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Session :
Mathematical modeling
35E00750 Logistics Systems and Analytics

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Data-driven vs. model-based analytics

	Data-driven analytics (DDA)	Model-based analytics (MBA)
Associated disciplines	Machine learning, statistics, information visualization	Operations research, management science, statistics
Primary objectives of models	Representation of relations in observed data and prediction of new data points, prediction of the effects of decisions, generation of decision recommendation	Representation of the structure and/or dynamics of a system, prediction of the effects of decisions, generation of decision recommendations
Role of data	Selection of model structure and parameters	Calibration of model parameters
Main mechanism in model selection	Quantitative assessment of model performance	Expert judgements regarding the feasibility of assumptions
Main types of data	Big data of various types (quantitative data, structured/unstructured text, images, etc.	Qualitative expert judgement, quantitative data, output of predictive models
Planning horizon	Short	Short to long

Slide from a public presentation from Dr. Eeva Vilkkumaa, Assistant Professor of Management Science at Aalto University School of Business

Learning objectives

- **Understand the fundamentals of mathematical modeling**
- **Understand the fundamentals of linear programing**
- **Understand the applications of mathematical modeling in Logistics and Supply Chain Management**

Slide courtesy of Dr. Ir. Paul Buijs, Assistant Professor, Department of Operations, Faculty of Economics and Business, University of Groningen

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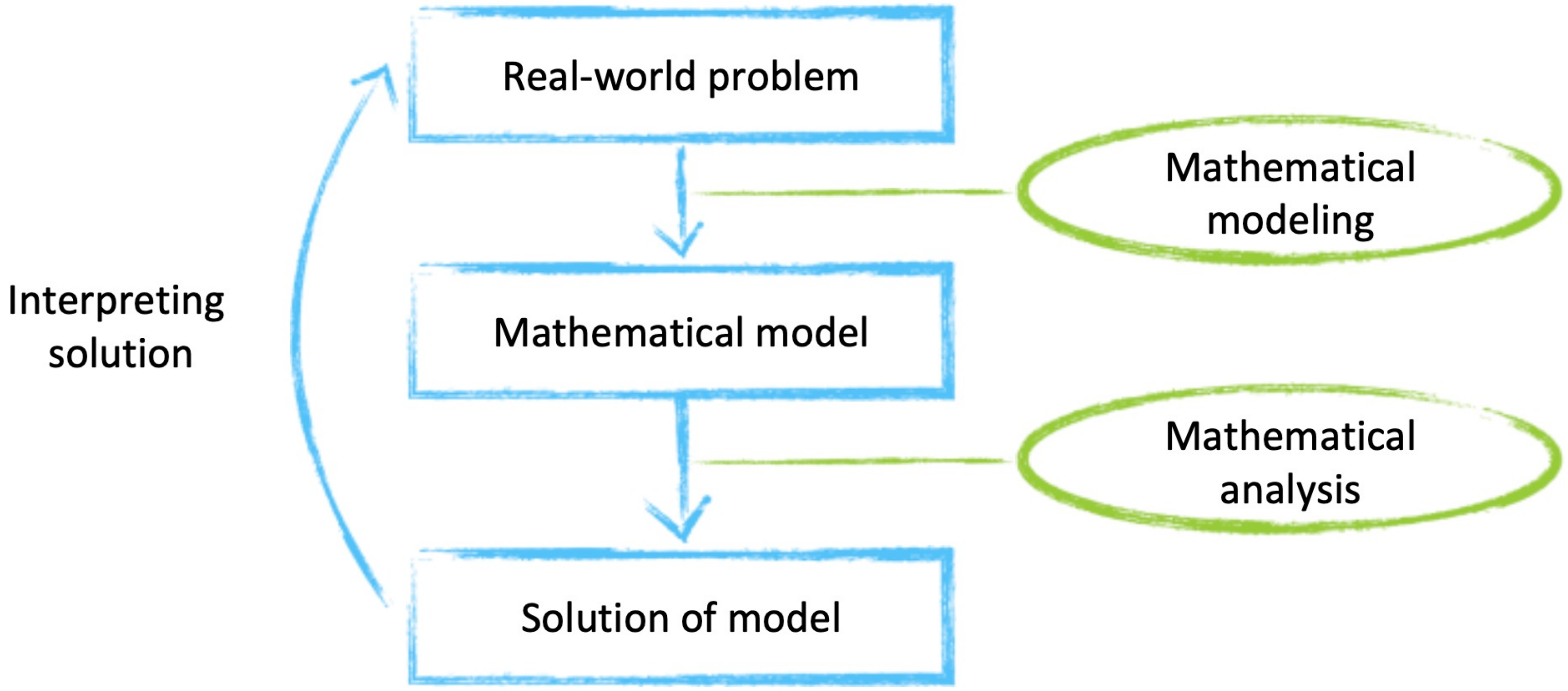
Mathematical modeling

Mathematical modeling steps



Terminologies

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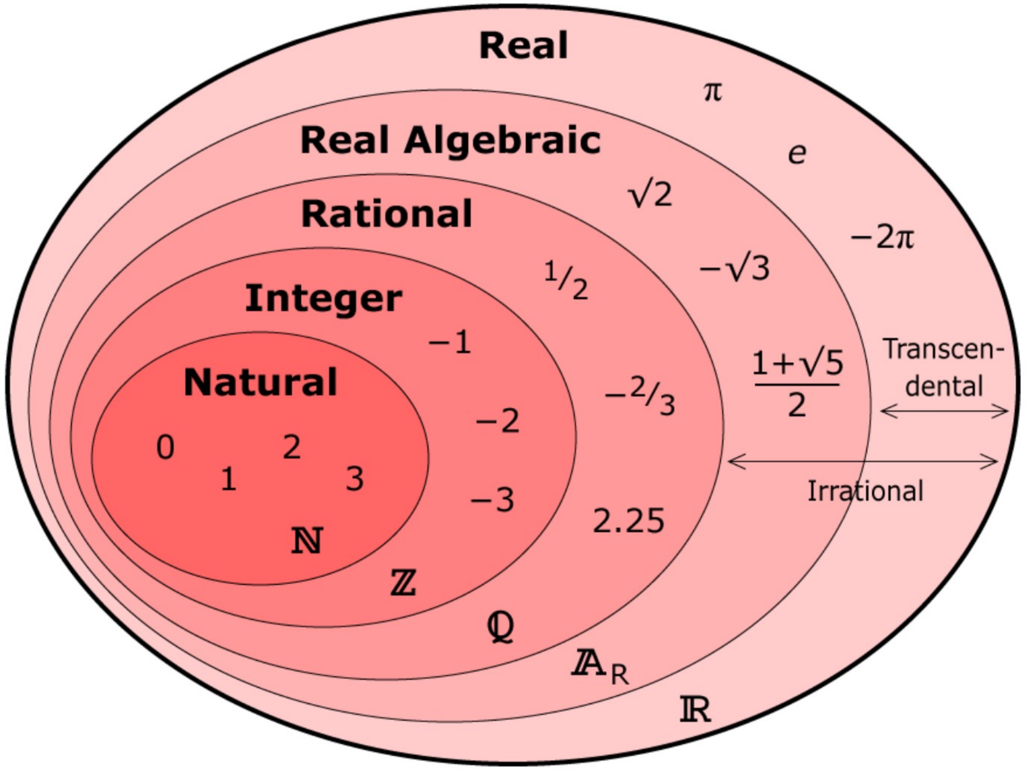
- **Decision variables**
  - Mathematical description of the set of decisions to be made
- **Parameters**
  - What input data are known and needed for making the decisions?
- **Objectives**
  - A measure to rank alternative solutions
  - What do you want to achieve? Express this mathematically by using your decision variables and parameters
- **Constraints**
  - Limitations on the values of the decision variables
  - Develop mathematical relationships to describe constraints

Types of mathematical models

- **Linear Programming**
  - Variables can take real numbers
- **Integer Programming**
  - Variables can only take integer values
- **Binary Programming**
  - Variables can only take the value of 0 or 1
- **Mixed Integer programming**
  - Some variables are constrained to be integer values

Valid range of a variable



Valid range of a variable

- **Binary programing:** $\boldsymbol{x} \in \{\mathbf{0}, \mathbf{1}\}$

- **Integer programming:** $\boldsymbol{x} \in \mathbb{N}, \mathbb{Z}$
  - Either non-negative: $\mathbb{N}$ is the set containing all non-negative integers: {0, 1, 2, 3,…}
  - Or all integer numbers: $\mathbb{Z}$ is the set containing zero, all positive integers, and all negative integers

- **Linear programing:** $\boldsymbol{x} \in \mathbb{R}$
  - $\mathbb{R}$ is the set containing all rational numbers and irrational numbers (such as $\sqrt{2}$ and π)

Feasible vs. infeasible solution

- **A feasible solution satisfies all the constraints**
  - That is, any point within the feasible region
  - Note that sometimes a feasible solution may not exist at all

- **Feasible region is a convex area**
  - All points on the constraint lines that form the boundary of the region are feasible solutions

Finding optimal solution(s)

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- **Optimal versus non-optimal**
  - Exact algorithms give an optimal solution
  - Heuristics are simple procedures guided by common sense that are meant to provide feasible but not necessarily optimal solutions to difficult problems

- **An optimal solution can be found in a corner point, or on a constraint line between two corner points**

- **Any point in the interior of the feasible region cannot be an optimal solution**

Set notations

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- $\boldsymbol{A = \{a, b, c\}}$ **for a set "A" contains the elements "a", "b", and "c"**
  - $a \in A$: denoting that a is an element of A
  - $A \ni a$: denoting that A has a as an element
  - $4 \notin A$: denoting that 4 is not an element of A
  - $\{a, b\} \subseteq A$: denoting that the set {a, b} is a subset of A

- **Using set builder notation**
  - $S = \{1, 2, 3, \ldots, n\}$
  - $S = \{x | 1 \leq x \leq n\}$
  - Where the "|" means "such that," or s.t.

Summation

$$1 + 2 + 3 + 4 + 5 = \sum_{i=1}^{5} i$$

$$3^2 + 4^2 + \cdots + 10^2 = \sum_{n=1}^{10} n^2$$

$$\sum_{j \in J} x_{ij} = 1 \quad \forall i \in I$$

Other useful notations

- **Quantifiers**
  - ∀ (universal quantifier) means "for all"
  - ∃ (existential quantifier) means "there exists"

- **Example:** $\forall \boldsymbol{z} \in \mathbb{Z} \ \exists \ \boldsymbol{z}' \in \mathbb{Z} \ \boldsymbol{s.t.} \ \boldsymbol{z}' > \boldsymbol{z}$
  - For every z that is an integer number, there exists another integer number z' that is larger than z.

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Linear programming
Model formulation

Steps to formulate a model

- **Read the problem, then read it again!**
- **Step 1: Define decision variables**
  - 1a. Decision needs to be made on?
    - *Express this by using, for example,* x_1, x_2 *(Clearly explaining each variable)*
  - 1b. Indicate valid range of all variables
    - *Binary, integer, real; (non-)negative?*
- **Step 2: Define objective function**
  - 2a. What do you want to achieve? Choose between minimize and maximize
  - 2b. Express this mathematically using variables
- **Step 3: Formulate all constraints**
  - Develop mathematical relationships to describe constraints (using either $<$, $>$, $=$, $\leq$, or $\geq$)

“Real-world” problem

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- **Suppose that a factory produces two types of products in a production week that contains 60 hours:**
    - It can produce 1 box of product A in 6 hours
    - It can produce 1 box of product B in 5 hours
- **A week’s production is stored in a stockroom on-site, with an effective capacity of 150 m^3**
    - One box of product A takes up 10 m^3 of storage space; that of B takes up 20 m^3
- **The profit contribution of a box of product A is €500**
    - The only customer of product A will accept no more than 8 boxes per week
- **The profit contribution of a box of product B is €450**
    - Has no limit on the amount that can be sold

How many boxes of each product type should be produced each week in order to maximize the total profit?

Step 1: Decision variables

The “real-world” problem

- **Suppose that a factory produces two types of products in a production week that contains 60 hours:**
  - It can produce 1 box of product A in 6 hours
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  - Has no limit on the amount that can be sold

How many boxes of each product type should be produced each week in order to maximize the total profit?

The model

- **Step 1a. What are the variables?**

x_A= number of boxes of product A produced per week

x_B= number of boxes of product B produced per week

- Step 1b. Indicate the valid range of all variables

x_A and x_B are non-negative

Step 2: Objective function

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The "real-world" problem

- **Suppose that a factory produces two types of products in a production week that contains 60 hours:**
  - It can produce 1 box of product A in 6 hours
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- **The profit contribution of a box of product B is €450**
  - Has no limit on the amount that can be sold

How many boxes of each product type should be produced each week in order to maximize the total profit?

The model

- **Step 2a. What do you want to achieve?**
  - Produce a number of boxes of products A and B such that total profit is maximized
- **Step 2b. Express mathematically**

$$Maximize\ Z = 500x_A + 450x_B$$

Step 3: Formulate constraints

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The "real-world" problem

- Suppose that a factory produces two types of products in a production week that contains 60 hours:
  - It can produce 1 box of product A in 6 hours
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  - Has no limit on the amount that can be sold

How many boxes of each product type should be produced each week in order to maximize the total profit?

The model

- **Production capacity constraint:**

$$6x_A + 5x_B \leq 60$$

- **Storage capacity constraint:**

$$10x_A + 20x_B \leq 150$$

- **Demand constraint:**

$$x_A \leq 8$$

- **Non-negativity constraints:**

$$x_A \geq 0, x_B \geq 0$$

Complete Linear Programing model

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The "real-world" problem

- **Suppose that a factory produces two types of products in a production week that contains 60 hours:**
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- **The profit contribution of a box of product B is €450**
  - Has no limit on the amount that can be sold

How many boxes of each product type should be produced each week in order to maximize the total profit?

The model

$$Maximize\ Z = 500x_A + 450x_B$$

s.t.

$$6x_A + 5x_B \leq 60$$

$$10x_A + 20x_B \leq 150$$

$$x_A \leq 8$$

$$x_A \geq 0, x_B \geq 0$$

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Solving linear programming models graphically

Complete LP model

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$$Maximize\ Z = 500x_A + 450x_B$$

s.t.

$6x_A + 5x_B \leq 60$

$10x_A + 20x_B \leq 150$

$x_A \leq 8$

$x_A \geq 0, x_B \geq 0$

Representing constraints graphically

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$$Maximize\ Z = 500x_A + 450x_B$$

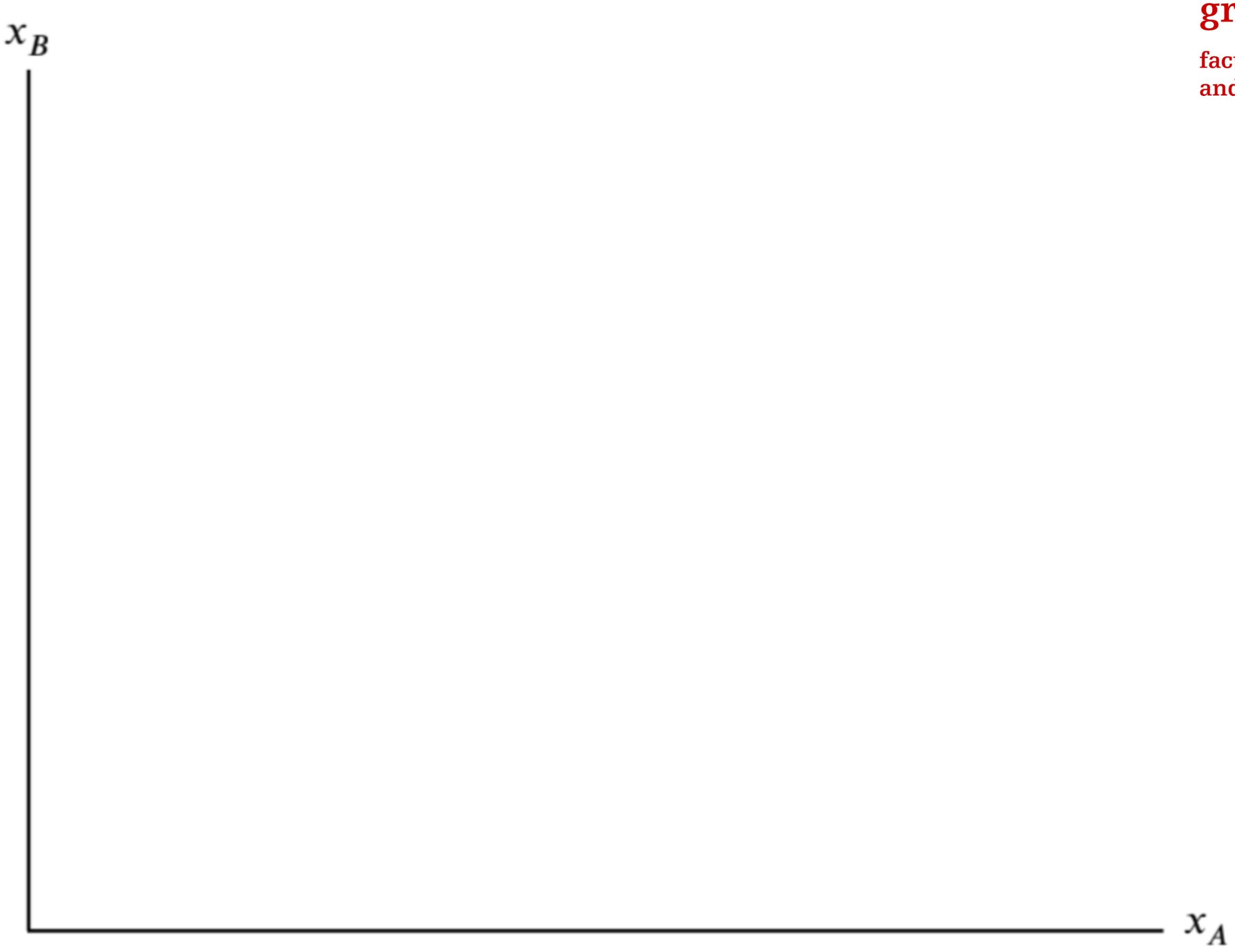
s.t.

$$6x_A + 5x_B \leq 60$$

$$10x_A + 20x_B \leq 150$$

$$x_A \leq 8$$

$$x_A \geq 0, x_B \geq 0$$



Representing constraints graphically

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$$Maximize\ Z = 500x_A + 450x_B$$

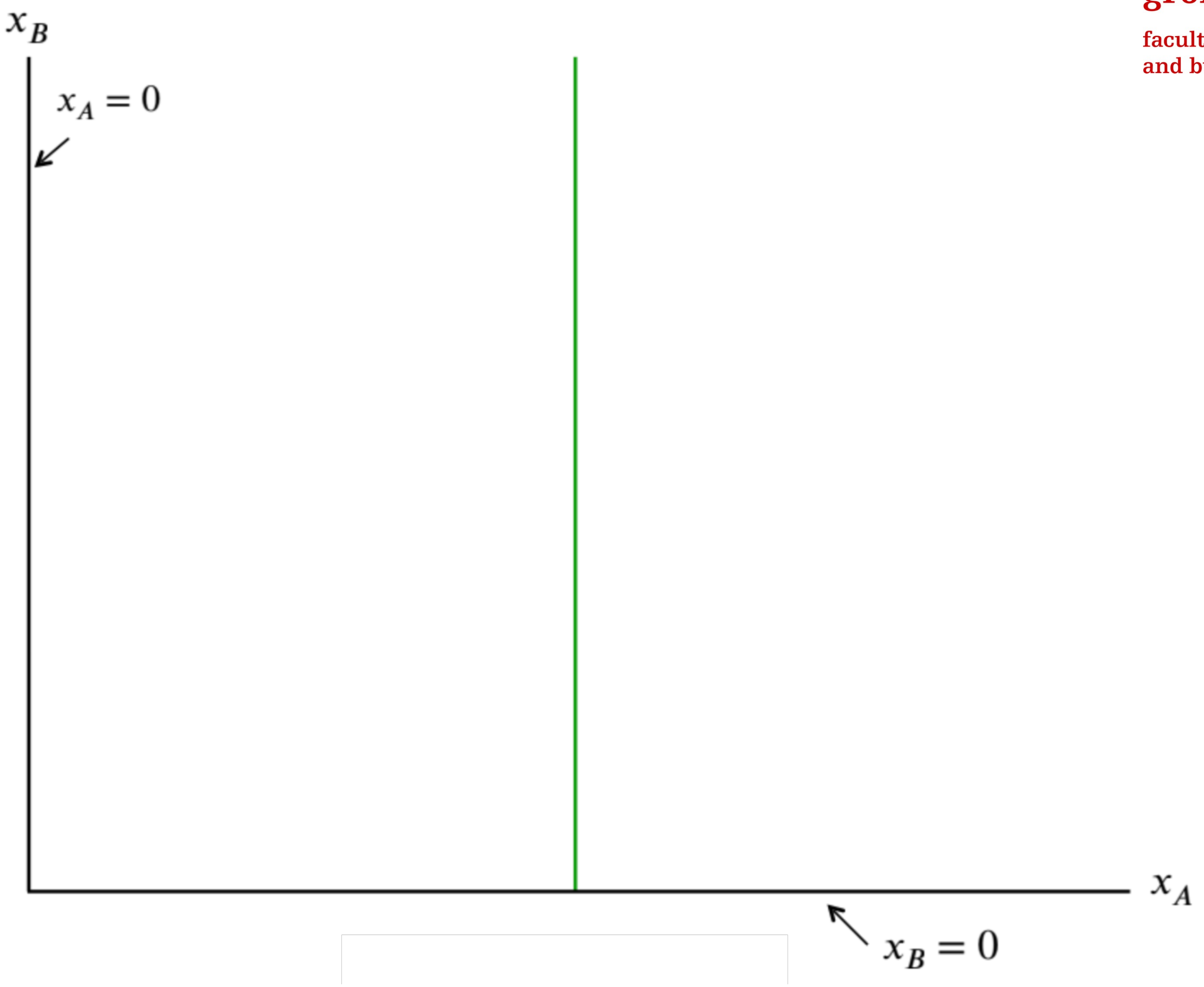
s.t.

$$6x_A + 5x_B \leq 60$$

$$10x_A + 20x_B \leq 150$$

$$x_A \leq 8$$

$$\boldsymbol{x_A \geq 0, x_B \geq 0}$$



Representing constraints graphically

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$$Maximize\ Z = 500x_A + 450x_B$$

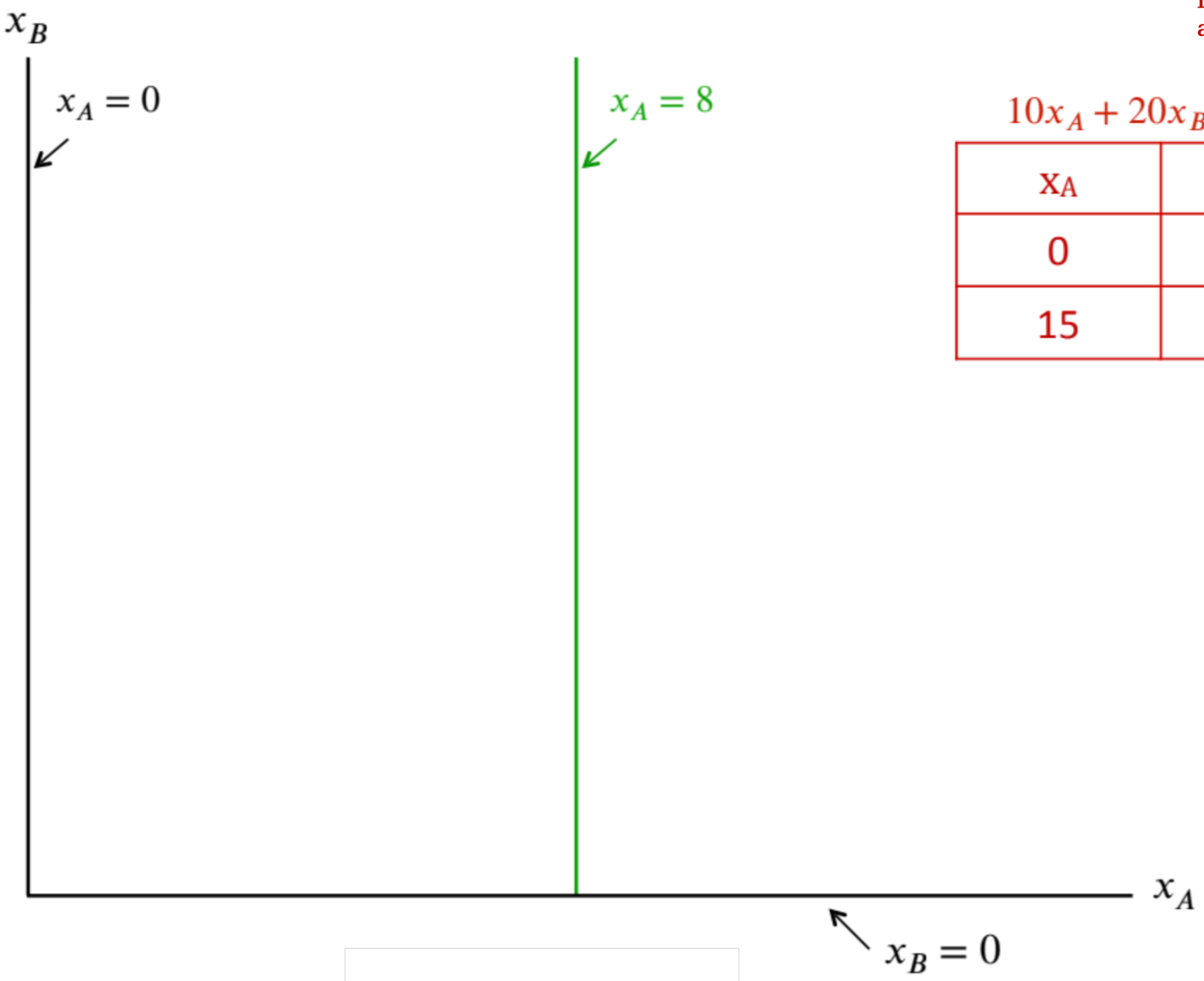
s.t.

$6x_A + 5x_B \leq 60$

$\mathbf{10x_A + 20x_B \leq 150}$

$\mathbf{x_A \leq 8}$

$\mathbf{x_A \geq 0, x_B \geq 0}$



$10x_A + 20x_B = 150$

xA	xB
0	7.5
15	0

Representing constraints graphically

$$Maximize\ Z = 500x_A + 450x_B$$

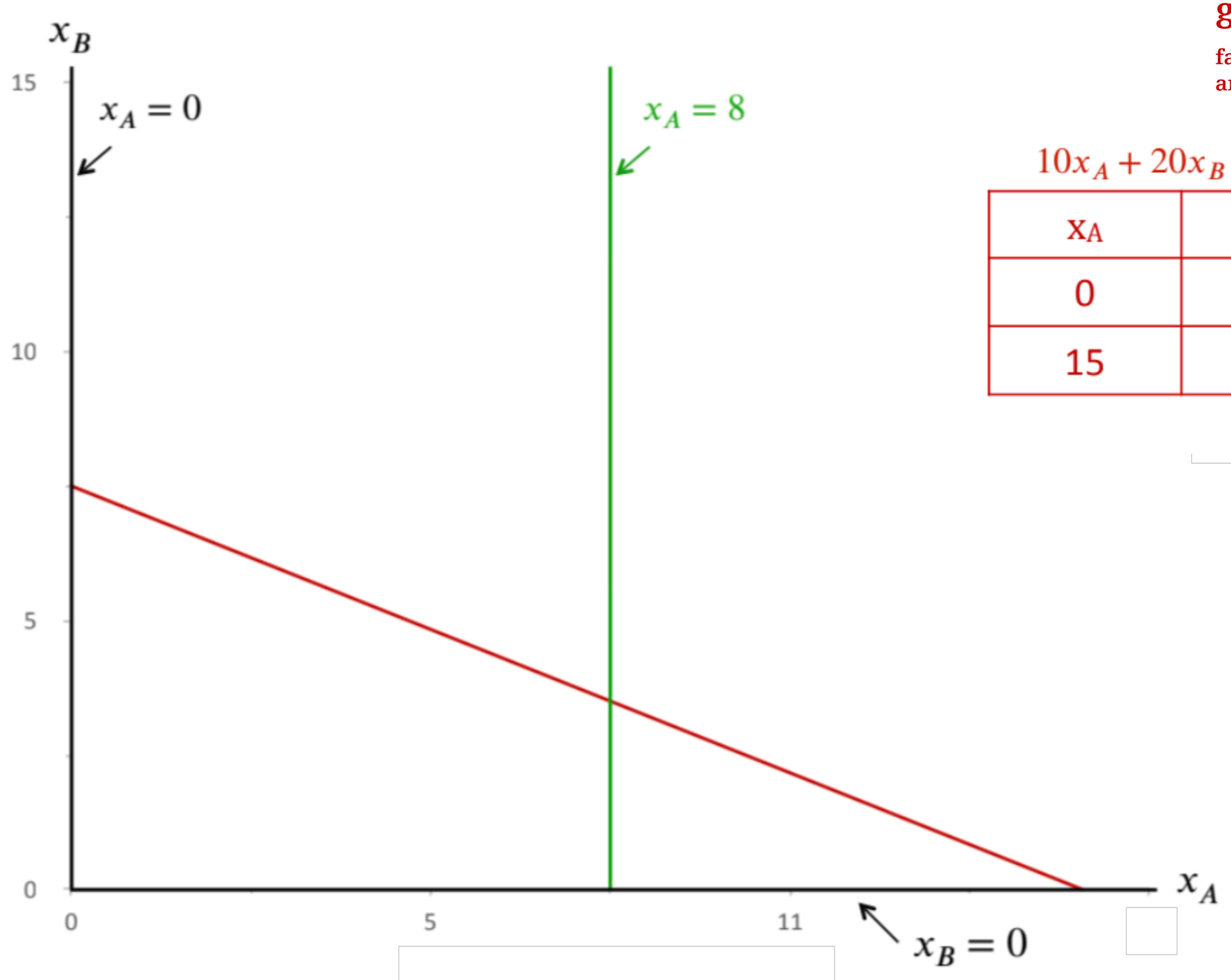
s.t.

$6x_A + 5x_B \leq 60$

$\mathbf{10x_A + 20x_B \leq 150}$

$\mathbf{x_A \leq 8}$

$\mathbf{x_A \geq 0, x_B \geq 0}$



$10x_A + 20x_B = 150$

x_A	x_B
0	7.5
15	0

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Representing constraints graphically

$$Maximize\ Z = 500x_A + 450x_B$$

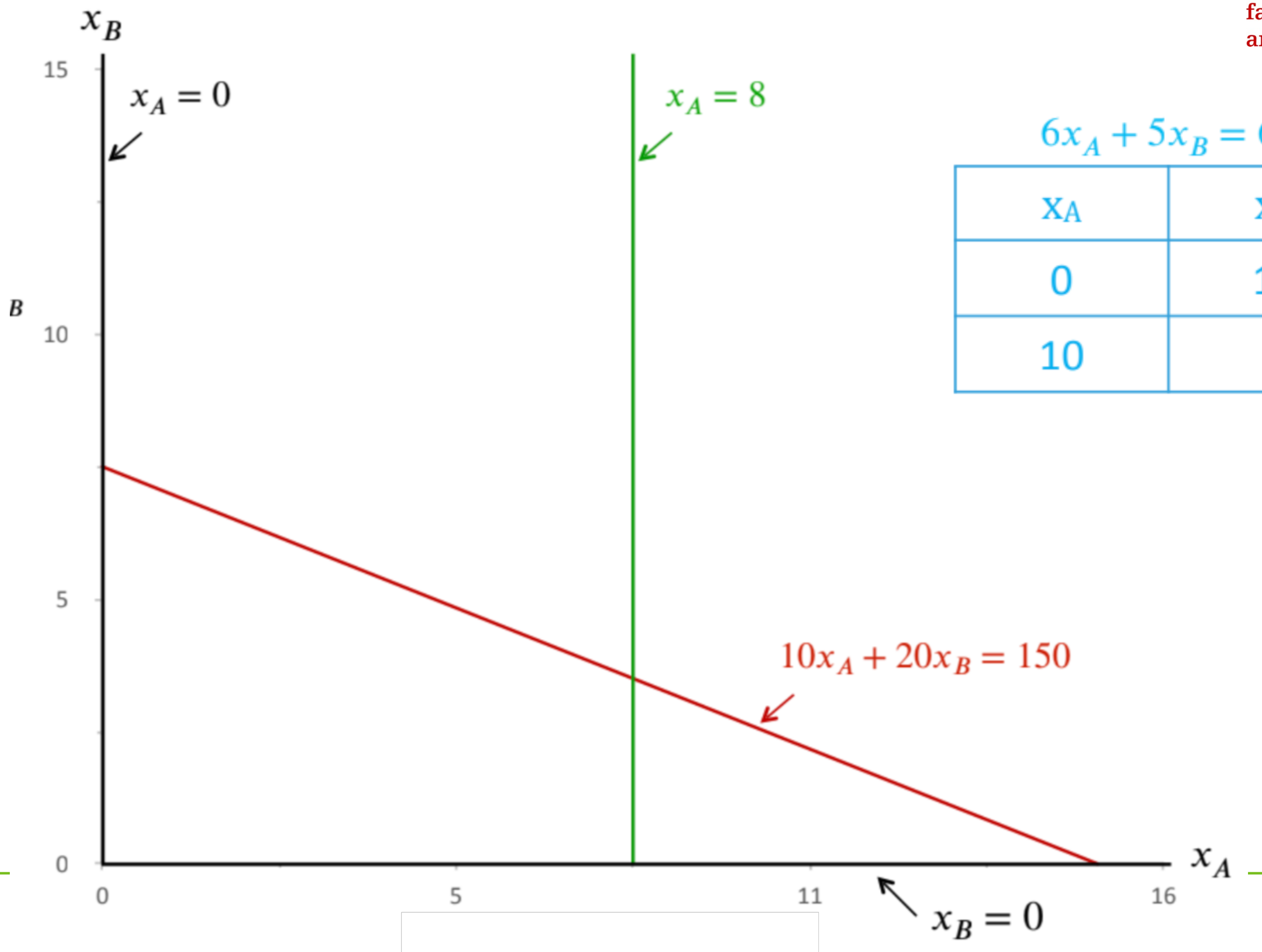
s.t.

$6x_A + 5x_B \le 60$

$10x_A + 20x_B \le 150$

$x_A \le 8$

$x_A \ge 0, x_B \ge 0$



$6x_A + 5x_B = 60$

x_A	x_B
0	12
10	0

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$$Maximize\ Z = 500x_A + 450x_B$$

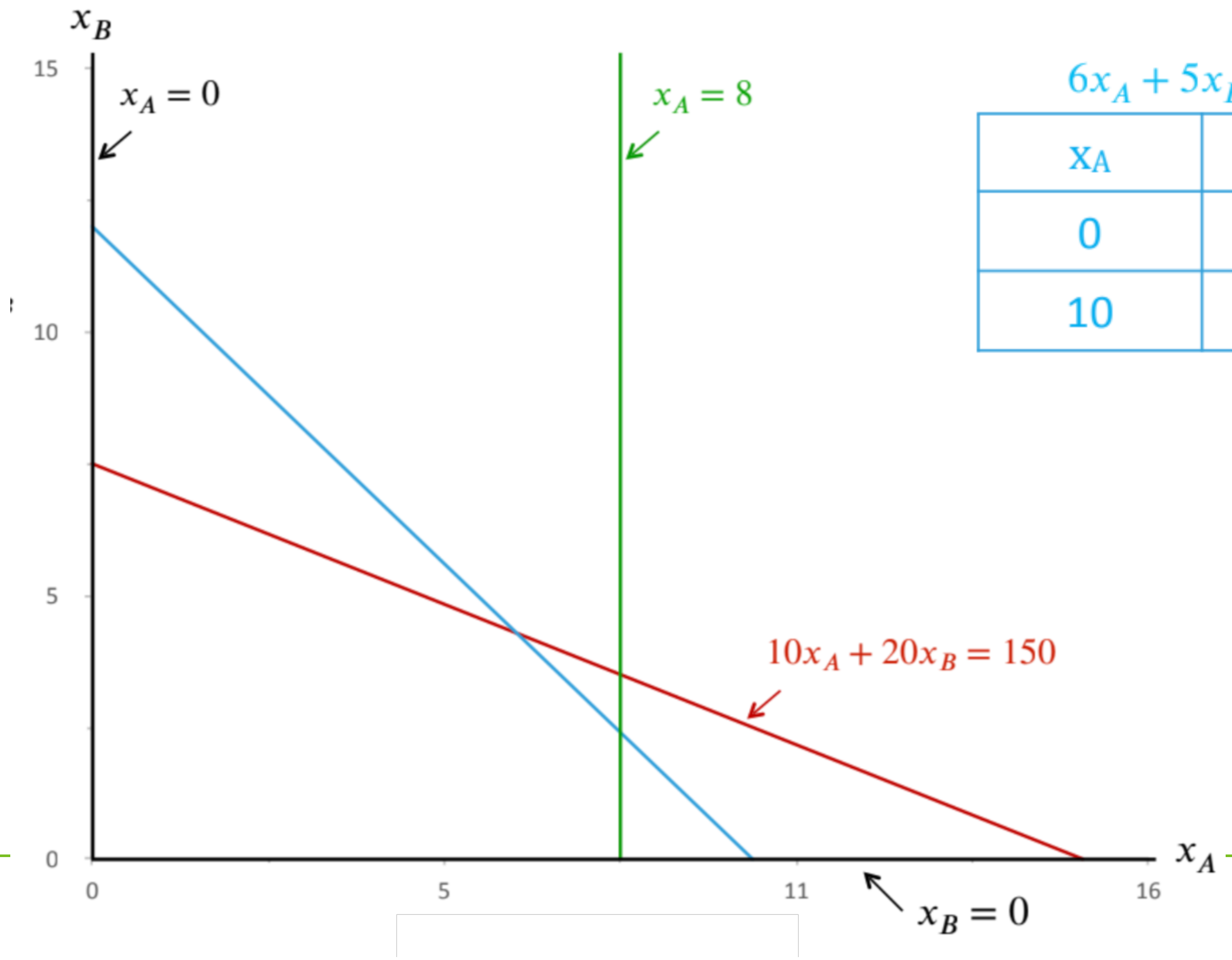
s.t.

$6x_A + 5x_B \leq 60$

$10x_A + 20x_B \leq 150$

$x_A \leq 8$

$x_A \geq 0, x_B \geq 0$



$6x_A + 5x_B = 60$

x_A	x_B
0	12
10	0

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Representing constraints graphically

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$$Maximize\ Z = 500x_A + 450x_B$$

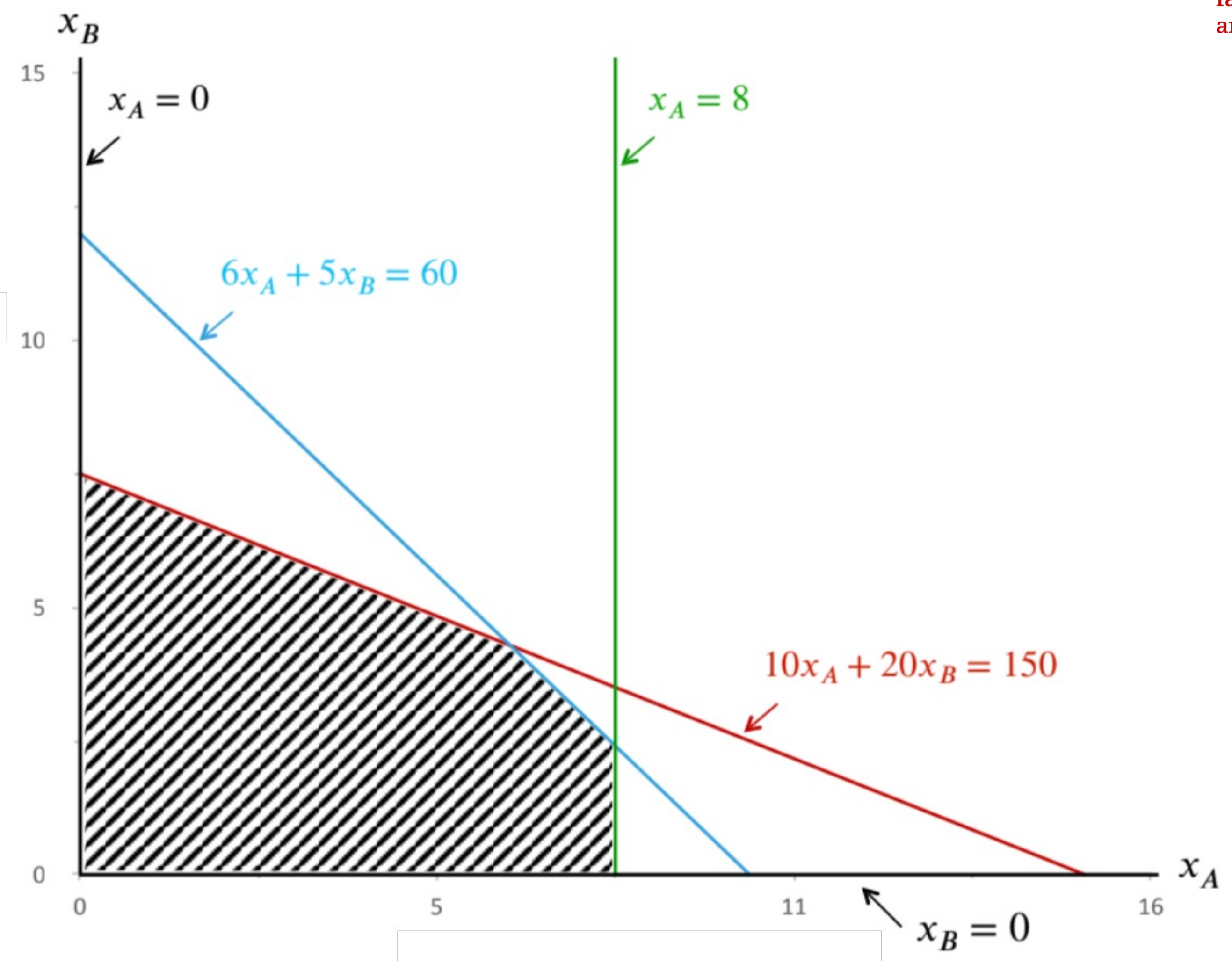
s.t.

$6x_A + 5x_B \leq 60$

$10x_A + 20x_B \leq 150$

$x_A \leq 8$

$x_A \geq 0, x_B \geq 0$



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$$Maximize\ Z = 500x_A + 450x_B$$

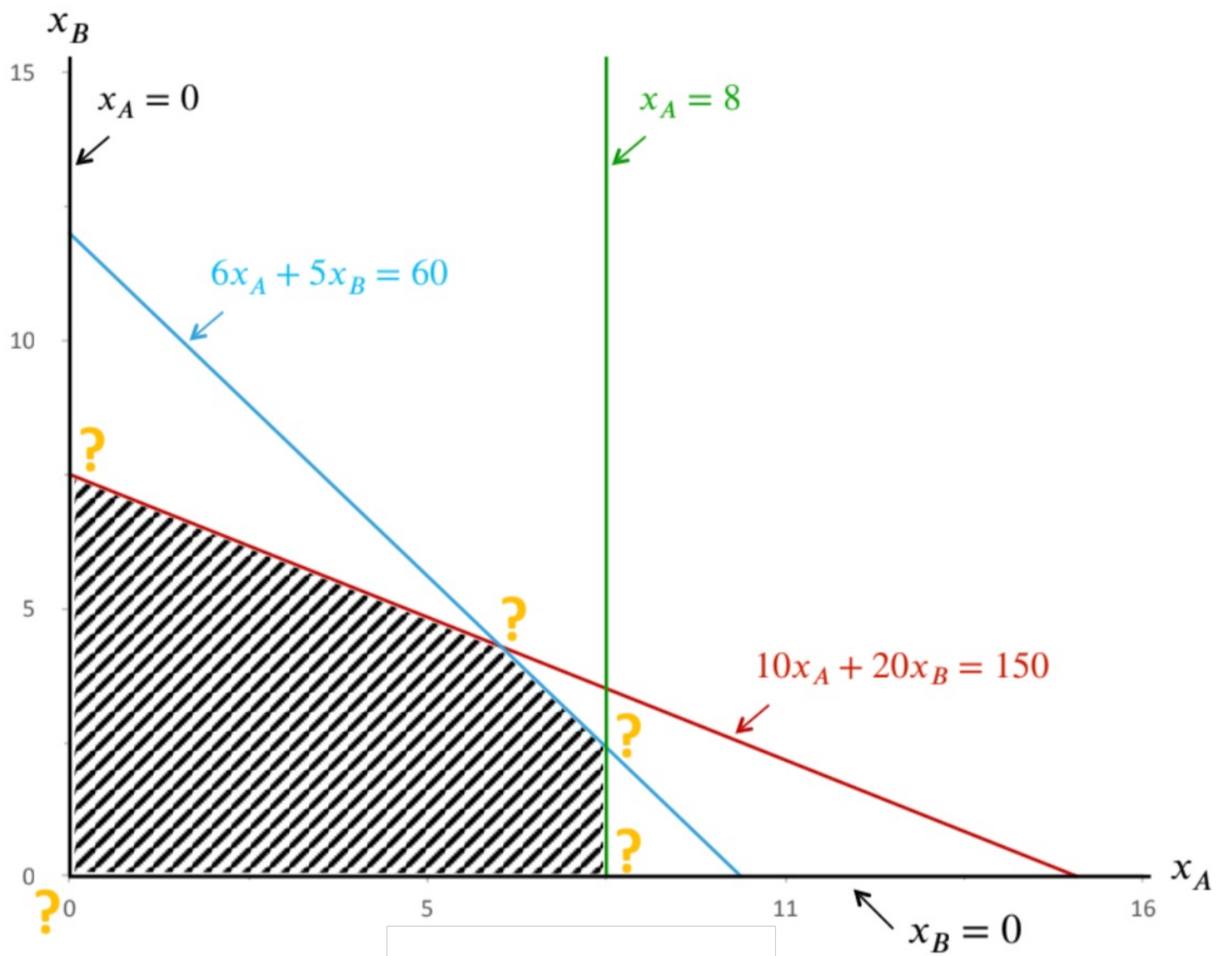
s.t.

$6x_A + 5x_B \leq 60$

$10x_A + 20x_B \leq 150$

$x_A \leq 8$

$x_A \geq 0, x_B \geq 0$



Representing constraints graphically

$$Maximize\ Z = 500x_A + 450$$

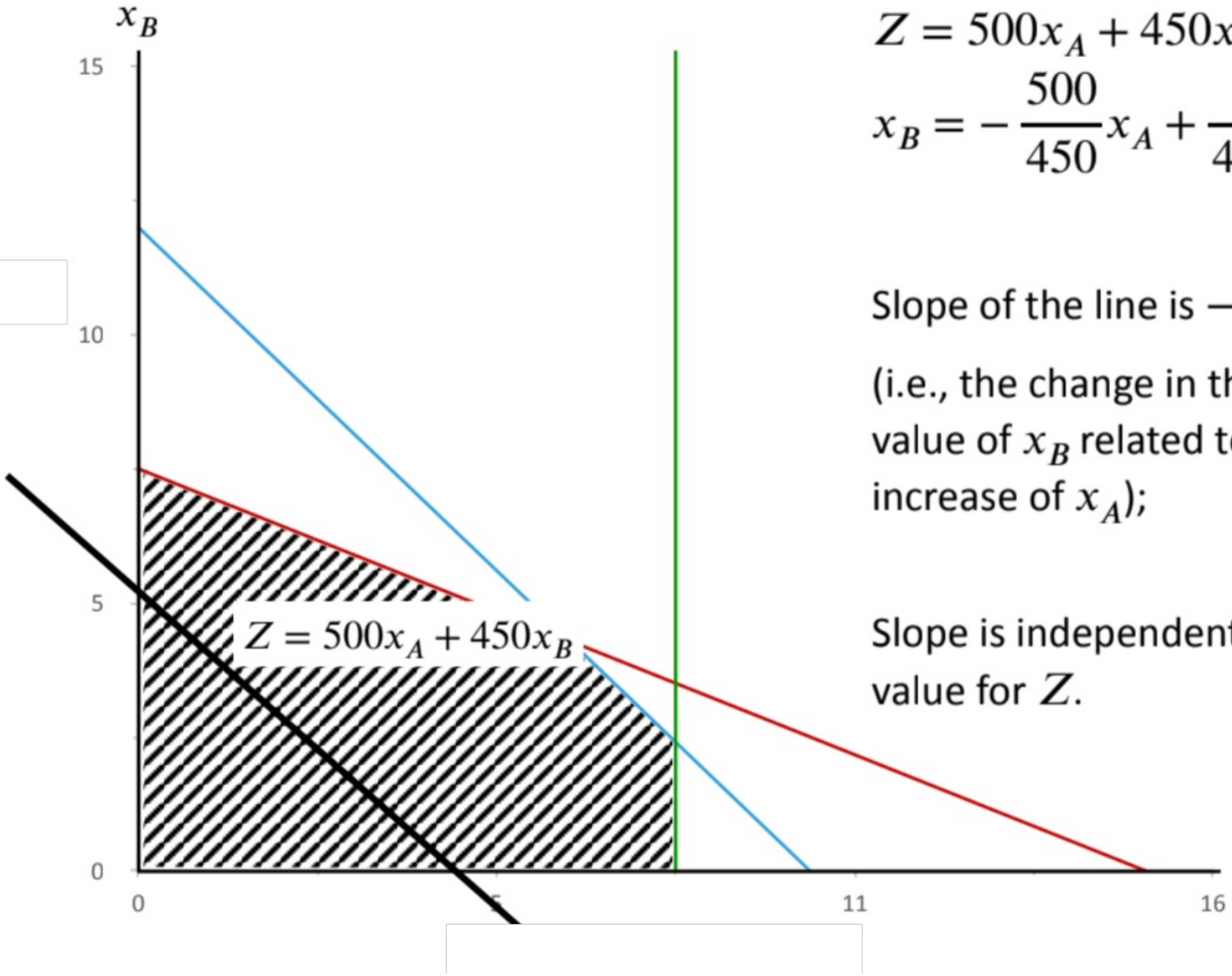
s.t.

$6x_A + 5x_B \leq 60$

$10x_A + 20x_B \leq 15$

$x_A \leq 8$

$x_A \geq 0, x_B \geq 0$



$$Z = 500x_A + 450x_B$$

$$x_B = -\frac{500}{450}x_A + \frac{1}{450}Z$$

Slope of the line is $-\frac{500}{450}$ (i.e., the change in the value of x_B related to unit increase of x_A);

Slope is independent of value for Z.

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Representing constraints graphically

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$$Maximize\ Z = 500x_A + 450x_B$$

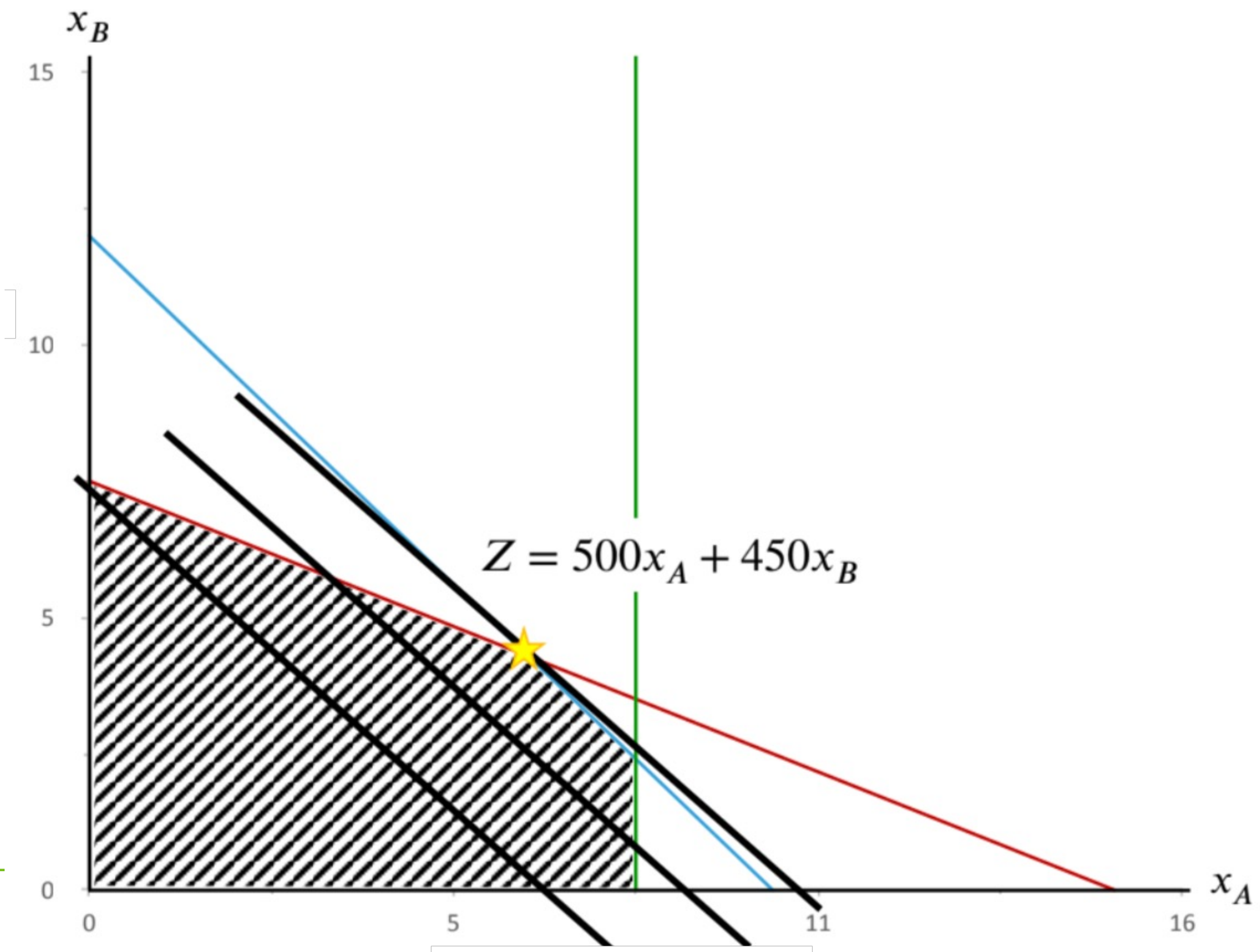
s.t.

$6x_A + 5x_B \leq 60$

$10x_A + 20x_B \leq 150$

$x_A \leq 8$

$x_A \geq 0, x_B \geq 0$



Representing constraints graphically

$$\textit{Maximize } Z = 500x_A + 450x_B$$

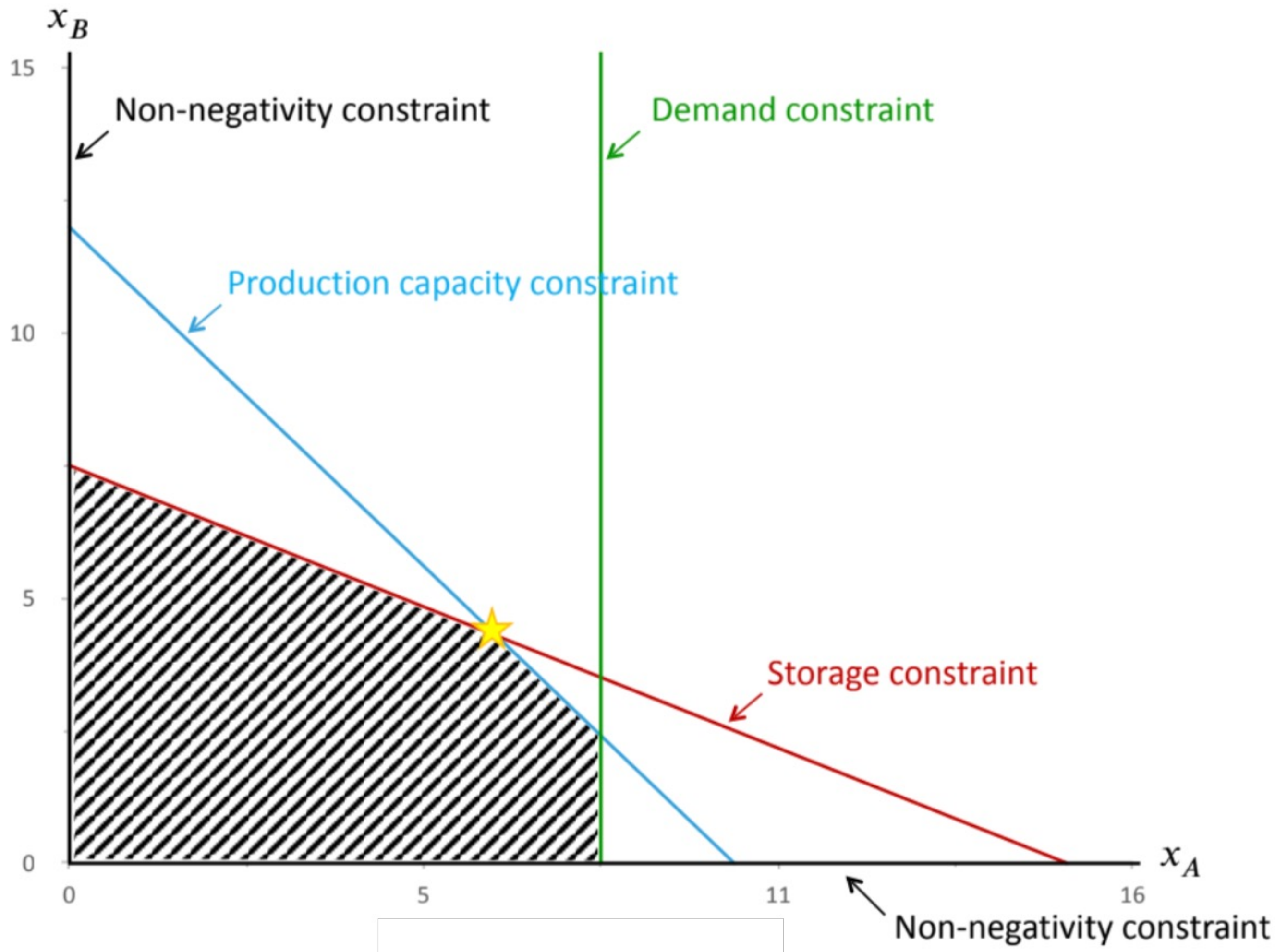
s.t.

$6x_A + 5x_B \leq 60$

$10x_A + 20x_B \leq 150$

$x_A \leq 8$

$x_A \geq 0, x_B \geq 0$



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Compute optimal solution

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The optimal solution lies in the intersection of

$6x_A + 5x_B = 60$ and $10x_A + 20x_B = 150$

$$6x_A + 5x_B = 60$$

$$24x_A + 20x_B = 240$$

$$10x_A + 20x_B = 150 \quad -$$

$$14x_A = 90$$

$$x_A = \frac{90}{14} = 6.43$$

Compute optimal solution

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The optimal solution lies in the intersection of

$$6x_A + 5x_B = 60 \text{ and } 10x_A + 20x_B = 150$$

$$6x_A + 5x_B = 60$$

$$24x_A + 20x_B = 240$$

$$10x_A + 20x_B = 150 \quad -$$

$$14x_A = 90$$

$$x_A = \frac{90}{14} = 6.43$$

$$6x_A + 5x_B = 60$$

$$5x_B = 60 - 6x_A$$

$$x_B = 12 - \left(\frac{6}{5}\right)x_A$$

$$x_B = 12 - \left(\frac{6}{5}\right)6.43 = 4.29$$

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Representing constraints graphically

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$$Maximize\ Z = 500x_A + 450x_B$$

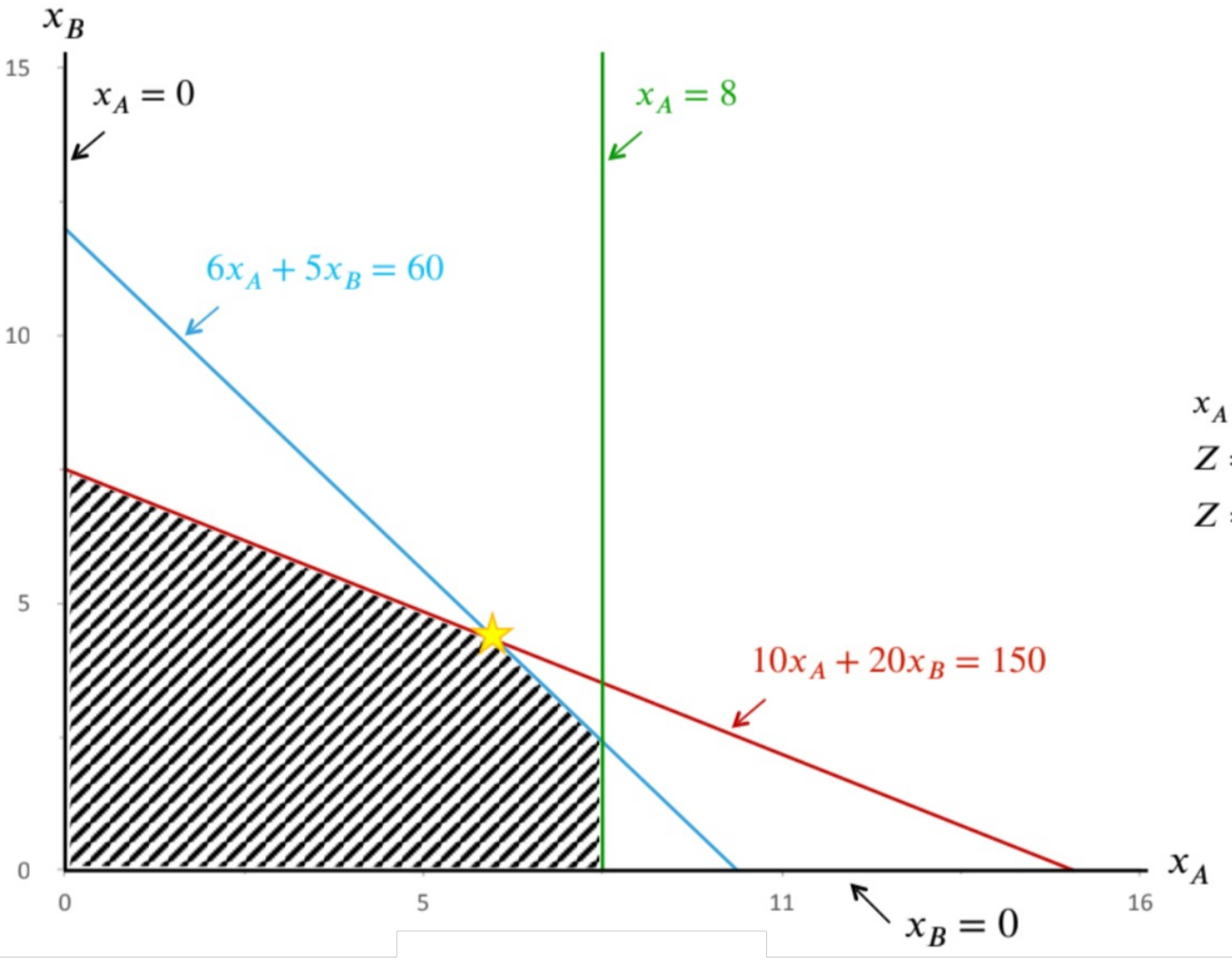
s.t.

$x_A \leq 8$

$10x_A + 20x_B \leq 150$

$6x_A + 5x_B \leq 60$

$x_A \geq 0,\ x_B \geq 0$



$x_A = 6.43;\ x_B = 4.29$

$Z = 500x_A + 450x_B$

$Z = 5142.86$

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Solving Linear Programing Models using Excel

Complete LP model

- **Previous factory example**

$$Maximize\ Z = 500x_A + 450x_B$$

s.t.

$6x_A + 5x_B \leq 60$

$10x_A + 20x_B \leq 150$

$x_A \leq 8$

$x_A \geq 0, x_B \geq 0$

Load the Solver Add-in

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Applies To: Excel 2016, Excel 2013, Excel 2010, Excel 2007

The Solver Add-in is a Microsoft Office Excel add-in program that is available when you install Microsoft Office or Excel.

To use the Solver Add-in, however, you first need to load it in Excel.

1. In Excel 2010 and later goto **File > Options**

   NOTE: For Excel 2007, click the **Microsoft Office Button** , and then click **Excel Options**.

2. Click **Add-Ins**, and then in the **Manage** box, select **Excel Add-ins**.
3. Click **Go**.
4. In the **Add-Ins available** box, select the **Solver Add-in** check box, and then click **OK**.

   NOTES:

   - If the **Solver Add-in** is not listed in the **Add-Ins available** box, click **Browse** to locate the add-in.
   - If you get prompted that the Solver Add-in is not currently installed on your computer, click **Yes** to install it.

5. After you load the Solver Add-in, the **Solver** command is available in the **Analysis** group on the **Data** tab.

https://support.microsoft.com/en-us/office/load-the-solver-add-in-in-excel-612926fc-d53b-46b4-872c-e24772f078ca

Set up the model in Excel

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$$\begin{aligned} Max \quad & 500x_A + 450x_B \\ s.t. \quad & 6x_A + 5x_B \leq 60 \\ & 10x_A + 20x_B \leq 150 \\ & x_A \leq 8 \\ & x_A \geq 0, x_B \geq 0 \end{aligned}$$

Objective function

Variable names

The right-hand side of the 3 constraints, e.g., constraint (1) “$6x_A + 5x_B \leq 60$”

The left-hand side of the 3 constraints, e.g., constraint (1) “$6x_A + 5x_B \leq 60$”

	A	B	C	D	E	F	G
1		XA	XB				RHS
2	Profit	500	450				
3					Output cells		
4	Constraint (1)	6	5		= SUMPRODUCT(B4:C4;$B$9:$C$9)	<=	60
5	Constraint (2)	10	20		= SUMPRODUCT(B5:C5;$B$9:$C$9)	<=	150
6	Constraint (3)	1			= SUMPRODUCT(B6:C6;$B$9:$C$9)	<=	8
7							
8					Total profit:		
9	Changing cells				=SUMPRODUCT(B2:C2;B9:C9)		

Apply the Excel Solver

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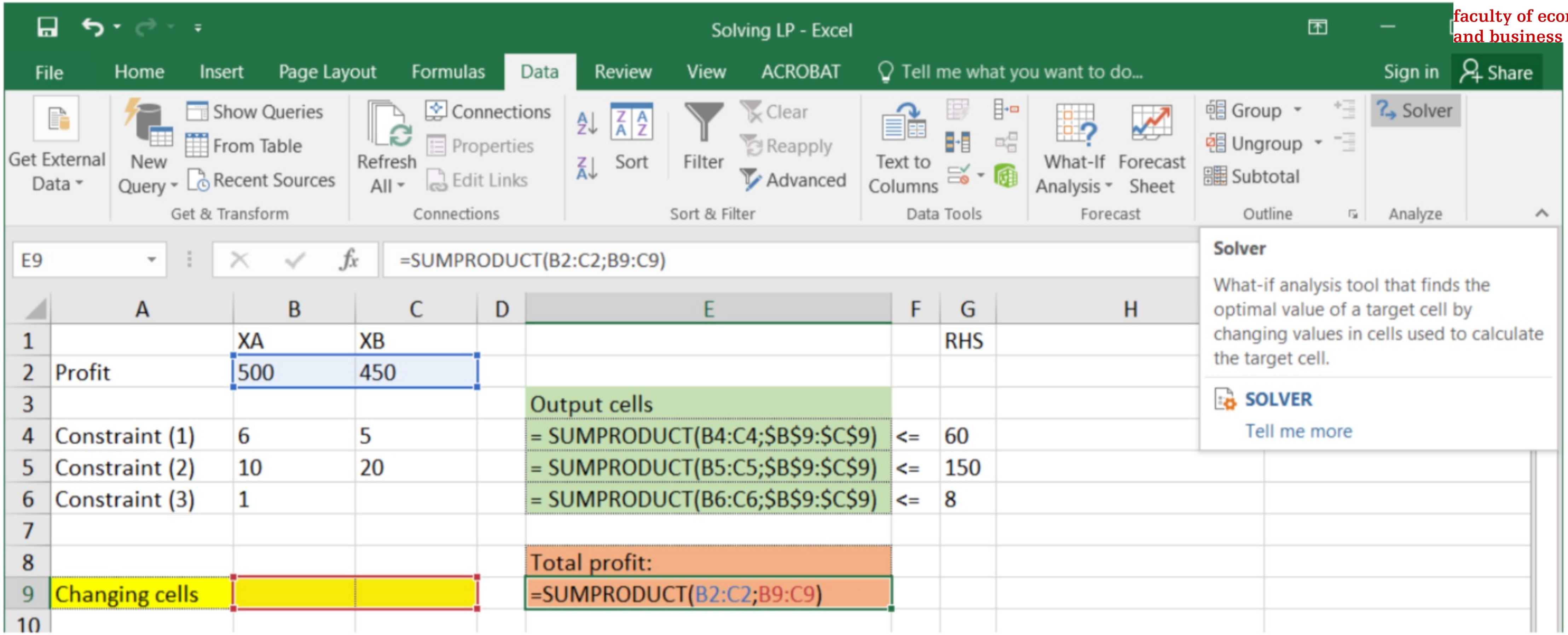

Excel Solver Parameters

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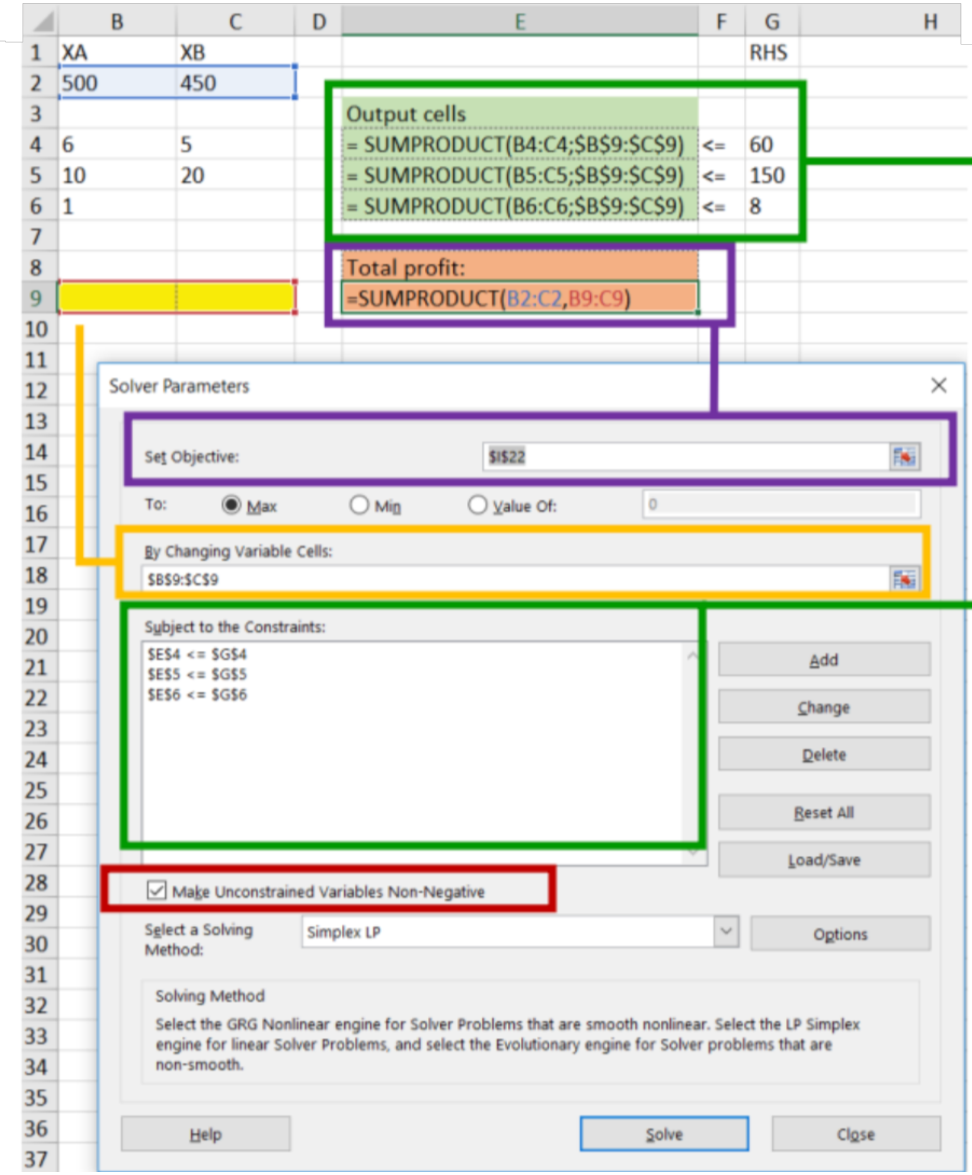
Select the objective value

Indicate cells that can be changed



Add the constraints

To ensure $x_A \geq 0, x_B \geq 0$

See/Interpret results

Production and storage constraints have no slack; only demand constraint has

	A	B	C	D	E	F	G
1		XA	XB				RHS
2	Profit	500	450				
3					Output cells		
4	Constraint (1)	6	5		60	<=	60
5	Constraint (2)	10	20		150	<=	150
6	Constraint (3)	1			6,428571429	<=	8
7							
8					Total profit:		
9	Changing cells	6,4285714	4,2857143		5142,857143		

Optimal values for x_A and x_B

Value of objective function Z

Integer variables in Excel

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- In the factory example, decision variables x_A and x_B are continuous variables, as specified in

$$x_A \geq 0, x_B \geq 0$$

- What if we want the decision variables to be integers?

$$x_A, x_B \geq 0 \text{ and integers}$$

- Excel Solver has options for integer and binary variables

Integer variables in Excel

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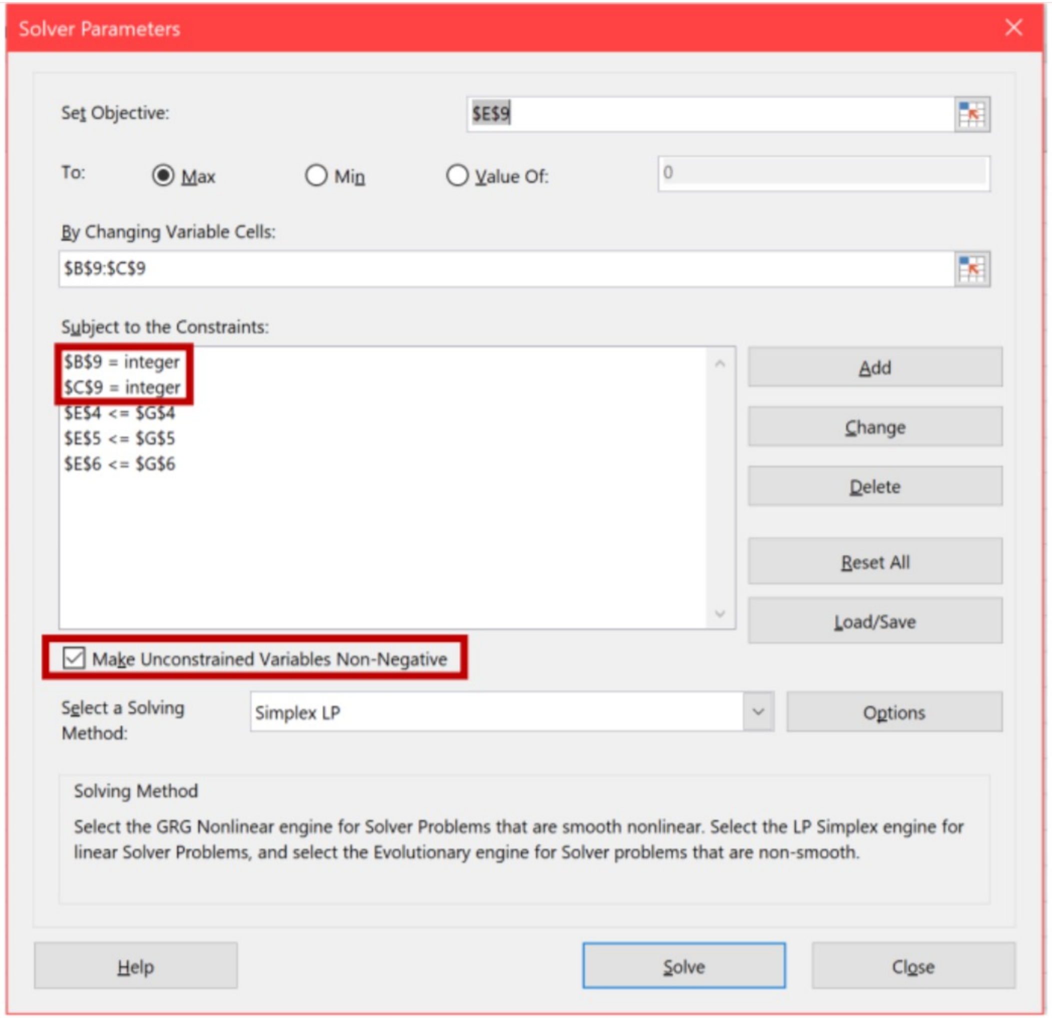
To ensure $x_A, x_B \geq 0$ and integers



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Integer variables in Excel

- **The optimal values of the decision variables are now integer**

	XA	XB				RHS
Profit	500	450				
				Output cells		
Constraint (1)	6	5		58	<=	60
Constraint (2)	10	20		120	<=	150
Constraint (3)	1			8	<=	8
				Total profit:		
Changing cells	8	2		4900		

- **Note that the profit is also lower than before, which is due to the fact that we impose an additional restriction (to be integers) on the decision variables**

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LP in Operations & Supply Chain Management

Some applications of LP in Operations & Supply Chain Management

- Routing problems (lecture 6)
- Transshipment problem (lecture 7)
- Facility location problems (lecture 9)
- Scheduling (integer)
- Production planning

Discussion points

- **Think about what kind of data we need to have to find the optimal solutions**
  - Are they easy to measure/collect?
  - What are the limitations of those data?
  - What are the limitations of the models?

Discussion

Limitations of data

Limitations of model

How can we overcome those limitations?

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The Fresh Connection

First round will start after the lecture

- **Any questions before we start?**
- **Code for the game: C8Nh-z8fh**
- **Deploy the game!**
- **Good luck!**

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Thank you!

Questions?

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