Problem 4.1: Necessary Conditions for Least Squares

Consider the following unconstrained optimization problem P:

$$(P) \;:\; \text{min.} \quad ||Ax - b||_2^2 \tag{1}$$

where A is a matrix in $\mathbb{R}^{m\times n}$ and b is a vector in $\mathbb{R}^m$. This problem is typically called a *least-squares* problem when using the Euclidean norm, and it has several applications in regression analysis, optimal control, parameter estimation, data fitting, etc.

An extension of the problem P involves minimizing $||x||_2^2$ on top of the original objective. To solve this problem, we can use *regularization* which is a common scalarization technique to find solutions to bi-criterion problems. We will consider the following *regularized* least-squares problem

$$(RP) \;:\; \text{min.} \quad ||Ax - b||_2^2 + \delta||x||_2^2 \tag{2}$$

where the penalty term $\delta > 0$ controls the trade-off between the two objectives.

(a) Give brief interpretations of the problems (1) and (2).

(b) Find solutions for the problems (1) and (2) by writing the first-order necessary optimality conditions. Justify why these conditions are also sufficient.

Problem 4.2: Optimality of Points in a Convex Problem

Consider the following convex optimization problem P:

$$(P) : \; \text{min.} \; (x_1 - 3)^2 + (x_2 - 2)^2 \tag{3}$$
$$\text{subject to: } x_1^2 + x_2^2 \leq 5 \tag{4}$$
$$x_1 + x_2 \leq 3 \tag{5}$$
$$x_1 \geq 0 \tag{6}$$
$$x_2 \geq 0 \tag{7}$$

Let S denote the feasible set defined by the constraints (4) − (7), and let $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x) = (x_1 - 3)^2 + (x_2 - 2)^2$ denote the objective function (3). Notice that both S and f are convex. Recall the following optimality condition for convex optimization problems presented in Lecture 4 (Corollary 4):

Let $S \subseteq \mathbb{R}^n$ be a convex set and $f : S \to \mathbb{R}$ a differentiable convex function on S. Then $\overline{x} \in S$ is optimal if and only if

$$\nabla f(\overline{x})^\top (x - \overline{x}) \geq 0, \text{ for all } x \in S \tag{8}$$

Using the condition (8), examine graphically if the following points are optimal for problem P:

(a) $\overline{x}_1 = (1, 2)$

(b) $\overline{x}_2 = (2, 1)$

Problem 4.3: Optimal Point of a Nonsmooth Convex Problem

Consider the following nonsmooth optimization problem P:

$$(P): \text{ min. } f(x) = \begin{cases} -\frac{3}{2}x + 6, & \text{if } 0 < x \leq 2 \\ -\frac{1}{2}x + 4, & \text{if } 2 \leq x \leq 4 \\ \frac{1}{4}x + 1 & \text{if } 4 \leq x \leq 8 \\ x - 5 & \text{if } x \geq 8 \end{cases} \quad (9)$$

$$\text{subject to: } x \in \mathbb{R}^+. \quad (10)$$

Let S denote the feasible set defined by the constraint (10), and let $f : \mathbb{R}^+ \to \mathbb{R}$ with $f(x)$ denote the objective function (9). Notice that both S and f are convex.

Characterize the subdifferential sets of f at points $\overline{x}_1 = 2$, $\overline{x}_2 = 4$, and $\overline{x}_3 = 8$. Use Corollary 3 from Lecture 4 to show that $\overline{x}_2 = 4$ is the unique optimal solution to the problem P. Corollary 3 states that a point $\overline{x} \in S$ is an optimal solution to P if and only if $0 \in \partial f(\overline{x})$, that is, f has a subgradient $\xi = 0$ at $\overline{x}$ that belongs to the subdifferential set $\partial f(\overline{x})$.