Linear systems and their models

- Impulse response and weighting function
- Transfer function matrices
- Transfer operator
- Input-output-models
- State-space representation
- Frequency functions
- (Discrete-time systems)

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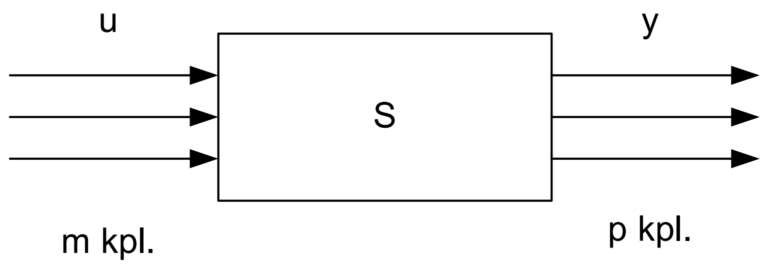
Impulse response and weighting function

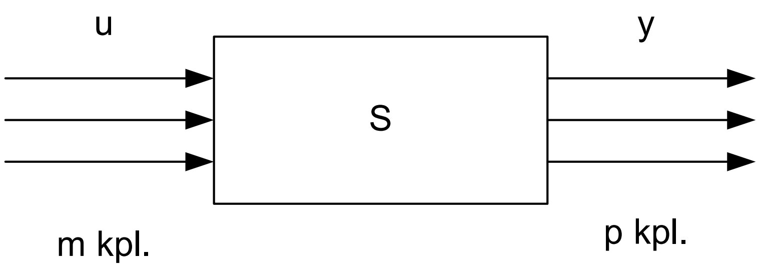


$$y(t) = \int_0^\infty g(\tau)u(t-\tau)d\tau$$

g the *weighting function*, which for linear systems is equal to the (unit) impulse response.

In multivariable case *g* is *p* x *m*-matrix, where the element (*k*, *j*) is the input response in channel *k*, when the impulse enters in input channel *j.*

Impulse response and weighting function, cont..



$$U(s) = L(u)$$
$$Y(s) = L(y)$$

$$G(s) = L(g) = \int_0^\infty e^{-st} g(t)dt \qquad Y(s) = G(s)U(s)$$

SISO: $G(s) = \dfrac{B(s)}{A(s)}$ dim(B)=n, dim(A)=m

$m > n$: G is "strictly proper"
$m = n$: G is "proper"
$m < n$: G is "non-proper"

$$\lim_{s \to \infty} G(s) = 0$$
strictly proper

Example:

Transfer function matrix

$$G(s) = \begin{bmatrix} 1/(s+1) & 2/(s+5) \end{bmatrix}$$ 2 inputs, 1 output

System at rest at time t=0 (initial conditions zero)

At time 0 in channel 1 a unit step and in channel 2 unit impulse. The output becomes

$$U(s) = \begin{bmatrix} 1/s \\ 1 \end{bmatrix} \qquad Y(s) = G(s)U(s) = \frac{1}{s(s+1)} + \frac{2}{s+5} = \frac{1}{s} - \frac{1}{s+1} + \frac{2}{s+5}$$

$$y(t) = 1 - e^{-t} + 2e^{-5t}$$

Notice also the final value theorem $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$

Transfer operator (derivative operator)

$$py(t) = \frac{d}{dt} y(t)$$

compare to discrete-time systems

$$qy(t) = y(t+1)$$

$$y(t) = G(p)u(t)$$

$$q^{-1}y(t) = y(t-1)$$

Note the ”connection” between the operator *p* in time domain with the laplace-variable *s*.

Same in discrete time (operator *q* and variable *z*).

Input-output representations

$$G(s) = \frac{B(s)}{A(s)}$$

$$A(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$$

$$B(s) = b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1} s + b_n$$

$$y(t) = \frac{B(p)}{A(p)} u(t) \quad \Leftrightarrow \quad A(p) y(t) = B(p) u(t)$$

the corresponding differential equation is

$$\frac{d^n}{dt^n} y(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} y(t) + \cdots + a_{n-1} \frac{d}{dt} y(t) + a_n y(t) =$$

$$b_1 \frac{d^{n-1}}{dt^{n-1}} u(t) + \cdots + b_n u(t)$$

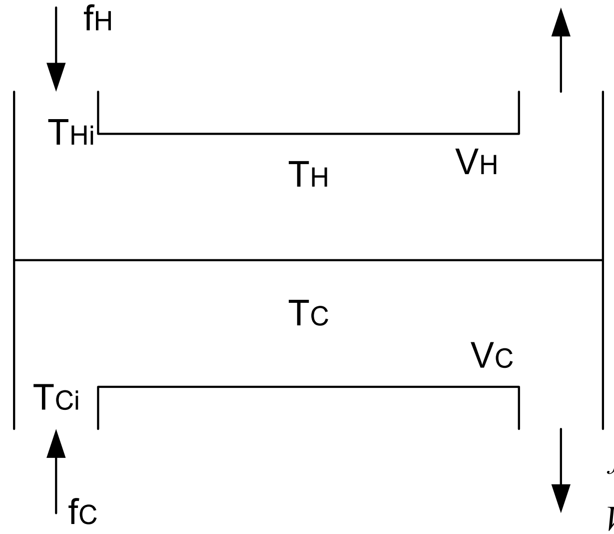
State-space representation

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

If $D = 0$, the system is ”strictly proper”, otherwise ”proper”

Example: a heat exchanger



$$V_C \frac{dT_C}{dt} = f_C\left(T_{C_i} - T_C\right) + \beta\left(T_H - T_C\right)$$

$$V_H \frac{dT_H}{dt} = f_H\left(T_{H_i} - T_H\right) - \beta\left(T_H - T_C\right)$$

$f = f_C = f_H = 0.01\ (\mathrm{m}^3/\mathrm{min}),\ \beta = 0.2\ (\mathrm{m}^3/\mathrm{min})$

$V_H = V_C = 1\ m^3$

$$\dot{x} = \begin{bmatrix} -(f+\beta)/V_C & \beta/V_C \\ \beta/V_H & -(f+\beta)/V_H \end{bmatrix} x + \begin{bmatrix} f/V_C & 0 \\ 0 & f/V_H \end{bmatrix} u \qquad \begin{aligned} u_1 &= T_{C_i} \\ u_2 &= T_{H_i} \end{aligned}$$

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From one representation to another

1. From state space to transfer function:

$$\dot{x} = px = Ax + Bu$$
$$y = Cx + Du$$

$$(pI - A)x(t) = Bu(t) \quad \Rightarrow \quad x(t) = (pI - A)^{-1}Bu(t)$$
$$y(t) = Cx(t) + Du(t) = \left[C(pI - A)^{-1}B + D\right]u(t) = G(p)u(t)$$

$$G(p) = C(pI - A)^{-1}B + D$$

Substituting *p* with *s* gives the transfer function matrix

Ex. Heat exchanger

$$\dot{x} = \begin{bmatrix} -0.21 & 0.2 \\ 0.2 & -0.21 \end{bmatrix} x + \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} u$$

$$y = x$$

$$G(p) = C(pI - A)^{-1} B + D$$

$$G(s) = I \begin{bmatrix} s+0.21 & -0.2 \\ -0.2 & s+0.21 \end{bmatrix}^{-1} \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} = \frac{0.01}{(s+0.01)(s+0.41)} \begin{bmatrix} s+0.21 & 0.2 \\ 0.2 & s+0.21 \end{bmatrix}$$

Canonical forms

SISO-model

$$G(p) = \frac{b_1 p^{n-1} + b_2 p^{n-2} + \cdots + b_{n-1} p + b_b}{p^n + a_1 p^{n-1} + \cdots + a_{n-1} p + a_n}$$

Realizations:

$$\dot{x}(t) = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} b_1 & b_2 & \cdots & b_{n-1} & b_n \end{bmatrix} x(t)$$

controllable canonical form

$$\dot{x}(t) = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} x(t) + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} x(t)$$

observable canonical form

The MIMO case is much more difficult and general algorithms complicated.

Esim. MISO (many inputs, one output)

$$y(t) = \frac{p+2}{p^2+2p+1}u_1(t) + \frac{1}{p^2+3p+2}u_2(t)$$

$$y(t) = \frac{1}{p^3+4p^2+5p+2}\begin{bmatrix} p^2+4p+4 & p+1 \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} -4 & 1 & 0 \\ -5 & 0 & 1 \\ -2 & 0 & 0 \end{bmatrix}x + \begin{bmatrix} 1 & 0 \\ 4 & 1 \\ 4 & 1 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}x$$

Realizations from inputs1 and 2 to output: same A- and C-mat-rices!

Of course Matlab can be used, Control System Toolbox commands

ss, tf
ss2tf
tf2ss
impulse
step

For example.

```
Gtf=tf(1,[1 2 3]);
Gss=ss(Gtf);
```

Properties of linear systems

- Solution of the state equation
- Controllability and observability
- Poles and zeros
- Stability
- Frequency response and frequency functions

Solution of the state equation

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

Note. A corresponding discrete-time system can be derived from this solution by assuming that the control signal remains constant between sampling instants (ZOH= zero order hold).

Change of the state variable

$x = T^{-1}\xi$ T is an invertible square matrix

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$\dot{x} = T^{-1}\dot{\xi} = AT^{-1}\xi + Bu$$
$$y = CT^{-1}\xi + Du$$

$$\dot{\xi} = TAT^{-1}\xi + TBu$$
$$y = CT^{-1}\xi + Du$$

new realization

The *similarity transformation*; matrices *A* and TAT^{-1} are similar.

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For a diagonal matrix

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

it holds

$$e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

The i:th component of the solution is

$$x_i(t) = e^{\lambda_i (t-t_0)} x_i(t_0) + \int_{t_0}^{t} e^{\lambda_i (t-\tau)} B_i u(\tau) d\tau$$

in which Bi is the *i*:th row of matrix *B*. The components xi do not depend on other components of *x*.
xi corresponds to the *mode* of λ_i

If *Ci* is the *i*:th column of the matrix *C* , it follows

$$y(t) = C_1 x_1(t) + C_2 x_2(t) + \cdots + C_n x_n(t) + Du(t)$$

The output can be seen as a weighted sum of the modes.

Controllability and observability

- Structural properties: stability, controllability (reachability), observability
- Describe how the system output depends on inputs and internal ”states”
- Kalman, 1960’s (pioneering period of modern control engineering)

State x^* is <u>controllable,</u> if there exist a control that drives the system in a finite time from the state x* to the origin of the state space (note the mistake in the textbook)

The system is controllable, if all states are controllable.

State $x^* \neq 0$ is <u>non-observable,</u> if

$$u(t) = 0,\ t \geq 0 \quad \text{and} \quad x(0) = x^* \quad \Rightarrow \quad y(t) \equiv 0,\ t \geq 0$$

The system is <u>observable</u> , if it lacks non-observable states.

The controllable states are the range of the linear map given by the controllability matrix

$$S(A,B) = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

The system is controllable, if the rank of the controllability matrix is full.

In the SISO case S(A,B) is a square matrix, which must have a non-zero determinant, in order the system to be controllable. More generally the rank of the matrix must be checked.

The non-observable states form the kernel of the linear map

$$O(A,C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad \text{(observability matrix)}$$

The system is observable, when the observability matrix has full rank. For a square matrix the determinant must be non-zero.

Note:

- If the realization can be transformed to the controllable canonical form, the system is controllable.
- If the realization can be transformed to the observable canonical form, the system is observable.
- A SISO-system is both controllable and observable, when there are no pole-zero cancellations in the calculation of the transfer operator (transfer function).
- When a state-space representation is both controllable and observable, it is the *minimal realization* of the system; there are no realizations of lower degree that would generate the same input-output behaviour.

Pole-placement:

$$\dot{x} = Ax + Bu$$

$$u = -Lx$$

$$\dot{x} = (A - BL)x$$

mxn-matrix *L* can be found such that arbitrary eigenvalues are obtained, if and only if

$$S(A,B) = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$ has full rank

Correspondingly, the *nxp*-matrix *K* can be found such that the matrix *A-KC* has arbitrary eigenvalues, if and only if

$$O(A,C)=\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full rank.

Stabilizability and Detectability

If the system is not controllable, still the controllable modes (eigenvalues) can be influenced by state feedback; non-controllable modes cannot be influenced.

But if non-controllable modes are (asymptotically) stable the system is called **stabilizable**.

Definition: The system (A,B,C) is **stabilizable**, if there exists a matrix L such that $A\text{-}BL$ is stable (all eigenvalues in the stability region = left half plane or inside the unit circle). The system is **detectable**, if there exists a matrix K such that $A\text{-}KC$ is (asymptotically) stable.

Poles and zeros

The eigenvalues of the system matrix *A* are important in the charcterization of the system behaviour (modes).

linear combinations $e^{p_i t}$

Definition: The **poles** are the eigenvalues of *A,* where *A* is the system matrix of the minimal realization. The dimension of a pole corresponds to the multiplicity of the corresponding eigenvalue. The **pole polynomial** is the characteristic polynomial of *A*.

$$\det(\lambda I - A)$$

To form realizations for MIMO systems is generally difficult. It would be nice to have a method to calculate the poles directly from the transfer function matrix G(s).

Note. SISO-systems do not have such problems; when all possible cancellations have been made, the pole polynomial is the denominator of the transfer function.

Matlab: Gss = minreal(ss(Gtf));

(starting from the transfer function matrix *Gtf* the command ss forms a state-space realization and then the command *minreal* forms the minimal realization)

But programs are only programs!

The *minors* are obtained as determinants of the sub-matrices.

Ex. The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

has nine minors

$$\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3 \qquad \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = -6 \qquad \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3$$

1, 2, 3, 4, 5, 6

The *largest* minors correspond to the largest sub-determinant (in the example the 2*x*2-cases)

Theorem: The pole polynomial is the least common denominator of all (not identically zero) minors. The poles of the system are the zeros of the pole polynomial.

Ex.

$$G(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

Minors:

$$\frac{2}{s+1}, \quad \frac{3}{s+2}, \quad \frac{1}{s+1}$$

$$\frac{2}{(s+1)^2} - \frac{3}{(s+1)(s+2)} = \frac{-s+1}{(s+1)^2(s+2)}$$

Pole polynomial $p(s) = (s+1)^2(s+2)$

Poles: -2 and –1 (multiplicity two)

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The realization can now be formed

$$x_1(t) = \frac{1}{p+1}u_1(t), \quad x_2(t) = \frac{1}{p+1}u_2(t), \quad x_3(t) = \frac{1}{p+2}u_2(t)$$

The order of the system is three (number of state variables)

which leads to

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 1 & 0 \end{bmatrix} x$$

```
g11=tf(2,[1 1]);
g12=tf(3,[1 2]);
g21=tf(1,[1 1]);
g22=tf(1,[1 1]);
Gtf=[g11 g12;g21 g22]
```

```
Gss1=ss(Gtf)
```

```
Gss2=minreal(ss(Gtf))
```

(the same result in this example case)

Note that the poles are the denominators of the transfer functions in the transfer function matrix. ”Minor analysis” is needed in the determination of pole multiplicities, which are again needed to form the minimal realization.

Zeros

The zero of a SISO-system is such *s*, which makes the value of the transfer function zero (to lose rank in the multivariable case). The zeros of a square matrix *G(s)* are the poles of $G^{-1}(s)$

Definition:
The system **zeros** (transmission zeros) are those *s*, for which the rank of the matrix

$$M(s) = \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix}$$

drops (is not full). The polynomial with these zeros *s*, is the **zero polynomial**.

For the system with equal number of inputs and outputs, the zero polynomial is det *M(s)*. In other cases the zeros can be determined directly from the transfer function matrix according to the following theorem.

Theorem: Form the maximal minors of G*(s)* normalized such that the denominators contain the pole polynomial. The **zero polynomial** of the system is the greatest common divisor of these. The **zeros** are the zeros of this polynomial.

Ex.

$$G(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

The maximal ”minors” $\det G(s) = \frac{-s+1}{(s+1)^2(s+2)}$

The pole polynomial (from the previous example)

$p(s) = (s+1)^2(s+2)$ which is already in the denominator

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The zero polynomial $z(s) = -s + 1$

and the system has one zero s = 1.

Note.

$$G^{-1}(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{(s+1)(s+2)}{-s+1} & -\frac{3(s+1)^2}{-s+1} \\ -\frac{(s+1)(s+2)}{-s+1} & \frac{2(s+1)(s+2)}{-s+1} \end{bmatrix}$$

The pole polynomial is (verify!) $-s+1$

Stability of linear systems

Definition: The **stability region** of a continuous system is the LHP = left half plane, imaginary axis excluded. For a discrete-time system the corresponding region is the inside of the unit circle.

For linear systems $$y(t) = \int_{-\infty}^{t} g(t-\tau)u(\tau)d\tau = \int_{0}^{\infty} g(\tau)u(t-\tau)d\tau$$

and so $$|y(t)| \leq \int_{0}^{\infty} |g(\tau)||u(t-\tau|d\tau$$

BIBO-stable, if $$\int_{0}^{\infty} |g(\tau)|\, d\tau < \infty$$

Result: A linear time-invariant system is BIBO-stable, if and only if its poles are in the stability region.

Result: A linear time-invariant system is asymptotically stable, if and only if the eigenvalues of the system matrix *A* are in yhe stability region. If the system is stable,all eigenvalues are in the stability region or on its boundary.

Definition: The system is **non-minimum phase**, if it has at least one zero outside the stability region. Otherwise the system is **minimum phase.**

Frequency response and frequency functions

Consider the transfer function matrix

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1m}(s) \\ G_{21}(s) & G_{22}(s) & \cdots & G_{2m}(s) \\ \vdots & \vdots & \vdots & \vdots \\ G_{p1}(s) & G_{p2}(s) & \cdots & G_{pm}(s) \end{bmatrix}$$

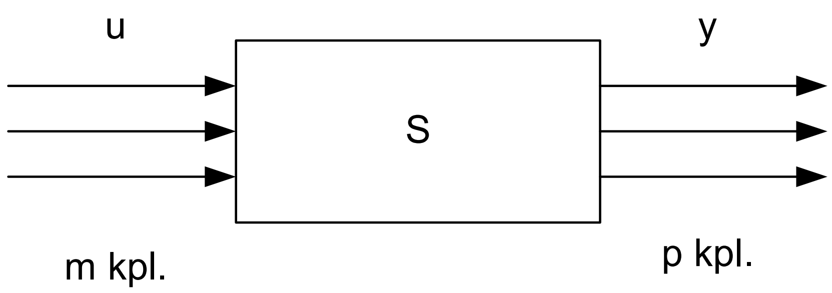
the input in channel *k* (other input channels zero)

$$u_k(t) = \cos(\omega t)$$

output in channel *j*

$$y_j(t) = A\cos(\omega t + \varphi) \qquad A = \left|G_{jk}(i\omega)\right| \qquad \varphi = \arg G_{jk}(i\omega)$$

How to define the gain of a MIMO-system?



The frequency response (Bode) of each channel separately does not give full information about the behaviour of the system (interconnections; different ”directions” of the multi-channel input function .

What about the eigenvalues of G? But they are defined only for square systems, and do not generally give a reliable view about the gain of a multivariable system.

Solution: *Singular values*

Consider the mapping

$$y = Ax$$

A p x m- matrix
x m x 1-vector
y p x 1-vector

(complex values are allowed)

Definition: The *hermitian matrix* (adjoint) of *A*, A^* or A^H is obtained by taking the transpose and then the complex conjugate of each term

Ex. $A = \begin{bmatrix} 1+i \\ 2-3i \end{bmatrix}, \quad A^* = \begin{bmatrix} 1-i & 2+3i \end{bmatrix}$

```
A=[1+i;2-3i]
A =

   1.0000 + 1.0000i
   2.0000 - 3.0000i

A'
  1.0000 - 1.0000i   2.0000 + 3.0000i

A.'
   1.0000 + 1.0000i   2.0000 - 3.0000i
```

Matlab: ’ means actually taking the conjugate transpose. For an ordinary transpose,write .’

For real matrices the shorter form is of course used also in the case of an ordinary transpose.

The familiar rules are valid, for example. $(AB)^* = B^* A^*$

Definition: A complex-valued matrix is *hermitian*, (self-adjoint), if

$A^* = A$ (cf. symmetric for real matrices)

Hermitian (and symmetric) matrices have real eigenvalues.

Definition: Let *A* be a hermitian matrix. It is *positive definite*, if the scalar $x^*Ax > 0$, for all non-zero vectors *x*.

$x^*Ax > 0$ (often written as $A > 0$)

Correspondingly, *A* is *negative definite*, if
$x^*Ax < 0$ (often written as $A < 0$)

A is *positive semidefinite*, if

$x^*Ax \geq 0$ $\quad\quad$ $A \geq 0$

A is *negative semidefinite,* if

$x^*Ax \leq 0$ $\quad\quad$ $A \leq 0$

Note that if *A* is hermitian, the square form x*Ax
is always real.

Result: A hermitian *A* is pos.def., when all eigenvalues are positive; pos. semidefinite, when all eigenvalues are non-negative.

Corresponding results are valid also in the case of negative definite matrices.

To check the positive definitness of a *symmetric*, *real* matrix: the *Sylvester rule*.

Let us return to study the gain of the map $y = Ax$

How ”big” is *y* when compared to *x*?

Well, the 2-norm of *x* is

$$\left| x \right| = \left(\sum_{i=1}^{m} \left| x_i \right|^2 \right)^{\frac{1}{2}} = \sqrt{x^* x}$$

So $$\left| y \right|^2 = \left| Ax \right|^2 = x^* A^* A x$$

The matrix $A^* A$ is hermitian, eigenvalues real

$\lambda_1, \lambda_2, \ldots, \lambda_m$; largest λ_1 and smallest λ_m

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Then $\lambda_m |x|^2 \leq x^* A^* A x \leq \lambda_1 |x|^2$ (Rayleigh-Ritz inequality)

and the definition follows

Definition: The *singular values* of *A* are $\sigma_i = \sqrt{\lambda_i}$

in which the values λ_i are the eigenvalues of $A^* A$

the largest eigenvalue is denoted as $\bar{\sigma}(A)$

and the smallest one as $\underline{\sigma}(A)$

$$\text{When } y = Ax, \text{ then} \qquad \underline{\sigma}(A) \leq \frac{|y|}{|x|} \leq \bar{\sigma}(A)$$

The gain of the matrix is between the smallest and largest singular value. The maximum (supremum) is a *norm.*

$$|A| = \overline{\sigma}(A)$$

The induced matrix norm corresponding to 2-vector norm

The singular values are generated naturally by the *singular value decomposition* (SVD):

Result: For any real or complex matrix *A* there always exists a factorization

$$A = U\Xi V^{*} \qquad \text{(SVD)}$$

A (n x m), U (n x n), V (m x m)

U and *V unitary* $UU^{*} = I, \quad VV^{*} = I$

(cf. *orthogonal* in the case of real matrices)

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Ξ (n x m) is a real matrix, and the singular values of A are located in the main diagonal in descending order. If A is complex, U and V are also complex; otherwise real.

The columns of U and V are the unit eigenvectors of AA^* and A^*A, respectively. They represent the *output* and *input* directions.

Matlab: $\left[U, Sigma, V\right] = svd(A)$

Frequency functions:

$$Y(i\omega) = G(i\omega)U(i\omega)$$

$$\underline{\sigma}(G(i\omega)) \leq \frac{|Y(i\omega)|}{|U(i\omega)|} \leq \overline{\sigma}(G(i\omega)) = |G(i\omega)|$$

The **gain** of the system at freq. ω is between the smallest and largest singular value of $G(i\omega)$

Corresponding to Bode diagrams in SISO systems the singular values are plotted as functions of frequency; the gain in all ”directions” is between the smallest and largest singular value.

Norm of the frequency function:

$$\|G\|_{\infty} = \max_{\omega} |G(i\omega)|$$ (the largest singular value)

$$\|y\| \le \|G\|_{\infty} \|u\|$$ $\|G\|_{\infty}$ is the system gain

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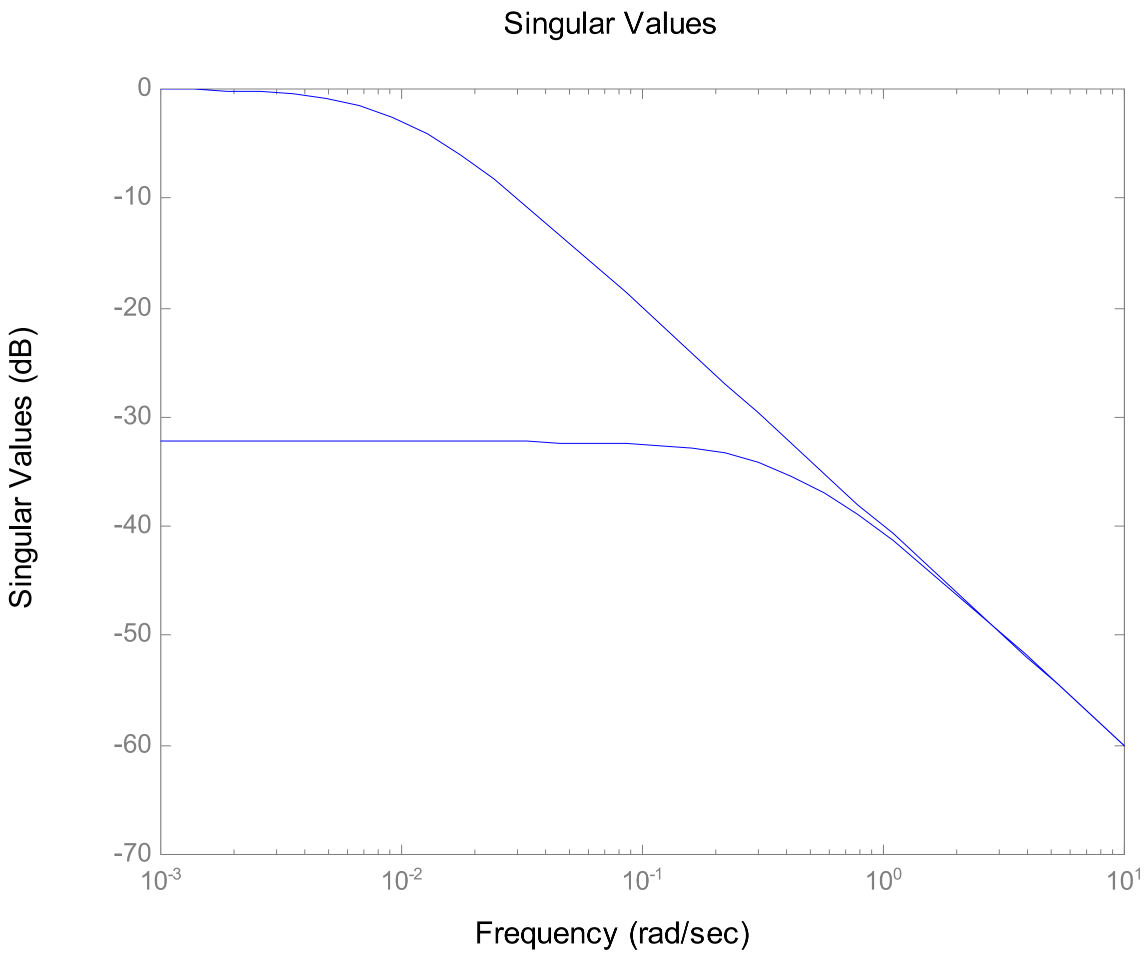
Ex. The calculation of singular values is difficult.
In Matlab: the *sigma* command is helpful.

```
A=[-0.21 0.2;0.2 -0.21];
B=0.01*eye(2);
C=eye(2);
D=zeros(2,2);
sigma(A,B,C,D)
```


Singular Values
Singular Values (dB)
0
-10
-20
-30
-40
-50
-60
-70
10-3
10-2
10-1
10^0
10^1
Frequency (rad/sec)

What about ”directions” ?

Transfer function

$$G(s) = I\begin{bmatrix} s+0.21 & -0.2 \\ -0.2 & s+0.21 \end{bmatrix}^{-1}\begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} = \frac{0.01}{(s+0.01)(s+0.41)}\begin{bmatrix} s+0.21 & 0.2 \\ 0.2 & s+0.21 \end{bmatrix}$$

Look at the frequency $s = 0$

```
g0=[0.21 0.2;0.2 0.21]/0.41;
[V,D]=eig(g0'*g0)
```

Result:

$$\begin{matrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{matrix} = V$$

input directions

$[0.7071 \;\; -0.7071]'$

$[0.7071 \;\; 0.7071]'$

$$\begin{matrix} 0.0006 & 0 \\ 0 & 1.0000 \end{matrix} = D$$

eigenvalues
0.0006 and 1.0

Singular value $\sqrt{1} = 1$ is related to the direction $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$

and $\sqrt{0.0006} = 0.024$ to $\begin{bmatrix} 1 & -1 \end{bmatrix}^T$

The input form $u_0 = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ has a very minor influence

to the output

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Useful Matlab-commands:

ss2zp, zp2ss, tf2zp, zp
tzero
pole
pzmap
lsim
eig, roots
bode
sigma
obsv, ctrb