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10.2 Finite element methods for linear elasticity problems

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**Remark.** The limit case $\text{div } u = 0$ comes true for incompressible materials.
10.2 Finite element methods for linear elasticity problems

In the incompressibility limit, also the finite element approximation strives for satisfying the *incompressibility condition*:

\[ \text{div } \mathbf{u}_h \rightarrow 0. \]
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This leads to locking, however, which can be seen in computations as an extremely slow convergence towards the exact solution – the closer to the limit, the lower the convergence rate. In the limit case, for a clamped body with linear elements, the approximation locks as fully as possible: \( \mathbf{u}_h = \mathbf{0} \).
10.2 Finite element methods for linear elasticity problems

In the incompressibility limit, also the finite element approximation strives for satisfying the \textit{incompressibility condition}: 

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\textbf{Remark.} The locking phenomena is not limited to linear elements only. However, for low order elements the locking effect takes a leading role – and this effect can be easily understood by considering the local approximations of the displacement:

\[
\ u_h|_{K} \in P_1(K) \quad \Rightarrow \quad \text{div} \ u_h|_{K} \in P_0(K),
\]

and now the condition \( \text{div} \ u_h \rightarrow 0 \) means that the displacement divergence, a piecewise constant function, should approach zero globally, enforcing the displacement to be a global constant (zero, due to clamped boundaries).
10.2 Finite element methods for linear elasticity problems

**Locking free reduced and bubble stabilized finite element method for the linear elasticity problem:** Let a three-dimensional body be under the loading \( b \in [L^2(\Omega)]^3, \ \Omega \subset \mathbb{R}^3 \). Find \( u_h \in U_h \subset U \) such that

\[
a_h(u_h, \hat{u}) = l(\hat{u}) \quad \forall \hat{u} \in U_h,
\]
10.2 Finite element methods for linear elasticity problems

**Locking free reduced and bubble stabilized finite element method for the linear elasticity problem:** Let a three-dimensional body be under the loading \( \mathbf{b} \in [L^2(\Omega)]^3, \ \Omega \subset \mathbb{R}^3 \). Find \( \mathbf{u}_h \in \mathbf{U}_h \subset \mathbf{U} \) such that

\[
a_h(\mathbf{u}_h, \hat{\mathbf{u}}) = l(\hat{\mathbf{u}}) \quad \forall \hat{\mathbf{u}} \in \mathbf{U}_h,
\]

with the reduced bilinear form, load functional and variational spaces

\[
a_h(\mathbf{u}_h, \hat{\mathbf{u}}) = \int_{\Omega} (2\mu \mathbf{\varepsilon}(\mathbf{u}_h) : \mathbf{\varepsilon}(\hat{\mathbf{u}}) + \lambda \mathbf{\Pi}_h \mathbf{\text{div}} \mathbf{u}_h \mathbf{\Pi}_h \mathbf{\text{div}} \hat{\mathbf{u}}) d\Omega,
\]

\[
l(\hat{\mathbf{u}}) = \int_{\Omega} \mathbf{b} \cdot \hat{\mathbf{u}} d\Omega + \int_{S_t} \mathbf{t} \cdot \hat{\mathbf{u}} dS,
\]

\[
\mathbf{U}_h = \{ \mathbf{v} \in [C(\Omega)]^3 \mid \mathbf{v}_{|S_u} = \mathbf{0}, \mathbf{v}_{|K} \in [P_k(K)]^3 \oplus \mathbf{B}(K) \}.
\]

\[
\mathbf{B}(K) = \{ \mathbf{b} = b_K \nabla q \mid q \in P_{k-1}(K) \},
\]

where \( k \geq 3 \), and the bubble function \( b_K \) is the product of the barycentric (or volume) coordinates of element \( K \).
10.2 Finite element methods for linear elasticity problems

The projection operator $\Pi_h : L^2(\Omega) \to Q_h$ is the $L^2$-projection,

$$(\Pi_h v, r) = (v, r) \quad \forall r \in Q_h,$$

to the space

$$Q_h = \{ \ r \in L^2(\Omega) \mid r|_K \in P_{k-1}(K) \ \}.$$
10.2 Finite element methods for linear elasticity problems

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**Remark.** Although for this method it must hold that \( k \geq 3 \), some other locking free lower order methods are available – both for three and two-dimensional problems.

**Remark.** The method above works also for two-dimensional problems for \( k \geq 2 \).
10.2 Finite element methods for linear elasticity problems

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Remark. The finite element approximation of the stress is computed in the method above as a reduced quantity

\[
\sigma_h(u_h) = 2\mu \varepsilon(u_h) + \lambda \Pi_h \text{div} u_h \ I
\]

where \( p_h := \lambda \Pi_h \text{div} u_h \ I \) is often referred as ”pressure” of the mixed formulation.
10.2 Finite element methods for linear elasticity problems

The error analysis for the reduced and bubble stabilized elements follows a nonstandard route: the problem is written in a mixed form (cf. Chapter 8.X). Ellipticity condition or stability condition will be proved in the mixed form by utilizing so called Babuska–Brezzi condition which is often called the inf–sup condition. In the displacement formulation above, an inconsistency term pops out due to projection:

$$0 \neq a_h(u, \hat{u}) - l(\hat{u}) = a_h(u, \hat{u}) - a(u, \hat{u})$$

$$= \int_{\Omega} \lambda (\Pi_h \text{div } u \Pi_h \text{div } \hat{u} - \text{div } u \text{ div } \hat{u}) \, d\Omega,$$

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which will not be present in the corresponding mixed formulation, however.

After all, the error estimate of the method is optimal (with respect to the polynomial order and the regularity of the exact solution; cf. Chapter 5.2):

\[ \left\| u - u_h \right\|_1 + \left\| p - p_h \right\|_1 \leq c_k h^k \left( \left\| u \right\|_{k+1} + \left\| p \right\|_k \right). \]
Element stiffness matrix. In terms of local shape functions, the local finite element approximations can be written in the form

\[ \begin{align*}
    u_h(x) |_{K^{(e)}} &= \sum_{i=1}^{n^{(e)}} N_i(\xi(x)) u_i^{(e)} =: Nd_u^{(e)} \\
    v_h(x) |_{K^{(e)}} &= \sum_{i=1}^{n^{(e)}} N_i(\xi(x)) v_i^{(e)} =: Nd_v^{(e)} \\
    w_h(x) |_{K^{(e)}} &= \sum_{i=1}^{n^{(e)}} N_i(\xi(x)) w_i^{(e)} =: Nd_w^{(e)}
\end{align*} \]
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\[ u_h(x)|_{K(e)} = \sum_{i=1}^{n(e)} N_i(\xi(x)) u_i^{(e)} =: N d_u^{(e)} \]

\[ v_h(x)|_{K(e)} = \sum_{i=1}^{n(e)} N_i(\xi(x)) v_i^{(e)} =: N d_v^{(e)} \]

\[ w_h(x)|_{K(e)} = \sum_{i=1}^{n(e)} N_i(\xi(x)) w_i^{(e)} =: N d_w^{(e)} \]

Accordingly, three node variables are present in each node, one in each global direction:

\[ u_i^{(e)} = (u_i^{(e)}, v_i^{(e)}, w_i^{(e)}) \]
10.2 Finite element methods for linear elasticity problems

**Element stiffness matrix.** In terms of local shape functions, the local finite element approximations can be written in the form

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    u_h(x)_{|K(e)} &= \sum_{i=1}^{n(e)} N_i(\xi(x)) u_i^{(e)} =: Nd_u^{(e)} \\
    v_h(x)_{|K(e)} &= \sum_{i=1}^{n(e)} N_i(\xi(x)) v_i^{(e)} =: Nd_v^{(e)} \\
    w_h(x)_{|K(e)} &= \sum_{i=1}^{n(e)} N_i(\xi(x)) w_i^{(e)} =: Nd_w^{(e)}
\end{align*}
\]

Accordingly, three *node variables* are present in each node, one in each global direction:

\[
    \mathbf{u}_i^{(e)} = (u_i^{(e)}, v_i^{(e)}, w_i^{(e)})
\]

**Remark.** In three-dimensional problems, elements are often *tetrahedra* or *hexahedra* which have (Lagrange) nodes at vertices (linear), edge midpoints (quadratic) etc. – and interior bubbles inside.

3D elements in NX Nastran software
10.2 Finite element methods for linear elasticity problems

It is often useful to write the local element degree of freedom vector in the form

$$\mathbf{d}^{(e)} = \begin{bmatrix} u_1^{(e)} & v_1^{(e)} & w_1^{(e)} & u_2^{(e)} & v_2^{(e)} & w_2^{(e)} & \cdots & u_{n(e)}^{(e)} & v_{n(e)}^{(e)} & w_{n(e)}^{(e)} \end{bmatrix}^T$$
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\end{bmatrix}^T
\]

and, accordingly, the local element shape function matrix in the form

\[
N^{(e)} = \begin{bmatrix}
N_1 & 0 & 0 & N_2 & 0 & 0 & N_{n(e)} & 0 & 0 \\
0 & N_1 & 0 & 0 & N_2 & 0 & \cdots & 0 & N_{n(e)} & 0 \\
0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_{n(e)}
\end{bmatrix}.
\]
It is often useful to write the local element degree of freedom vector in the form

\[ \mathbf{d}^{(e)} = \begin{bmatrix} u_1^{(e)} & v_1^{(e)} & w_1^{(e)} & u_2^{(e)} & v_2^{(e)} & w_2^{(e)} & \cdots & u_{n(e)}^{(e)} & v_{n(e)}^{(e)} & w_{n(e)}^{(e)} \end{bmatrix}^T \]

and, accordingly, the local element shape function matrix in the form

\[ \mathbf{N}^{(e)} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_{n(e)} & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \cdots & 0 & N_{n(e)} & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_{n(e)} \end{bmatrix}. \]

This way, the local finite element approximations can be written in the form

\[ \mathbf{u}_h (\mathbf{x})_{|K^{(e)}} = \mathbf{N}^{(e)} \mathbf{d}^{(e)}, \]

and the entries in the different parts of the stiffness matrix and force vector will be automatically positioned correctly in the element contributions.
With this notation, the element stiffness matrix is of the form

\[ K_b^{(e)} = \int_{K^{(e)}} B^{(e)^T} D B^{(e)} \, d\Omega \]
10.2 Finite element methods for linear elasticity problems

With this notation, the element stiffness matrix is of the form

\[ K_b^{(e)} = \int_{K^{(e)}} B^{(e)T} D B^{(e)} \, d\Omega \]

**Remark.** When computing the stiffness matrix entries in terms of the local coordinate system, i.e., in the reference element, it should be noticed that the integral and the derivatives must be transformed appropriately. In particular, the differential operator in the \( B \)-matrix needs to be transformed by using the chain rule since the shape functions are expressed in terms of the local shape functions:

\[
B^{(e)} = L_{xyz} N^{(e)}, \quad L_{xyz} = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 \\
0 & 0 & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0
\end{bmatrix}, \quad N^{(e)} = N^{(e)}(\xi, \eta, \zeta).
\]
For 2D problems as the *plane strain* elasticity problem, the element stiffness matrix is of the same form

\[ K_b^{(e)} = \int_{K^{(e)}} B^{(e)T} D B^{(e)} \, d\Omega, \]

whereas the *constitutive matrix*, differential operator and shape function matrix are of the form

\[ D = \begin{bmatrix} 2\mu + \lambda & \lambda & 0 \\ \lambda & 2\mu + \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}, \]

\[ B^{(e)} = L_{xy} N^{(e)}, \]

\[ L_{xy} = \begin{bmatrix} \partial / \partial x & 0 \\ 0 & \partial / \partial y \\ \partial / \partial y & \partial / \partial x \end{bmatrix}, \quad N^{(e)} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \ldots & N_{n^{(e)}} & 0 \\ 0 & N_1 & 0 & N_2 & \ldots & 0 & N_{n^{(e)}} \end{bmatrix}. \]
For the elasticity problem, the ellipticity of the energy (cf. Break exercise 10) can be proven by utilizing so called Korn’s inequalities:

**Korn’s first inequality.** Let \( \Omega \subset \mathbb{R}^d \) be an open bounded domain with a piecewise smooth boundary. Then there exist a positive constant \( c = c(\Omega) \) such that

\[
\| \varepsilon(\nu) \|_0^2 + \| \nu \|_0^2 \geq c \| \nu \|_1^2 \quad \forall \nu \in [H^1(\Omega)]^d.
\]

**Korn’s second inequality.** Let \( \Omega \subset \mathbb{R}^3 \) be an open bounded domain with a piecewise smooth boundary, and let \( S_u \subset \partial \Omega \) have a positive two-dimensional measure. Then there exist a positive constant \( c = c(\Omega, S_u) \) such that

\[
\| \varepsilon(\nu) \|_0^2 \geq c \| \nu \|_1^2 \quad \forall \nu \in [H^1_{S_u}(\Omega)]^3.
\]

**Remark.** It is essentially the Korn’s inequalities that guarantee a unique solution for the elasticity problem.
QUESTIONS?

ANSWERS”

LECTURE BREAK!