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In Preparation for Session 3: Mathematical modeling

35E00750 Logistics Systems and Analytics

Dr. Tri M. Tran
Assistant Professor of Operations Management
University of Groningen
https://www.rug.nl/staff/tri.tran/

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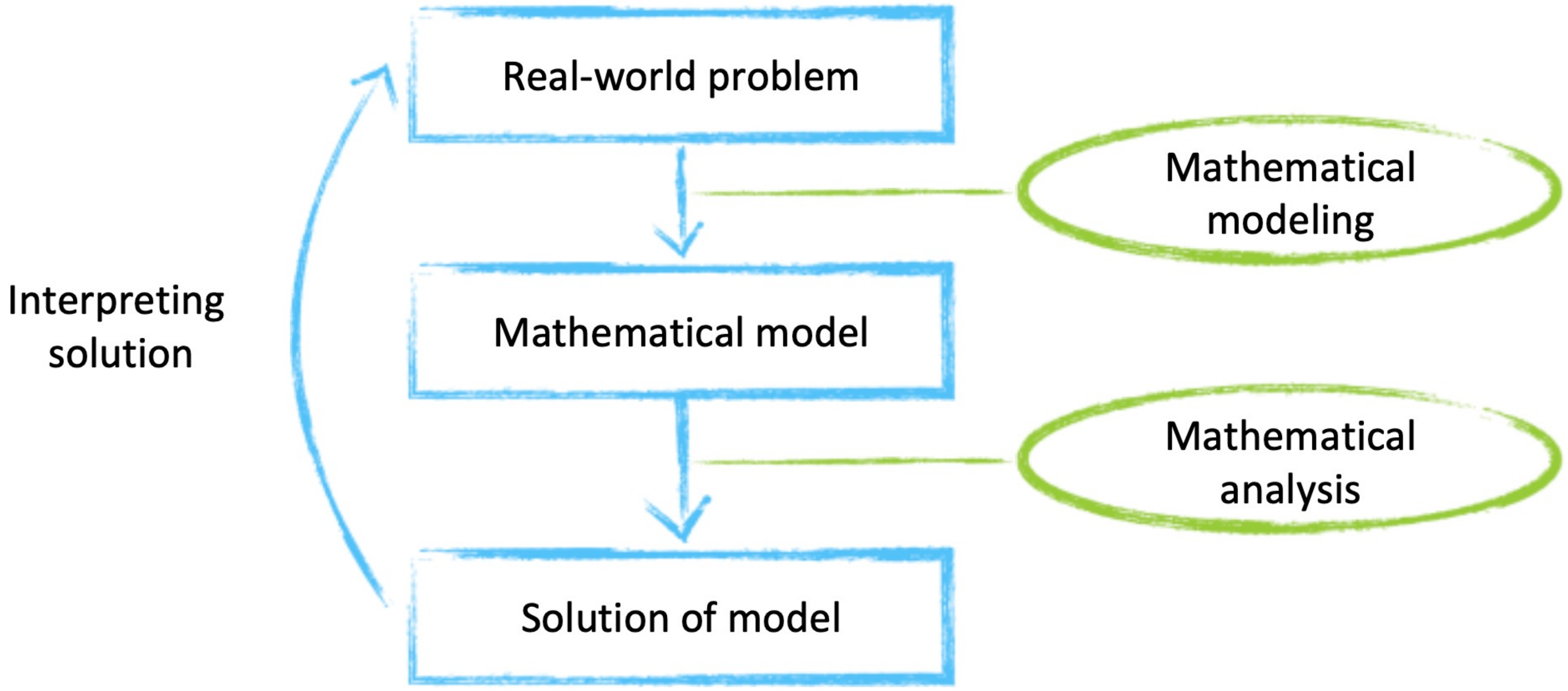
Mathematical modeling

Slide courtesy of Dr. Ir. Paul Buijs, Assistant Professor, Department of Operations, Faculty of Economics and Business, University of Groningen

Mathematical modeling steps



Terminologies

- **Decision variables**
  - Mathematical description of the set of decisions to be made
- **Parameters**
  - What input data are known and needed for making the decisions?
- **Objectives**
  - A measure to rank alternative solutions
  - What do you want to achieve? Express this mathematically by using your decision variables and parameters
- **Constraints**
  - Limitations on the values of the decision variables
  - Develop mathematical relationships to describe constraints

Types of mathematical models

- **Linear Programming**
  - Variables can take real numbers
- **Integer Programming**
  - Variables can only take integer values
- **Binary Programming**
  - Variables can only take the value of 0 or 1
- **Mixed Integer programming**
  - Some variables are constrained to be integer values

Valid range of a variable

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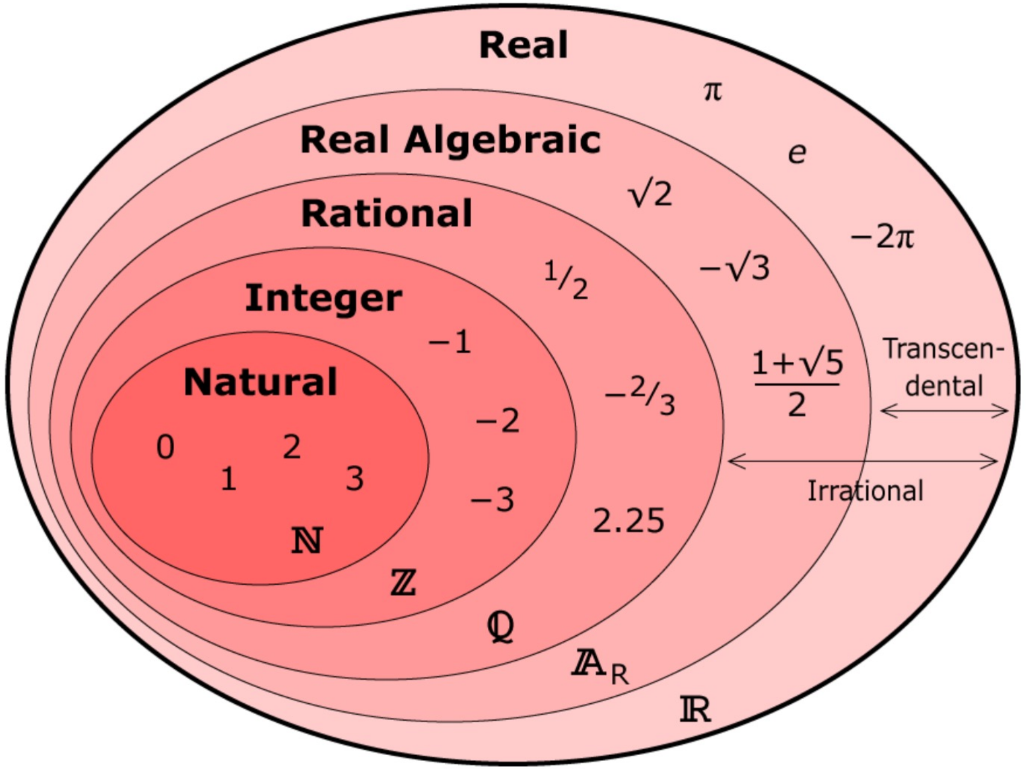

Valid range of a variable

- **Binary programing:** $\boldsymbol{x} \in \{\mathbf{0}, \mathbf{1}\}$

- **Integer programming:** $\boldsymbol{x} \in \mathbb{N}, \mathbb{Z}$
  - Either non-negative: $\mathbb{N}$ is the set containing all non-negative integers: {0, 1, 2, 3,...}
  - Or all integer numbers: $\mathbb{Z}$ is the set containing zero, all positive integers, and all negative integers

- **Linear programing:** $\boldsymbol{x} \in \mathbb{R}$
  - $\mathbb{R}$ is the set containing all rational numbers and irrational numbers (such as $\sqrt{2}$ and π)

Feasible vs. infeasible solution

- **A feasible solution satisfies all the constraints**
  - That is, any point within the feasible region
  - Note that sometimes a feasible solution may not exist at all

- **Feasible region is a convex area**
  - All points on the constraint lines that form the boundary of the region are feasible solutions

Finding optimal solution(s)

- **Optimal versus non-optimal**
  - Exact algorithms give an optimal solution
  - Heuristics are simple procedures guided by common sense that are meant to provide feasible but not necessarily optimal solutions to difficult problems

- **An optimal solution can be found in a corner point, or on a constraint line between two corner points**

- **Any point in the interior of the feasible region cannot be an optimal solution**

Set notations

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- **$A = \{a, b, c\}$ for a set "A" contains the elements "a", "b", and "c"**
  - $a \in A$: denoting that a is an element of A
  - $A \ni a$: denoting that A has a as an element
  - $4 \notin A$: denoting that 4 is not an element of A
  - $\{a, b\} \subseteq A$: denoting that the set {a, b} is a subset of A

- **Using set builder notation**
  - $S = \{1, 2, 3, \dots, n\}$
  - $S = \{x | 1 \leq x \leq n\}$
  - Where the "|" means "such that," or s.t.

Summation

$$1 + 2 + 3 + 4 + 5 = \sum_{i=1}^{5} i$$

$$3^2 + 4^2 + \cdots + 10^2 = \sum_{n=1}^{10} n^2$$

$$\sum_{j \in J} x_{ij} = 1 \quad \forall i \in I$$

Other useful notations

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- **Quantifiers**
  - ∀ (universal quantifier) means "for all"
  - ∃ (existential quantifier) means "there exists"

- **Example:** $\forall \boldsymbol{z} \in \mathbb{Z}\ \exists\ \boldsymbol{z}' \in \mathbb{Z}\ \boldsymbol{s.t.}\ \boldsymbol{z}' > \boldsymbol{z}$
  - For every z that is an integer number, there exists another integer number z' that is larger than z.

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Linear programming
Model formulation

Steps to formulate a model

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- **Read the problem, then read it again!**
- **Step 1: Define decision variables**
  - 1a. Decision needs to be made on?
    - *Express this by using, for example, x_1, x_2 (Clearly explaining each variable)*
  - 1b. Indicate valid range of all variables
    - *Binary, integer, real; (non-)negative?*
- **Step 2: Define objective function**
  - 2a. What do you want to achieve? Choose between minimize and maximize
  - 2b. Express this mathematically using variables
- **Step 3: Formulate all constraints**
  - Develop mathematical relationships to describe contraints (using either $<$, $>$, $=$, $\leq$, or $\geq$)

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“Real-world” problem

- **Suppose that a factory produces two types of products in a production week that contains 60 hours:**
  - It can produce 1 box of product A in 6 hours
  - It can produce 1 box of product B in 5 hours
- **A week’s production is stored in a stockroom on-site, with an effective capacity of 150 m^3**
  - One box of product A takes up 10 m^3 of storage space; that of B takes up 20 m^3
- **The profit contribution of a box of product A is €500**
  - The only customer of product A will accept no more than 8 boxes per week
- **The profit contribution of a box of product B is €450**
  - Has no limit on the amount that can be sold

How many boxes of each product type should be produced each week in order to maximize the total profit?

Step 1: Decision variables

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The "real-world" problem

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How many boxes of each product type should be produced each week in order to maximize the total profit?

The model

- **Step 1a. What are the variables?**

x_A = number of boxes of product A produced per week

x_B = number of boxes of product B produced per week

- Step 1b. Indicate the valid range of all variables

x_A and x_B are non-negative

Step 2: Objective function

The "real-world" problem

- **Suppose that a factory produces two types of products in a production week that contains 60 hours:**
  - It can produce 1 box of product A in 6 hours
  - It can produce 1 box of product B in 5 hours
- **A week's production is stored in a stockroom on-site, with an effective capacity of 150 m^3**
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- **The profit contribution of a box of product B is €450**
  - Has no limit on the amount that can be sold

How many boxes of each product type should be produced each week in order to maximize the total profit?

The model

- **Step 2a. What do you want to achieve?**
  - Produce a number of boxes of products A and B such that total <u>profit</u> is <u>maximized</u>
- **Step 2b. Express mathematically**

$$Maximize\ Z = 500x_A + 450x_B$$

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Step 3: Formulate constraints

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The "real-world" problem

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  - Has no limit on the amount that can be sold

How many boxes of each product type should be produced each week in order to maximize the total profit?

The model

- **Production capacity constraint:**

$$6x_A + 5x_B \leq 60$$

- **Storage capacity constraint:**

$$10x_A + 20x_B \leq 150$$

- **Demand constraint:**

$$x_A \leq 8$$

- **Non-negativity constraints:**

$$x_A \geq 0, x_B \geq 0$$

Complete Linear Programing model

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The "real-world" problem

- **Suppose that a factory produces two types of products in a production week that contains 60 hours:**
  - It can produce 1 box of product A in 6 hours
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  - Has no limit on the amount that can be sold

How many boxes of each product type should be produced each week in order to maximize the total profit?

The model

$$Maximize\ Z = 500x_A + 450x_B$$

s.t.

$$6x_A + 5x_B \leq 60$$

$$10x_A + 20x_B \leq 150$$

$$x_A \leq 8$$

$$x_A \geq 0, x_B \geq 0$$

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Solving linear programming models graphically

Complete LP model

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$$Maximize\ Z = 500x_A + 450x_B$$

s.t.

$6x_A + 5x_B \leq 60$

$10x_A + 20x_B \leq 150$

$x_A \leq 8$

$x_A \geq 0, x_B \geq 0$

Representing constraints graphically

$$Maximize\ Z = 500x_A + 450x_B$$

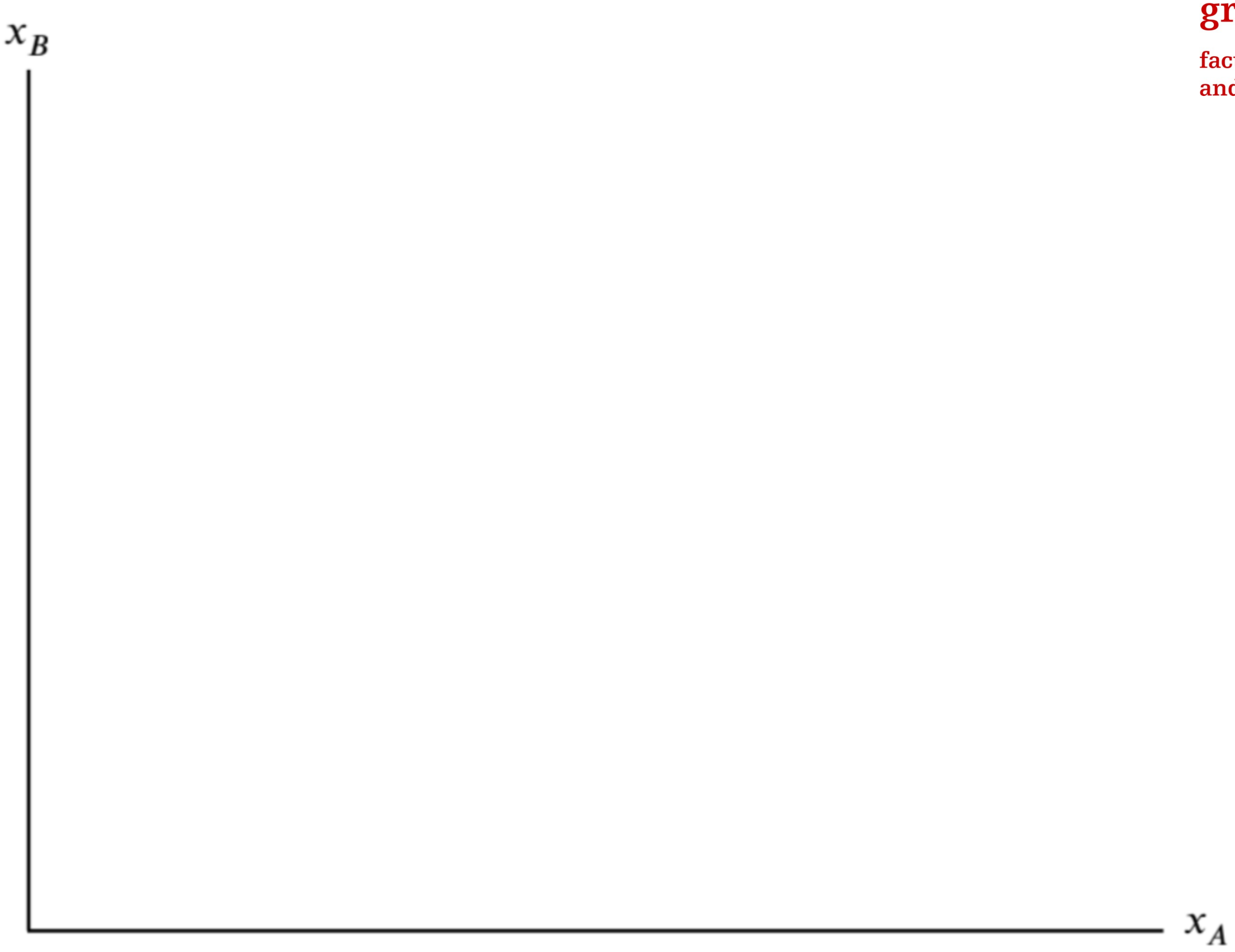
s.t.

$$6x_A + 5x_B \leq 60$$

$$10x_A + 20x_B \leq 150$$

$$x_A \leq 8$$

$$x_A \geq 0, x_B \geq 0$$



Representing constraints graphically

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$$Maximize\ Z = 500x_A + 450x_B$$

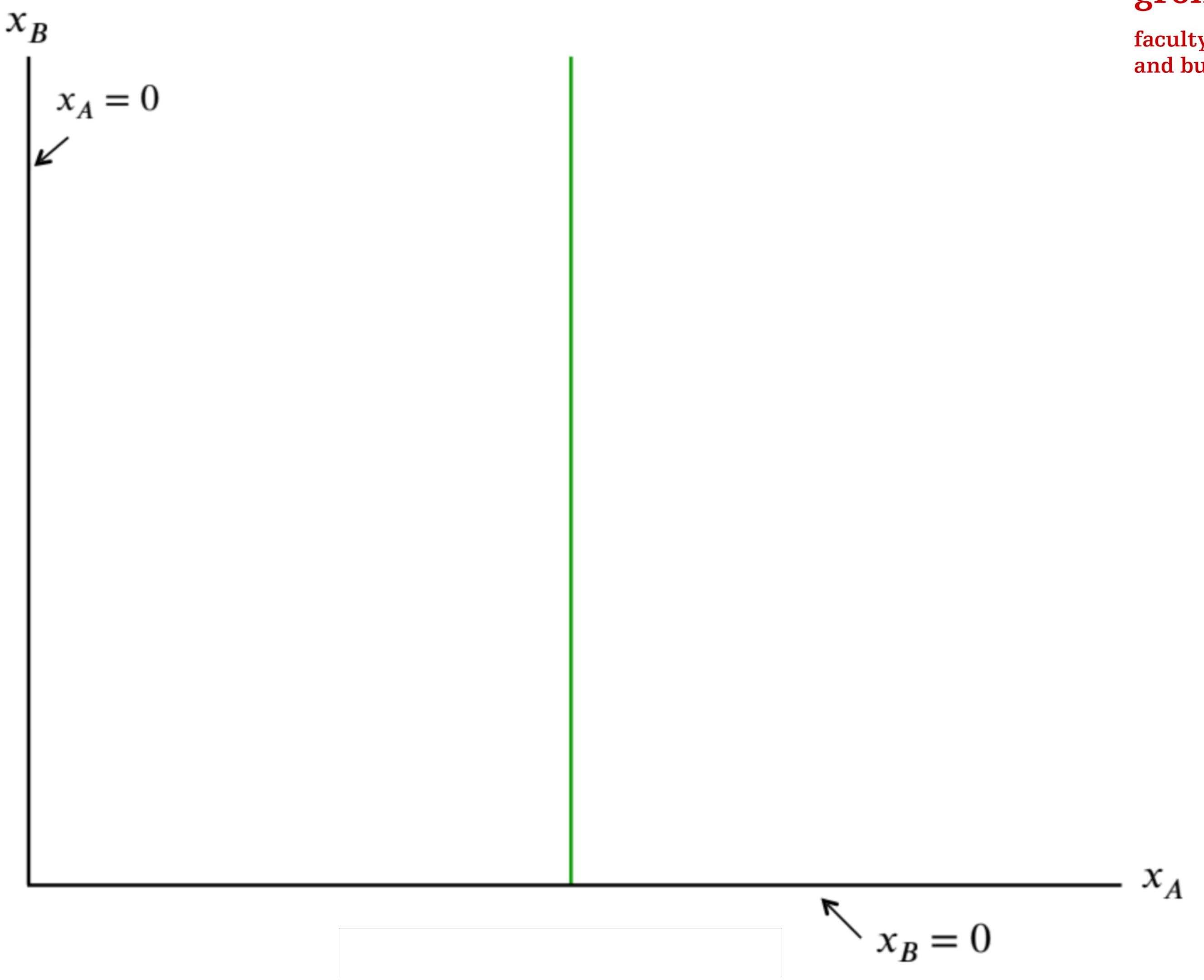
s.t.

$6x_A + 5x_B \leq 60$

$10x_A + 20x_B \leq 150$

$x_A \leq 8$

$\boldsymbol{x_A \geq 0, x_B \geq 0}$



Representing constraints graphically

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$Maximize\ Z = 500x_A + 450x_B$

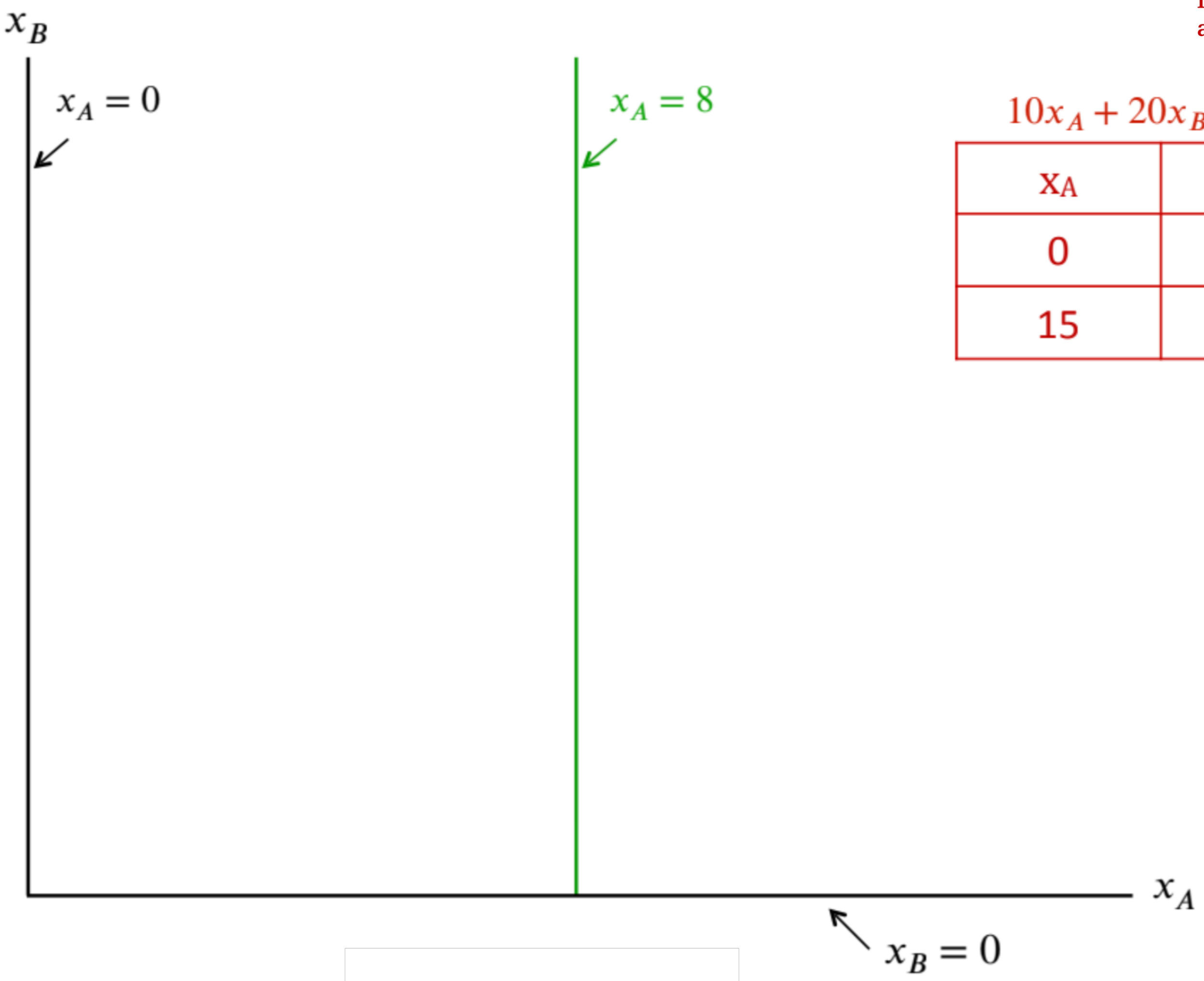
s.t.

$6x_A + 5x_B \leq 60$

$\mathbf{10x_A + 20x_B \leq 150}$

$\mathbf{x_A \leq 8}$

$\mathbf{x_A \geq 0, x_B \geq 0}$



$10x_A + 20x_B = 150$

x_A	x_B
0	7.5
15	0

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Representing constraints graphically

$$Maximize\ Z = 500x_A + 450x_B$$

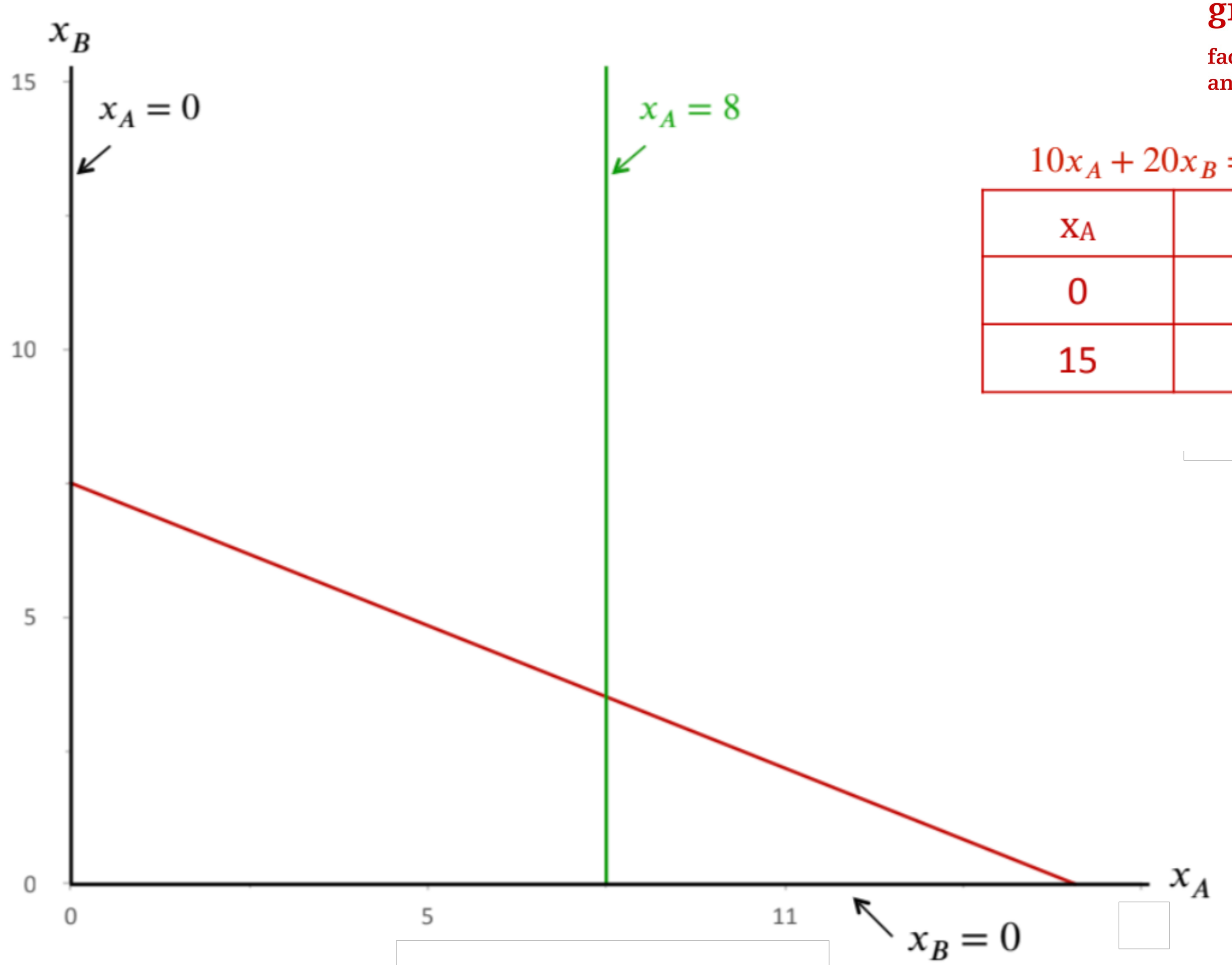
s.t.

$6x_A + 5x_B \leq 60$

$\mathbf{10x_A + 20x_B \leq 150}$

$\mathbf{x_A \leq 8}$

$\mathbf{x_A \geq 0, x_B \geq 0}$



$10x_A + 20x_B = 150$

x_A	x_B
0	7.5
15	0

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Representing constraints graphically

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$$Maximize\ Z = 500x_A + 450x_B$$

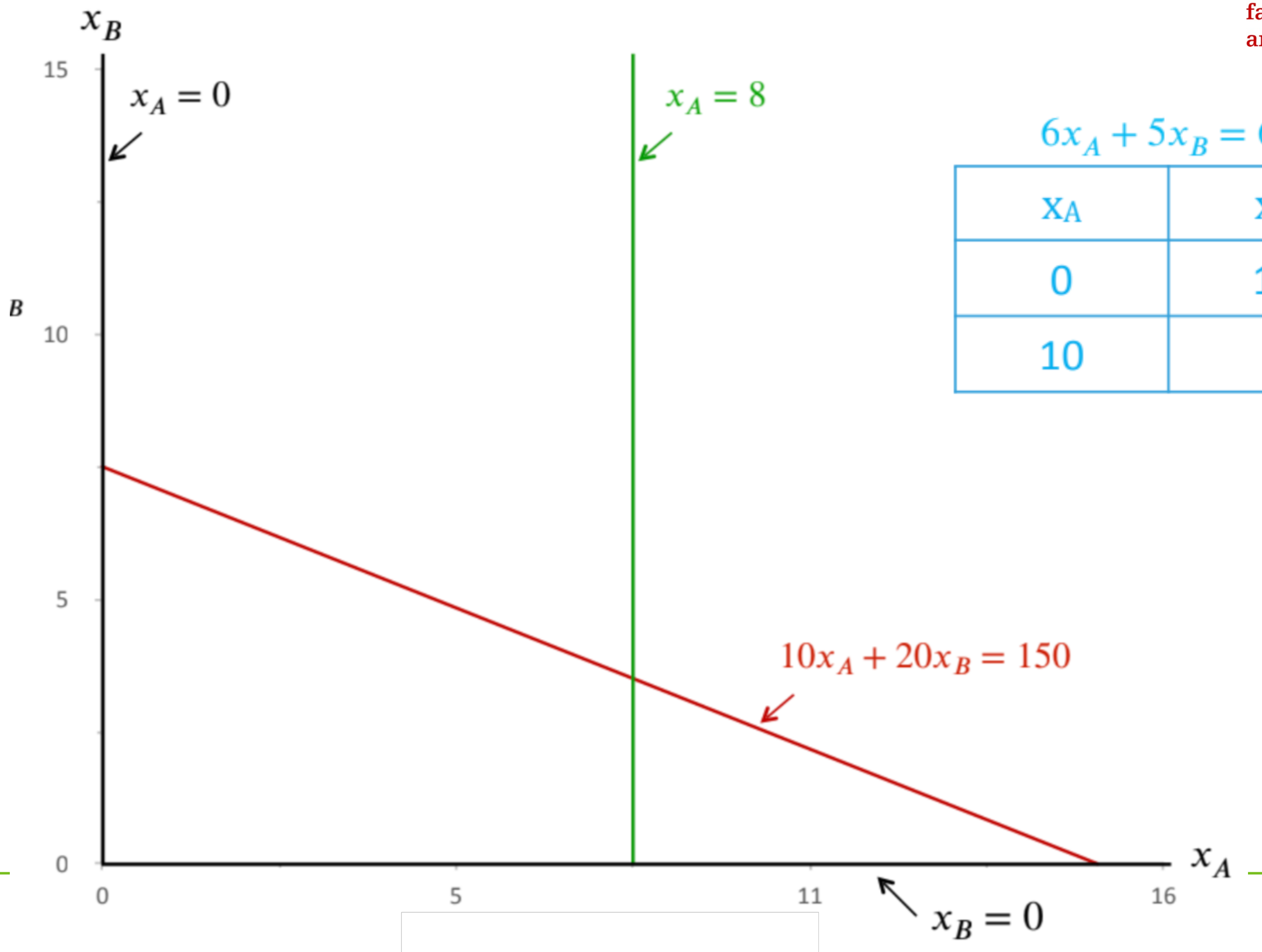
s.t.

$6x_A + 5x_B \leq 60$

$10x_A + 20x_B \leq 150$

$x_A \leq 8$

$x_A \geq 0, x_B \geq 0$



$6x_A + 5x_B = 60$

x_A	x_B
0	12
10	0

Representing constraints graphically

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$$Maximize\ Z = 500x_A + 450x_B$$

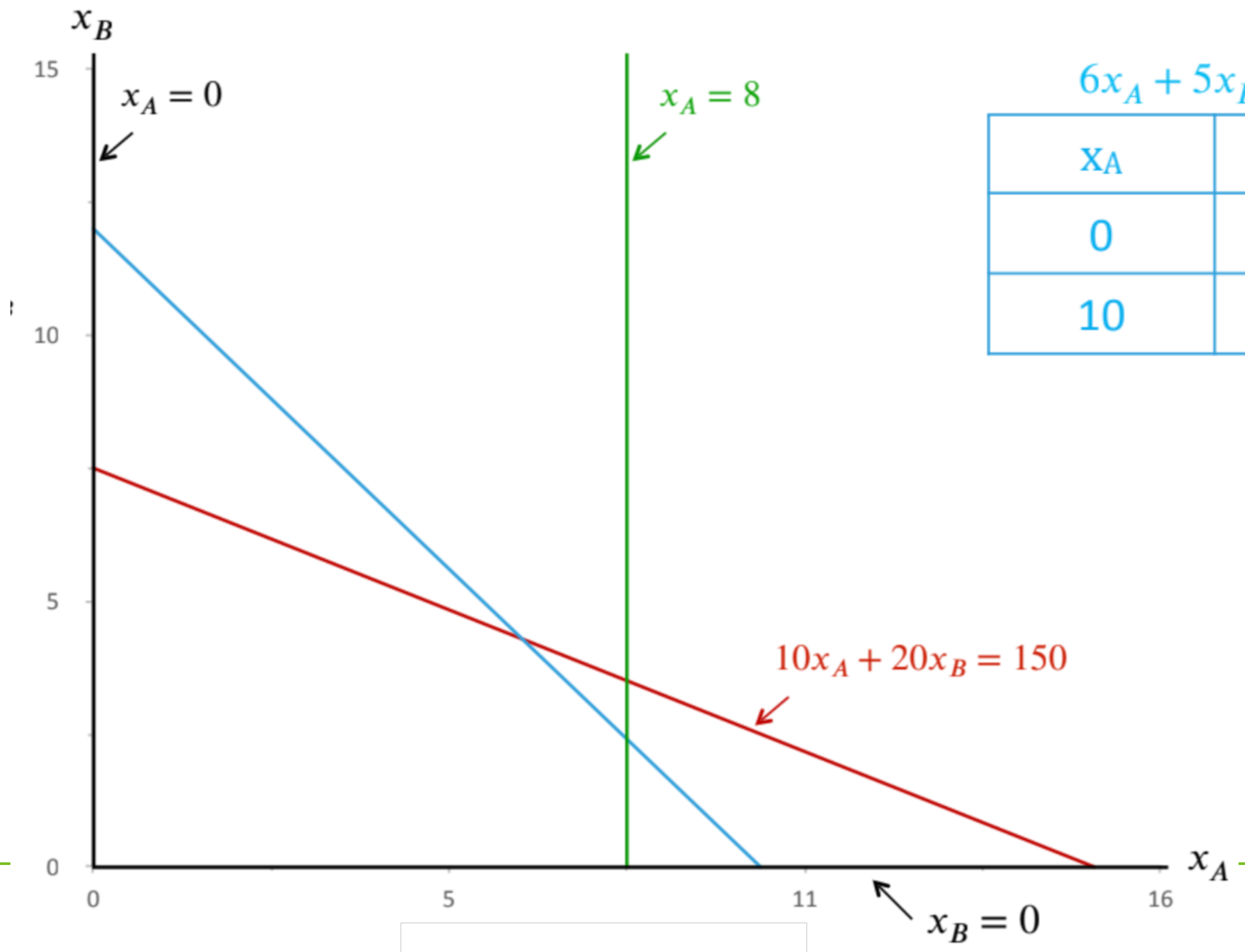
s.t.

$6x_A + 5x_B \leq 60$

$10x_A + 20x_B \leq 150$

$x_A \leq 8$

$x_A \geq 0, x_B \geq 0$



$6x_A + 5x_B = 60$

x_A	x_B
0	12
10	0

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Representing constraints graphically

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$$Maximize\ Z = 500x_A + 450x_B$$

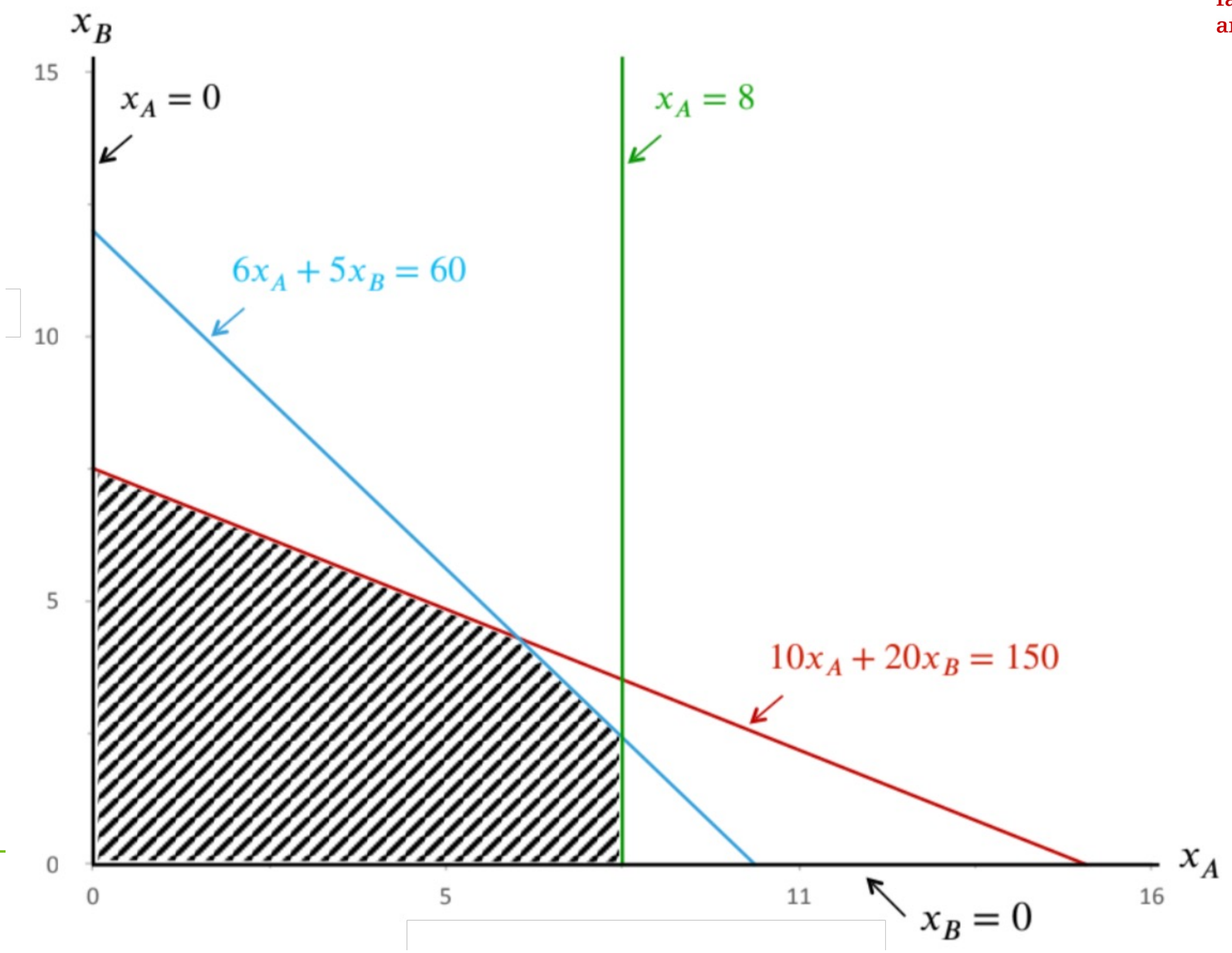
s.t.

$6x_A + 5x_B \leq 60$

$10x_A + 20x_B \leq 150$

$x_A \leq 8$

$x_A \geq 0, x_B \geq 0$



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Representing constraints graphically

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$$Maximize\ Z = 500x_A + 450x_B$$

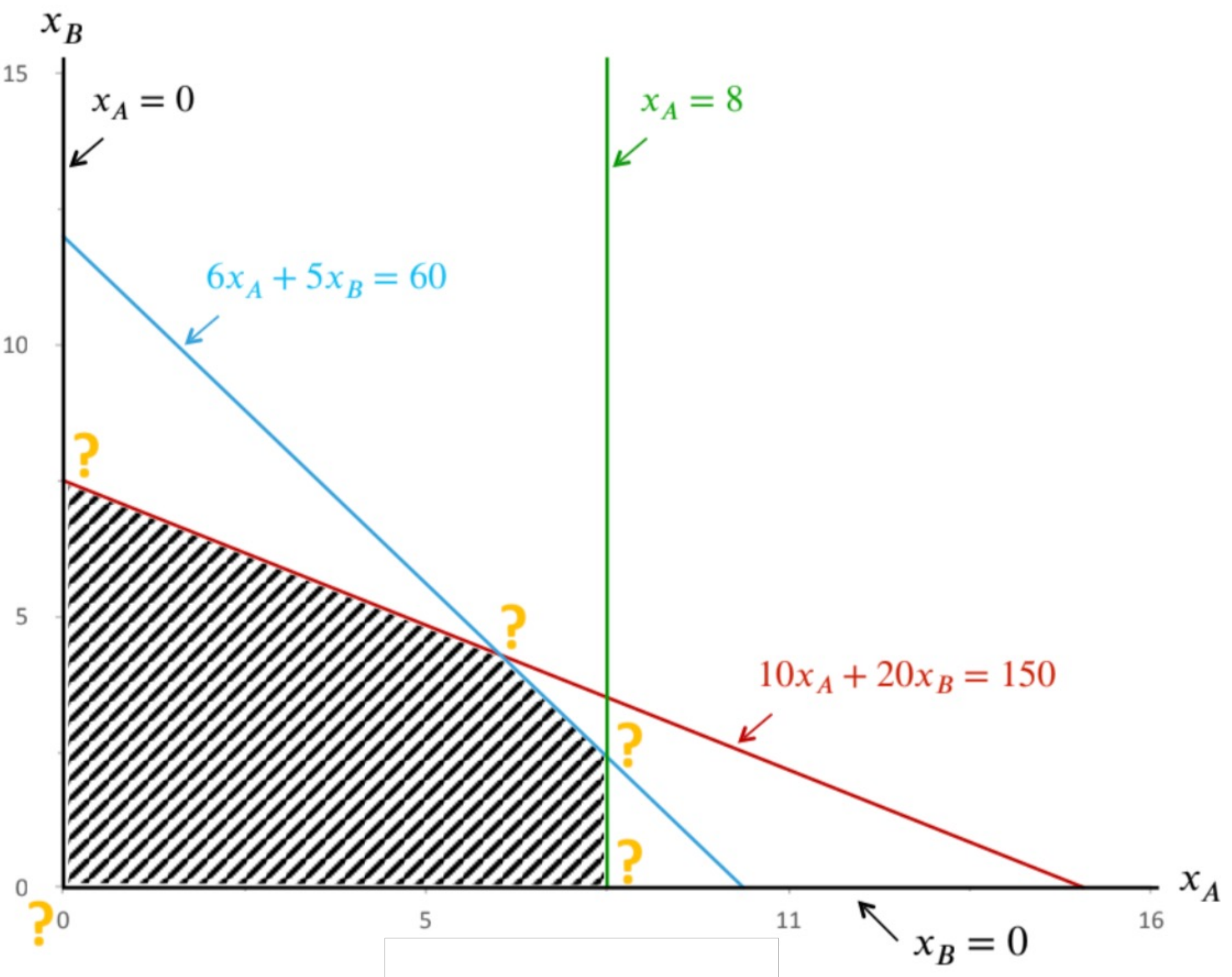
s.t.

$6x_A + 5x_B \leq 60$

$10x_A + 20x_B \leq 150$

$x_A \leq 8$

$x_A \geq 0, x_B \geq 0$



Representing constraints graphically

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$Maximize\ Z$
$= 500x_A + 450$

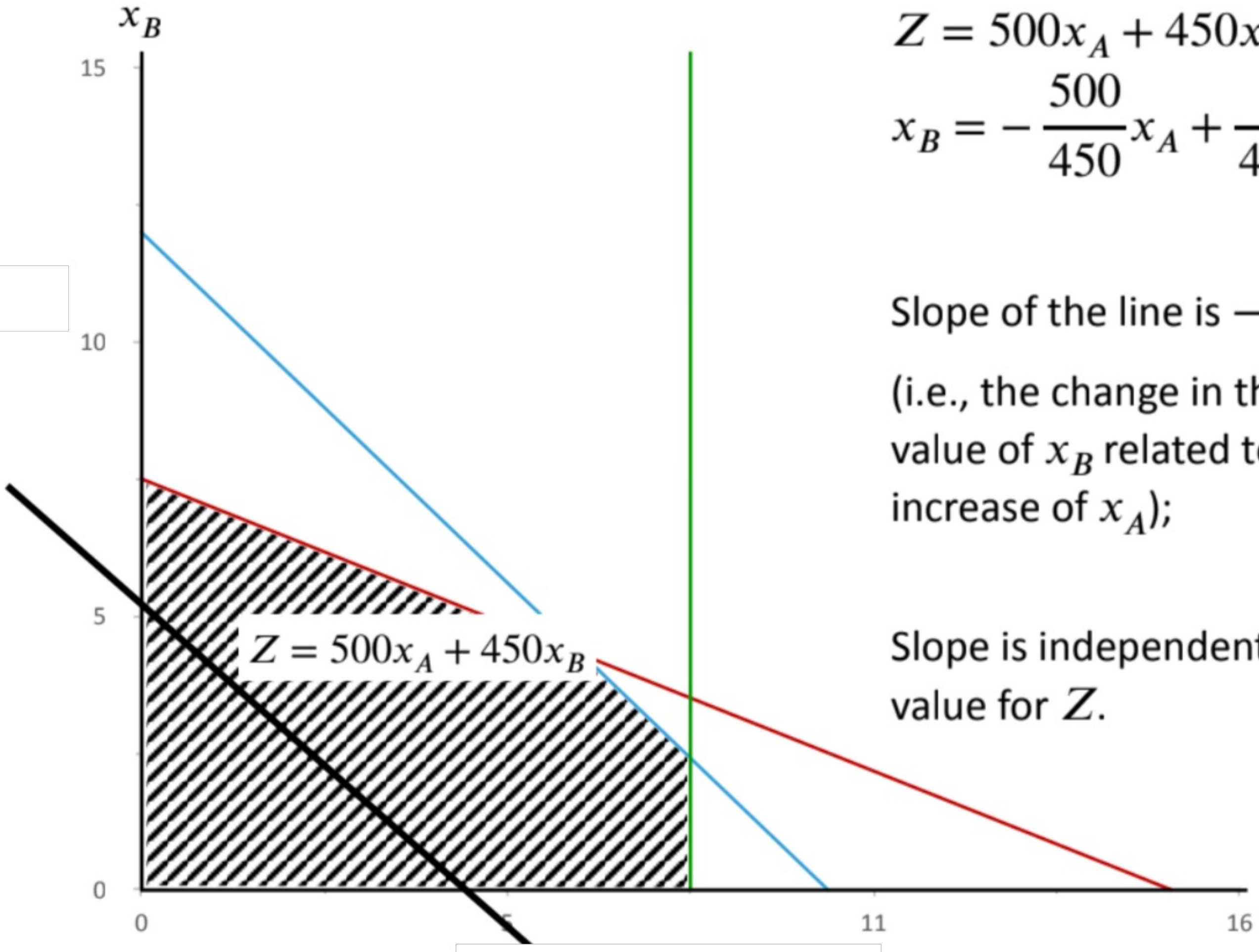
s.t.

$6x_A + 5x_B \leq 60$

$10x_A + 20x_B \leq 15$

$x_A \leq 8$

$x_A \geq 0, x_B \geq 0$



$$Z = 500x_A + 450x_B$$

$$x_B = -\frac{500}{450}x_A + \frac{1}{450}Z$$

Slope of the line is $-\frac{500}{450}$ (i.e., the change in the value of x_B related to unit increase of x_A);

Slope is independent of value for Z.

Representing constraints graphically

$$Maximize\ Z = 500x_A + 450x_B$$

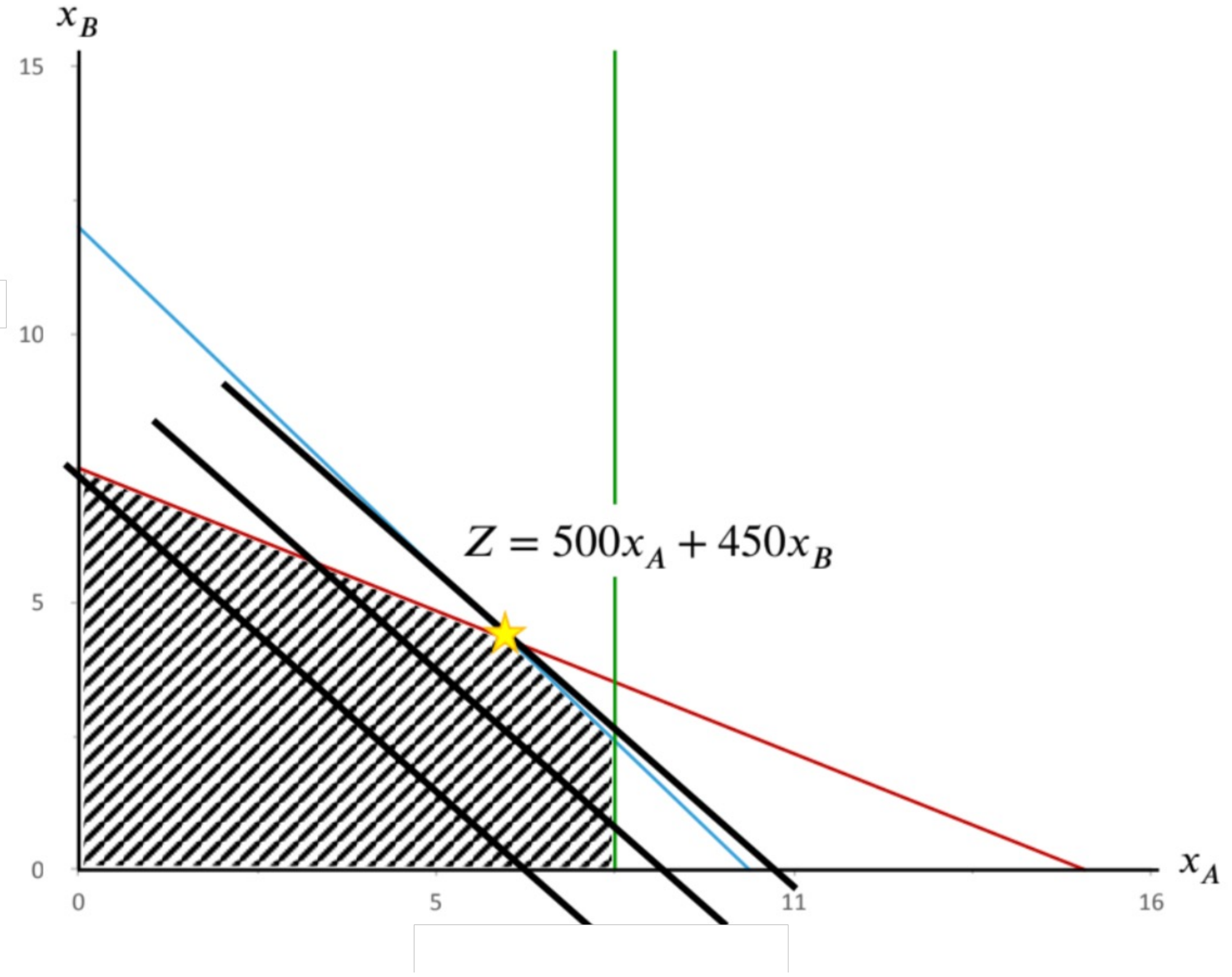
s.t.

$$6x_A + 5x_B \leq 60$$

$$10x_A + 20x_B \leq 150$$

$$x_A \leq 8$$

$$x_A \geq 0, x_B \geq 0$$



Representing constraints graphically

$$Maximize\ Z = 500x_A + 450x_B$$

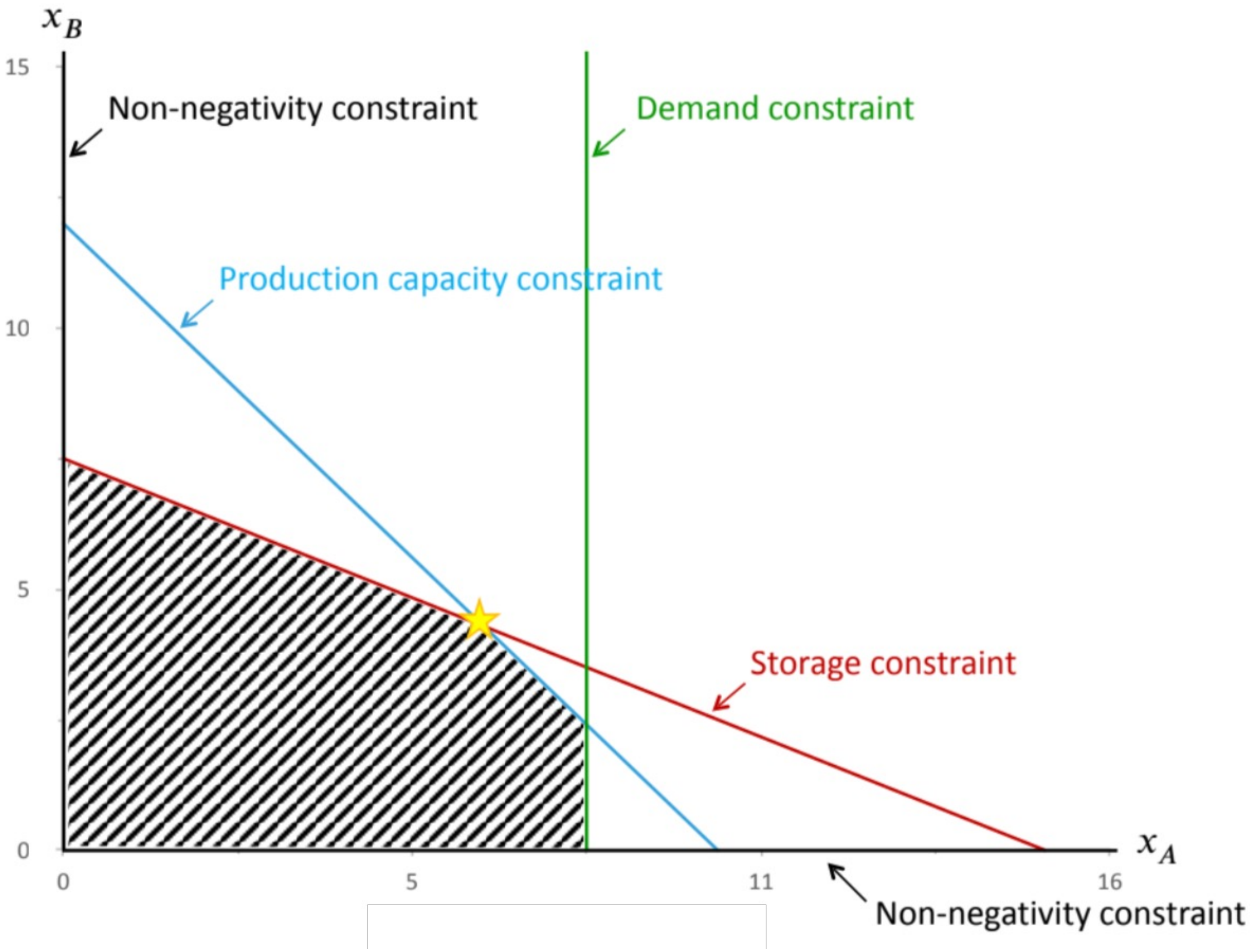
s.t.

$$6x_A + 5x_B \leq 60$$

$$10x_A + 20x_B \leq 150$$

$$x_A \leq 8$$

$$x_A \geq 0, x_B \geq 0$$



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Compute optimal solution

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The optimal solution lies in the intersection of

$$6x_A + 5x_B = 60 \text{ and } 10x_A + 20x_B = 150$$

$$6x_A + 5x_B = 60$$

$$24x_A + 20x_B = 240$$

$$10x_A + 20x_B = 150 \quad -$$

$$14x_A = 90$$

$$x_A = \frac{90}{14} = 6.43$$

Compute optimal solution

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The optimal solution lies in the intersection of

$6x_A + 5x_B = 60$ and $10x_A + 20x_B = 150$

$$6x_A + 5x_B = 60$$

$$24x_A + 20x_B = 240$$

$$10x_A + 20x_B = 150 \quad -$$

$$14x_A = 90$$

$$x_A = \frac{90}{14} = 6.43$$

$$6x_A + 5x_B = 60$$

$$5x_B = 60 - 6x_A$$

$$x_B = 12 - \left(\frac{6}{5}\right)x_A$$

$$x_B = 12 - \left(\frac{6}{5}\right)6.43 = 4.29$$

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Representing constraints graphically

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$Maximize\ Z = 500x_A + 450x_B$

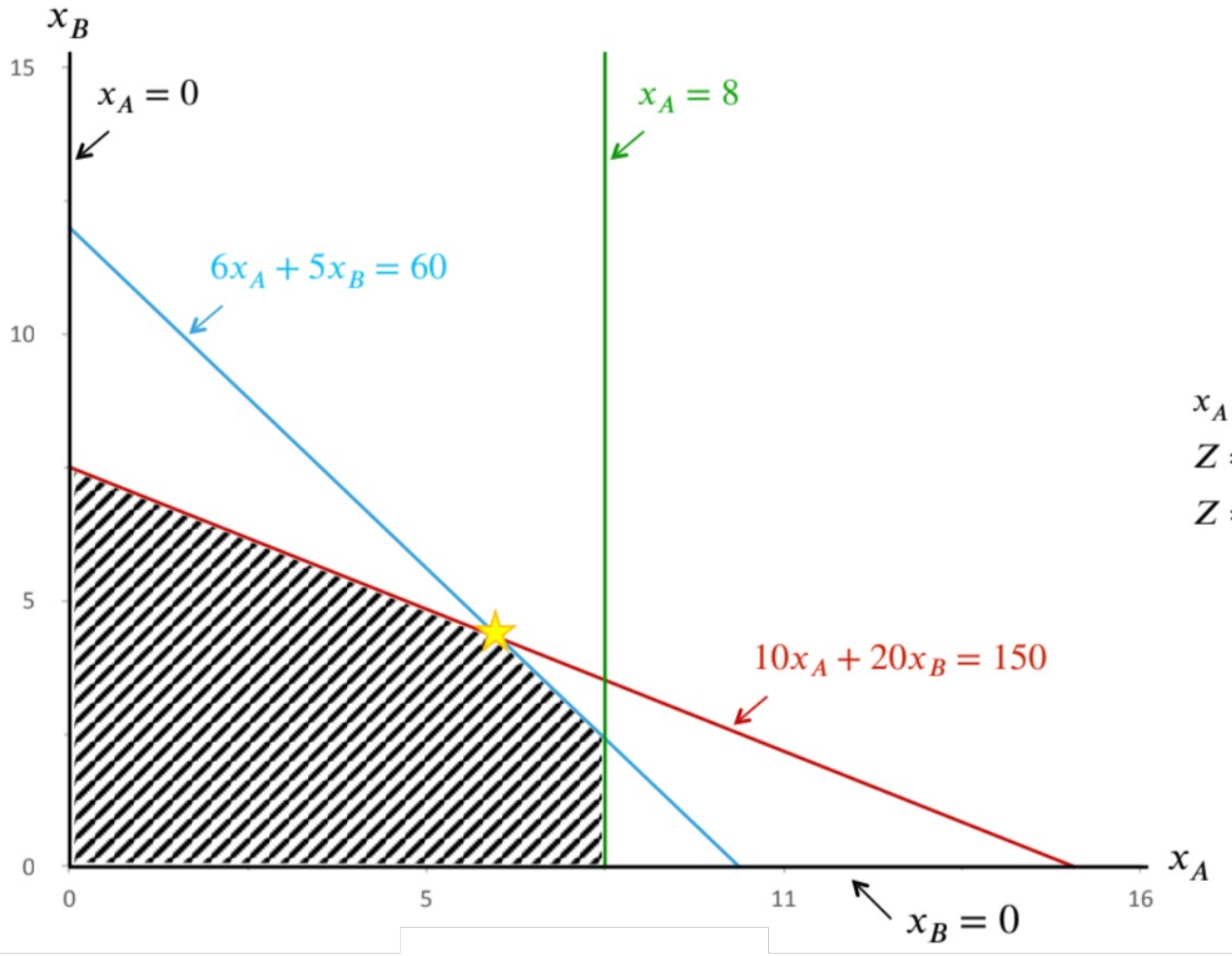
s.t.

$x_A \leq 8$

$10x_A + 20x_B \leq 150$

$6x_A + 5x_B \leq 60$

$x_A \geq 0,\ \ x_B \geq 0$



$x_A = 6.43;\ x_B = 4.29$

$Z = 500x_A + 450x_B$

$Z = 5142.86$

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Thank you!

Questions?

Dr. Tri M. Tran
tri.tran@aalto.fi